Double Theodolite Pibal Evaluation by Computer

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ABSTRACT

The position of a pilot balloon, which is being followed by two theodolites, can be computed by using all four theodolite angles to make an optimum estimate of the balloon's position and also indicate the probable magnitude of the error in its position. The method described does not fail when the balloon is near or over the base-line, and it is suitable for digital computers.

1. Introduction

When making measurements of upper winds by pilot-balloon, two theodolites are sometimes used to follow the balloon, and so determine its position accurately at any particular moment, without having to make any assumptions about its rate of ascent or the presence of vertical currents. Each of the two theodolite readings consists of an elevation and an azimuth angle, which together define a ray from the theodolite to the balloon. These two rays should intersect at the balloon.

The balloon's position in space at any particular instant can be specified by 3 coordinates. However, the two theodolites give 4 coordinates from which the balloon's position can be determined. Therefore, these 4 coordinates must satisfy a certain relationship; if this relationship is fulfilled, the two rays will intersect. If it is not fulfilled, the two rays will not intersect, in which case the usual method of computation can give two different computed heights for the balloon.

Unfortunately, the cases in which the rays do not intersect are somewhat frequent. There are several possible reasons for this, e.g., the stop-watches at the two theodolites may not be properly synchronized, an error of 1 deg, 5 deg or 10 deg may be made in reading the scale, or one theodolite may not be correctly adjusted, leveled and oriented, and mistakes of this nature are difficult to correct (though an error of 10 deg in a scale reading will in many cases be obvious from inspection of the readings). However, even if none of these faults is present, an exact agreement may not be obtained because of limitations of accuracy of the theodolites. If readings are made to the nearest 0.1 deg, then the balloon may be anywhere within a cone of semi-vertical angle 0.05 deg whose axis coincides with the ray defined by the elevation and azimuth angles, and whose vertex is at the theodolite. Thus the balloon may be anywhere within the region of intersection of the two cones whose axes are defined by the theodolite angles, and consequently one can only give a "most probable" location for the balloon. If the two cones do not intersect, and no other errors are obvious, one can only assume that the vertical angles of the cones (corresponding to the angular errors present) must be increased until they do intersect.

2. Principle of method

So now the problem is to find the region of intersection of the two cones, and the balloon's most probable location in this region. Somewhere, the two rays will approach within a certain shortest distance of each other, so that they can be joined by a certain line of minimum length which we can refer to as the "short line." Further, if the short line is divided internally in the ratio of the diameters of the cones where it intersects them, then this point of division is the point most likely to be in the region of intersection, and can be taken to be the balloon's most probable position.

A vector method has been found to be a satisfactory means of solving this problem. Each of the two rays can be described by a position vector, of known direction but unknown length, drawn from its respective theodolite, A (#1) or D (#2). (See Fig. 1.) The short line BC will be perpendicular to both rays. Therefore the direction of BC can be determined. There now remain 3 unknowns, viz., the lengths of BC and of the two rays, #1 (AB) from theodolite #1 and #2 (DC) from theodolite #2. Now the position of C relative to A can be expressed in two different ways: the vector sum of AB and BC, or the sum of AD and DC. These two vector sums must agree, and so we get a vector equation, equivalent to 3 scalar equations, with 3 unknowns. The solution gives all the information required to estimate the balloon's most probable position, even when the balloon is over the base-line.

3. Mathematical details

Let AB, or $r_1(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})$, be the position vector of the junction of ray #1 and the short line, relative to theodolite #1.

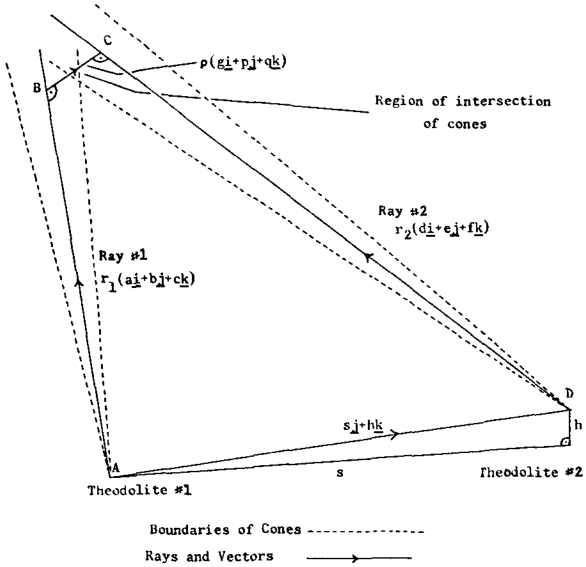

FIG. 1. Diagram of rays and vectors.

- Let DC, or $r_2(d\mathbf{i}+e\mathbf{j}+f\mathbf{k})$, be the position vector of the junction of ray #2 and the short line, relative to theodolite #2.
- Let BC, or $\rho(g\mathbf{i}+p\mathbf{j}+q\mathbf{k})$, represent the short line, relative to its junction with ray #1, ρ being its length.
- a, b, c, d, e, f, g, p and q are the appropriate direction cosines, and r_1 and r_2 are the lengths of the rays.
- AD, or $s\mathbf{j}+h\mathbf{k}$, represents the base-line, s being its horizontal length and h the height of theodolite #2 above theodolite #1.

a, b, c, d, e and f can be found from:

$$a=\cos El_1 \sin Az_1,\quad b=\cos El_1 \cos Az_1,\quad c=\sin El_1,$$
$$d=\cos El_2 \sin Az_2,\quad e=\cos El_2 \cos Az_2,\quad f=\sin El_2,$$

provided that theodolite #1 has its 360 deg azimuth mark set on #2, and #2 has its 180 deg mark set on #1, El_1, Az_1, El_2 and Az_2 being the elevation and azimuth angles measured from theodolites #1 and #2.

As the short line is perpendicular to each of the rays, its scalar product with each ray will be zero.

$$\therefore \quad ag+bp+cq=0$$
$$\text{and} \quad dg+ep+fq=0.$$

These can be solved to give g, p and q, under the condition that $g^2+p^2+q^2=1$ (as they are direction cosines for one line.)

If $\alpha=ce-bf$, $\beta=af-cd$, $\gamma=bd-ae$, $\Delta=(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}}$, then $g=\alpha/\Delta$, $p=\beta/\Delta$, $q=\gamma/\Delta$.

The vector AC can be expressed in two different ways, as AB+BC or AD+DC, and these must be equivalent.

$$\therefore \quad r_1(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})+\rho(g\mathbf{i}+p\mathbf{j}+q\mathbf{k}) = s\mathbf{j}+h\mathbf{k}+r_2(d\mathbf{i}+e\mathbf{j}+f\mathbf{k}).$$

Equating components,

$$ar_1-dr_2+g\rho=0,$$
$$br_1-er_2+p\rho=s,$$
$$cr_1-fr_2+q\rho=h.$$

These can be solved to give r_1, r_2 and ρ, which will be respectively

$$[s(dq-fg)+h(ge-dp)]/D,\quad [s(aq-cg)+h(bg-ap)]/D,$$

and

$$[h(bd-ae)+s(af-cd)]/D=(s\beta+h\gamma)/D,$$

where

$$D=\begin{vmatrix} a & -d & g \\ b & -e & p \\ c & -f & q \end{vmatrix}=q\alpha+p\beta+q\gamma.$$

Finally, the desired position of the balloon will be given by

$$r_1(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})+[\rho r_1/(r_1+r_2)](g\mathbf{i}+p\mathbf{j}+q\mathbf{k}),$$

i.e.,

$$x=r_1a+[\rho r_1/(r_1+r_2)]g,$$
$$y=r_1b+[\rho r_1/(r_1+r_2)]p,$$
$$z=r_1c+[\rho r_1/(r_1+r_2)]q.$$

(The last equation may include an extra term to compensate for the earth's curvature if desired.)

4. Treatment of incomplete data

There will probably occur certain times during the ascent when one or both readings from one theodolite are missing. In this case, it is probably best to proceed to the next complete set of 4 coordinates. After the height of the balloon has been computed at every point having such a complete set of 4 readings, the height can be determined for other points by interpolation, and so the position calculated from the existing readings by the usual method for a single theodolite.

It might be argued that when there is only one angle missing, the remaining 3 may be enough to determine the balloon's position. However, experience has shown that very unreliable results are likely to follow adoption of this method. If, for instance, a complete set of 4 angles is obtained at any given instant, and these determine two rays which do not intersect, there are four different ways of choosing a combination of 3 angles from which to compute the balloon's position, and the positions given by the 4 possible computations may be vastly different. If the position is computed using different combinations of angles for successive times, an extremely erratic trajectory can result. In the case of the balloon being nearly over the base-line and its position determined from two azimuths and one elevation, there would be a small angle of intersection between the azimuths, and the accuracy of its determined position would be very low. In fact, in the extreme case of the balloon being right over the base-line, its position could not be determined at all from such a set of readings. For balloons rising at a rate of 150 m/min or more, the method of interpolation of heights has been found to give reasonably smooth trajectories.

Wind velocity data can be obtained from the trajectory of the balloon using any of the standard methods.

5. Conclusions

Methods previously described, such as those of Hansen and Taft, and of Weedfall and Jagodzinski, use the two azimuth angles to determine the balloon's horizontal coordinates. The latter are susceptible to significant error when the balloon comes close to the base-line, and are indeterminate when the balloon crosses it. Such cases are not covered by the older methods, and it is not always possible to choose one's base-line in such a way that these cases do not occur. The method described here does not fail when the balloon crosses the base-line, as it uses all four elevation and azimuth angles simultaneously. Further, the older methods do not indicate the magnitude of the discrepancy which may result from the use of four readings to determine three coordinates.

The error of the result depends on (1) the experimental error of the original readings and (2) the error inherent in the computer.

In the case of the IBM 709 computer, which has been used for this purpose, the latter error is considerably less than the former. Thus it is possible for the computer output to give an estimate of the accuracy of the balloon's position, in terms of the discrepancy resulting from the use of four readings, by printing the value of ρ, and also $\delta=0.05(r_1+r_2)/57.2958$, which should be the largest permissible value of $|\rho|$ if all the theodolite readings are accurate to the nearest 0.1 deg.

Comparison of computer output with a carefully-made graphical plot of the same ascent gave agreement of positions to within a couple of meters.

Once the computer program has been compiled, very little effort is needed to obtain the final results. For a 30-interval run, about 3 minutes are required for card-punching, plus another 2 minutes for checking if desired. On a 709 Computer, the computing time is about 6 seconds, costing 40 cents.

This method has not yet been adapted to manual and graphical techniques, and so it is at present unsuitable when quick results are required in the absence of a high speed computer.

REFERENCES

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Weedfall, R. O., and W. M. Jagodzinski, 1961: Comments on double-theodolite evaluations. *Bull. Amer. meteor. Soc.*, **42**, 322–324.