Bent-Over Vapor Plumes

G. T. CSANADY

Dept. of Mechanical Engineering, University of Waterloo, Ontario, Canada

(Manuscript received 30 July 1970)

ABSTRACT

A theory is developed to predict the behavior of bent-over continuous plumes of substantial moisture content in the atmosphere. Main interest is focussed on the prediction of condensation. It is found that in a well-mixed environment, under appropriate conditions, a finite portion of the plume may contain some liquid-phase water, beginning to condense at some distance from the source and re-evaporating at some larger distance. Plumes, which are initially saturated and contain some liquid, will begin to re-evaporate immediately on release, provided that the environment is not saturated and that the excess temperature of the plume is *smaller* than a given limit (of the order of 20F at 65% relative humidity). The direct dynamical effects of evaporation and condensation of plume path are found to be minor. In a *stratified* environment the calculations become more complex but the above conclusions are fundamentally unchanged (except perhaps that there may be several finite portions of the plume exhibiting condensation).

1. Introduction

A problem of some current interest is the behavior of continuous plumes in the atmosphere under conditions giving rise to condensation, cloud and fog formation, and "rainout" of droplets which may contain acids or solid particles. Certain industrial plants (e.g., cement or smelter works) may produce heavily moisture-laden effluents prone to such problems. Electric power stations which "scrub" their effluent gases to remove undesirable gaseous constituents, such as SO₂, may exchange their pollution problem for those involving condensation. The most important sources of this particular environmental nuisance, however, are probably the cooling towers of large electric power stations relying on the atmosphere for the disposal of their waste heat. In cooling towers with direct contact between cooling water and air a large portion of the waste heat is absorbed as latent heat of evaporation, so that a large stream of air with high vapor content and a relatively low excess temperature leaves the cooling tower and enters the atmosphere. Nuclear power stations with their relatively greater waste heat load and otherwise "clean" operation may be particularly plagued by vapor-plume side effects.

As compared to a chimney plume which remains "dry" throughout, the principal new problem with a heavily moisture-laden plume is to predict whether or where condensation will occur. In addition, there may be important dynamical side effects; release of latent heat by condensation may significantly increase the buoyancy of a hot plume, while the weight of liquid droplets suspended in it may depress it. In the case of industrial plumes it would be important to know whether such effects significantly modify the height of the plume above ground level, and with it ground level concentrations of pollutants.

Previous theoretical work on plumes and thermals falls broadly into two categories: one, an "integral" kind of approach in which the *total* buoyancy, momentum, etc., in a thermal or plume section is considered; two, a detailed consideration of the flow pattern by means of a differential equation, whether solved analytically or numerically. The first approach is quite adequate to predict some of the more important main features of plume behavior [for example, the mean path of a chimney plume in a uniform wind (Slawson and Csanady, 1967, 1970)], and has been used widely and successfully in the past, particularly since the publication of Morton *et al.* (1956). The present paper constitutes a further application of this approach, this time to the bent-over vapor plume problem.

The only major contribution to the theory of *continuous* vapor plumes on this basis seems to be that of Morton (1957), who considered vertical vapor plumes in a quiescent atmosphere. This paper is a direct sequel to the analysis of Morton *et al.*, the key step in which is the assumption that there is an identifiable "plume" into which there is an influx ("entrainment") from the environment. The exact way in which properties are distributed within the plume is then of relatively little significance and the simple device may be adopted of assuming excess temperature, vertical velocity, etc., constant for a cross section of a plume (a "top hat" profile, i.e., constant property within the plume, zero without).

A short distance downstream of the source, in a moderately strong wind, atmospheric chimney or cooling tower plumes become nearly horizontal. The dynamics of such *bent-over* plumes differs somewhat from that of the vertical plumes studied by Morton (1957), partly because atmospheric turbulence becomes important in the mixing process at some stage in the

plume's growth. The theory of *dry* plumes in such circumstances has been dealt with extensively in Slawson and Csanady (1967, 1970), on the basis of the "integral" approach, and the results were found to provide a satisfactory framework for the understanding of complex observed phenomena in plume rise. The present paper is an extension of this work to plumes with high moisture content and is addressed primarily to the prediction of where condensation occurs in nearly-horizontal chimney or cooling tower plumes, and how this can be avoided.

2. Conservation laws

Consider a slice of an identifiable "plume" into which there is some influx of ambient air ("entrainment") (Fig. 1). Let the plume be (nearly) parallel to the x axis, along which the wind speed is a constant U . The density difference between plume and the ambient air may be considered negligible, except when acted upon by the force of gravity. If the plume radius is R , and if the influx (entrainment) velocity at the perimeter is v_e , conservation of mass requires that

$$U \frac{dR^2}{dx} = 2v_e R. \quad (1)$$

The entrainment velocity depends on what physical mechanism dominates the mixing process. In a plume of moderately strong buoyancy the turbulence responsible for entrainment of ambient air is often mainly generated by the plume's own vertical motion through its relatively quiet surroundings. For such a case it is usual to write

$$v_e = \alpha w, \quad (2a)$$

where w is the vertical component of plume velocity and α an "entrainment constant." When the vertical movement is slow, the turbulence naturally present in the wind dominates the entrainment; thus, according to the size of the plume element, the entrainment velocity may be

$$v_e = a_1 \epsilon^{1/3} R^{1/3}, \quad (2b)$$

or

$$v_e = a_2 v_m (L_t/R), \quad (2c)$$

where ϵ is the turbulent energy dissipation per unit mass, v_m the rms turbulence velocity, and L_t a turbulence length scale. Eq. (2b) applies if R is small compared to L_t (diffusion by eddies in the inertial subrange), while (2c) is applicable where R is large compared to L_t . There is reasonable evidence for the validity of these assumptions with regard to v_e in chimney plumes (Slawson and Csanady, 1970), and the three equations (2a), (2b) and (2c) characterize, respectively, the "initial," "intermediate" and "final" phase of plume growth.

Eq. (1) with (2b) and (2c) fully determine the rate of growth of a plume in the "intermediate" and "final" phases.

FIG. 1. Slice of nearly horizontal plume.

In the "initial" phase [Eq. (2a) valid] the upward velocity w also influences the rate of growth.

In the (vertical) equation of motion we also have to consider the weight of any *liquid* droplets suspended in the plume (in addition to the plume's buoyancy). If, following Morton (1957), we introduce the mass-fraction σ of liquid water present in the plume, the vertical momentum equation becomes

$$U \frac{d(R^2 w)}{dx} = R^2 b - R^2 g \sigma, \quad (3)$$

where b , the buoyant acceleration, is

$$b = g \frac{\rho_a - \rho_p}{\rho_p} = g \frac{T_p - T_a}{T_a} = \frac{g \Delta T}{T_a}, \quad (4)$$

with T_p , T_a being the absolute temperatures in the plume and the ambient atmosphere, g the acceleration of gravity, and ρ_p , ρ_a the corresponding densities. In specifying the conservation of energy we have to take into account the effect of atmospheric (potential) temperature changes in the vertical; this is best introduced through the Brunt-Väisälä frequency N , i.e.,

$$N^2 = \frac{g}{T_a} \frac{d\theta_a}{dz}, \quad (5)$$

where θ_a is the atmospheric potential temperature at level z . In an unstable atmosphere N^2 is negative. A new factor in the case of a vapor plume is the possibility of evaporation and/or condensation. If we neglect any "rainout" of liquid droplets, the rate of condensation is proportional to $d(R^2 \sigma)/dx$. This liberates latent heat affecting the energy balance. For small temperature differences the heat content of an element is proportional to its buoyant acceleration [see Eq. (4)], so that much as in the earlier work on dry plumes, but including the latent heat effect, we obtain the equation

$$U \frac{d(R^2 b)}{dx} = -R^2 w N^2 + \left(\frac{g L_v}{c_{pa} T_a} \right) U \frac{d}{dx} (R^2 \sigma), \quad (6)$$

where L_v is the latent heat of evaporation, and c_{pa} the specific heat of air at constant pressure.

Again using Eq. (1) the balance of water (liquid plus vapor) yields

$$U \frac{d}{dx} [R^2(q_p - q_a) + R^2\sigma] = -R^2 w \frac{dq_a}{dz}, \quad (7)$$

where q_p and q_a are the specific humidities in the plume and in the ambient air. The latter may again be a function of elevation.

The above set of four conservation laws (mass of air and vapor, momentum, and energy) for the five variables, R , w , b , σ and q_p , must be supplemented by one other relationship describing any phase changes in the plume. If the plume is dry everywhere, the extra relationship is $\sigma=0$ at all x . On the other hand, when liquid water is present, thermodynamic equilibrium between liquid and vapor may be assumed—with limitations well known from cloud physics, which in chimney and cooling tower plumes with their many impurities may not be serious. For a saturated or wet plume we may therefore use the Clausius-Clapeyron equation

$$\frac{dq_s}{dT} = \frac{q_s L_v}{R_v T^2}, \quad (8)$$

where R_v is the gas constant for vapor (at constant pressure); or we may rely on charts or tables giving the saturation specific humidity as a function of pressure and temperature, i.e.,

$$q_s = q_s(p, T). \quad (9)$$

In bent-over plumes total pressure variations are usually small.

The four conservation laws may be given a particularly simple form if the following "flux variables" are introduced:

$$\left. \begin{aligned} M &= UR^2 w && \text{(momentum flux)} \\ F &= UR^2 b && \text{(flux of buoyancy)} \\ S &= UR^2 \sigma g && \text{(flux of suspended liquid, i.e., its weight)} \\ H &= UR^2 \Delta q && \text{(flux of humidity)} \end{aligned} \right\}, \quad (10)$$

where $\Delta q = q_p - q_a$.

For the *initial* phase of plume rise [Eq. (2a) valid], we then obtain the relationships

$$\left. \begin{aligned} U \frac{dR^3}{dx} &= 3\alpha \frac{M}{U} \\ U \frac{dM}{dx} &= F - S \\ U \frac{d}{dx} (F - \lambda S) &= -MN^2 \\ U \frac{d}{dx} \left(H + \frac{1}{g} S \right) &= -MG \end{aligned} \right\}, \quad (11)$$

where we have written

$$G = dq_a/dz, \quad \lambda = L_v/(c_{pa} T_a). \quad (12)$$

3. Well-mixed environment

Fairly simple results are obtained for the case of a well-mixed environment, where, without thermal stratification or humidity gradient, we have

$$N^2 = G = 0. \quad (13)$$

Such conditions may be assumed to apply to a satisfactory degree of approximation over a first portion of the plume's rise (but excluding the immediate vicinity of the chimney), even if there are moderate vertical gradients present, as is well known from the initial behavior of "dry" plumes. The conservation laws for heat and vapor under such conditions become

$$\left. \begin{aligned} \frac{d}{dx} (F - \lambda S) &= 0 \\ \frac{d}{dx} \left(H + \frac{1}{g} S \right) &= 0 \end{aligned} \right\}. \quad (14)$$

Combining these two equations, we find that

$$\frac{d}{dx} (F + \lambda g H) = 0. \quad (15)$$

If the vapor in the plume is superheated and there are no liquid droplets in suspension, $\sigma=0$ everywhere and the above equations reduce to

$$\left. \begin{aligned} S &= 0 \\ H &= H_0 = \text{constant} \\ F &= F_0 = \text{constant} \end{aligned} \right\}. \quad (16)$$

If, on the other hand, both vapor and liquid-phase water are present, on the assumption of thermodynamic equilibrium and at essentially constant absolute pressure (i.e., supposing that any vertical displacements of the plume are small enough), we may write

$$q_p = q_s(T_p),$$

where q_s is the saturation specific humidity at absolute temperature T_p .

Thus, for given initial conditions the *excess* humidity, $\Delta q = (q_p - q_a)$, will be a single-valued function of the excess temperature, $\Delta T = T_p - T_a$. It then follows from Eq. (15), which is equivalent to

$$\left(\Delta T + \frac{L_v \Delta q}{c_{pa}} \right) R^2 = \text{constant}, \quad (15a)$$

that R^2 can also be expressed as a function of ΔT . By Eq. (14) it is then also possible to express σ as a function of ΔT , so that all three fluxes, F , S and H , can be

calculated for a given stage of cooling, as characterized by given ΔT .

The above suggests the use of ΔT as the independent variable in describing the development of a wet vapor plume. It is also clear that such a measure is equally valuable in the superheated phases of the plume, wherein by Eqs. (16) we have

$$\left. \begin{aligned} \Delta T R^2 &= \text{constant} \\ \Delta q R^2 &= \text{constant} \end{aligned} \right\} \quad (17)$$

Thus, the ratio $\Delta T/\Delta q$ is also constant in these phases, equal to its initial value. We conclude that, whether the plume is wet or dry, its radius, liquid vapor content and specific humidity may all be expressed as functions of the excess temperature ΔT , provided only that the environment is "well mixed" [Eq. (13) holds].

On dividing the first two of Eqs. (11) we also find a relationship between momentum flux and radius, which is valid regardless of the atmospheric temperature and humidity gradients:

$$M^2 = \int \frac{U}{\alpha} R^2 (F - S) dR. \quad (18)$$

With the aid of this result we can now also calculate M as a function of ΔT . In order to revert to a description of plume behavior as a function of distance downwind, the second of Eqs. (11) yields

$$x = \frac{UM}{F-S} - \int UM \frac{d}{d\Delta T} \left(\frac{1}{F-S} \right) d\Delta T + \text{constant}. \quad (19)$$

In a dry plume ($S=0$ everywhere, $F=F_0=\text{constant}$) the integral is zero. The vertical displacement of the plume is

$$z = \int w dt = \int \frac{M}{UR^2} \frac{dx}{U} = \frac{R}{\alpha} + \text{constant}, \quad (20)$$

having used the first of Eqs. (11), or, more directly, Eqs. (1) and (2a). Eqs. (19) and (20) hold for the "initial" phase of plume behavior; in the "intermediate" and "final" phases a simple relationship between R and x may be deduced from Eqs. (2b) and (2c), so that all variables are easily expressed as functions of x (as well as of ΔT). The constants in the above integrated relationships may be determined from the initial conditions without difficulty.

4. Some applications

To illustrate possible developments in a vapor plume in a well-mixed environment consider the following example (Fig. 2): at the source the excess temperature is 60F, the excess humidity $\Delta q_0 = 48.6 \times 10^{-3}$ (mass per mass), and the ambient conditions are 60F and $q_a = 8.6 \times 10^{-3}$ gm gm⁻¹ corresponding to 77% relative

FIG. 2. Plume behavior in temperature-humidity diagram.

humidity or a 56F wet-bulb temperature. The initial plume humidity corresponds to 75% relative humidity. As long as no condensation occurs, the excess humidity reduces from its value at the origin (point 0) linearly with excess temperature. This process would take the plume humidity q_p above the saturation value at point A; consequently, humidity at this point begins to decrease faster, along the saturation line, until point B is reached. In between points A and B some of the water content in the plume appears in liquid form. Further to the left of point B, i.e., after further dilution of vapor in the plume, it becomes superheated again and finally becomes indistinguishable from the environment (point E). We conclude that condensation occurs for a finite portion of the plume, stretching from the point where the excess temperature is ~ 37 F to where it reduces to approximately ~ 6 F.

The variation of plume radius with excess temperature may be calculated from Eqs. (15a) and (16), the former being valid between A and B. The result, in the form of a ratio of plume radius to original radius R_0 , is also shown in Fig. 2. The dotted line is what would be calculated without condensation. The variation of the buoyancy flux may be calculated from the first of Eqs. (14); this yields on integration

$$F = F_0 + \lambda S, \quad (21)$$

where F_0 is the initial value of the buoyancy flux. The value of S may be determined from the second of these

FIG. 3. Schematic picture of latent heat effect.

equations, i.e.,

$$\sigma = \frac{R_0^2}{R^2} \Delta q_0 - \Delta q. \quad (22)$$

At a plume temperature of 80F ($\Delta T = 20$) in this example, we find approximately $\sigma = 10^{-3}$, $F = 1.25F_0$, i.e., relatively modest influences from condensation. By Eq. (18) this yields an increase of upward momentum of a few percent for given plume radius, which by Eq. (2) also means at given height, as compared to a dry plume of otherwise the same characteristics.

Beyond point B all values revert to their dry plume equivalents (at the same height we will find the same radius, excess temperature and upward momentum as in a "dry" plume with the same initial conditions), except that the source distance is reduced by a constant amount. In other words, the portion of the plume downstream of a point corresponding to B in Fig. 2 is displaced toward the origin by a constant distance, the value of which is given by the integral in Eq. (19). The net outcome of this displacement is illustrated in Fig. 3, in a somewhat exaggerated form. In the example calculated above the maximum "extra" vertical velocity is a few percent (wet compared to dry plume) which would not, in all probability, produce an observable change of plume shape, given the practical difficulties of such observations. On the other hand, the presence of condensed water droplets from A to B should be conspicuous.

It may be of interest to remark here that the above arguments parallel the discussion of the mixing of different damp air masses by Brunt (1939, p. 47).

As a second application of the above results we shall try to determine under what conditions the (infinitesimal) liquid content of an initially just-saturated plume will tend to evaporate, rather than increase by condensation (again in a well-mixed environment). This problem has relevance to the behavior of cooling tower vapor plumes, which may be assumed to be saturated, and in which it is desirable to avoid subsequent condensation.

Conditions under which condensation is avoided in an initially just-saturated plume are illustrated in Fig. 4, in a temperature-humidity diagram similar to Fig. 2.

Since the liquid content of the plume is to be zero from start to finish, the temperature-humidity relationship is a straight line, from initial conditions O to environmental conditions E. In order to avoid condensation, the line OE must not intersect the locus of saturation specific humidity outside point O. In the limit, when the plume is initially on the verge of condensation, OE is tangent to the $q_s(T)$ locus. For this limit, we may calculate the minimum difference δT between the environmental temperature T and the saturation temperature corresponding to the environmental specific humidity q_a .

From the Clausius-Clapeyron equation

$$\frac{\Delta q}{\Delta T} = \frac{dq_s}{dT} \bigg|_0 = \frac{q_p L_v}{R_v T_p^2}, \quad (23)$$

which, regarding L_v and R_v constant over the small range of temperatures involved, may be integrated to give

$$\ln \left(1 - \frac{\Delta q}{q_p} \right) = - \frac{L_v}{R_v T_p} \left(\frac{\delta T + \Delta T}{T_p - \Delta T - \delta T} \right). \quad (24)$$

Substituting (23) for Δq , we may write this result as

$$\delta T = T_p \left/ \left[1 - \beta \ln \left(1 - \beta \frac{\Delta T}{T_p} \right) \right] \right. - \Delta T, \quad (25)$$

where $\beta = L_v / (R_v T_p)$ is a quantity close to 17.0 under ordinary atmospheric conditions (and small excess temperatures in the plume). For given values of ΔT and T_p we may eliminate δT from Eq. (25), which enables us to read from a psychrometric chart at what ambient relative humidity initial condensation will just be avoided. Conversely, for given δT (i.e., given relative humidity of the atmosphere) we can solve (25) for ΔT , which will be the *maximum* excess temperature permissible, if initial condensation is to be avoided. For example, with $T_p = 560R$ (100F), $\Delta T = 20F$, we find $\delta T = 13.2F$, corresponding to 65% relative humidity. Thus, if excess temperature of the plume is kept below 20F, no condensation will occur in the plume except when the relative humidity in the atmosphere exceeds 65%. In order to avoid condensation at higher ambient humidities, the excess temperature ΔT of the plume would have to be appropriately reduced.

5. Saturation in a stratified environment

We now return to the case of a stratified environment, $N^2 \neq 0$, $G \neq 0$. In order to model such environmental conditions in a simple way, we shall assume both temperature and humidity gradients to be constant over the height range of interest. Because our main interest is the prediction of the point where condensation first occurs, we assume $\sigma = 0$ in the conservation laws. The

second and third of Eqs. (11) then become

$$\left. \begin{aligned} U \frac{dM}{dx} &= F \\ U \frac{dF}{dx} &= -MN^2 \end{aligned} \right\}. \quad (26)$$

With $N^2 = \text{constant}$ and with initial conditions $M=0$, $F=F_0=UR_0^2 g \Delta T_0 / T_a$ at $x=0$, we find at once [as for dry plumes (Slawson and Csanady, 1970)] that

$$\left. \begin{aligned} F &= F_0 \cos \frac{Nx}{U} \\ M &= \frac{F_0}{N} \sin \frac{Nx}{U} \end{aligned} \right\}. \quad (27)$$

With $H=H_0=UR_0^2 \Delta q_0$ at $x=0$, the fourth of Eqs. (11) now yields

$$H = H_0 + \frac{G}{N^2} (F - F_0). \quad (28)$$

Substituting definitions we obtain for the variation of the plume's excess specific humidity the expression

$$\frac{\Delta q}{\Delta q_0} = \Lambda \frac{\Delta T}{\Delta T_0} + (1-\Lambda) \frac{R_0^2}{R^2}, \quad (29)$$

where we have written

$$\Lambda = \frac{F_0 G}{H_0 N^2} = \frac{\Delta T_0}{\Delta q_0} \left(\frac{dq_a/dz}{d\theta_a/dz} \right). \quad (30)$$

In order to again obtain a functional relationship $\Delta q(\Delta T)$, we have to determine the variation of R^2 with ΔT . From the first and third of Eqs. (11), we have, with $S=0$:

$$\frac{dF}{dx} = -\frac{M}{U} N^2 = -N \frac{U}{3\alpha} \frac{dR^3}{dx}, \quad (31)$$

which integrates to

$$F = F_0 - N^2 \frac{U}{3\alpha} (R^3 - R_0^3). \quad (32)$$

From this we obtain the desired relationship of ΔT to R , i.e.,

$$\Delta T = \frac{R_0^2}{R^2} \left(\Delta T_0 + \frac{R_0}{3\alpha} \frac{d\theta_a}{dz} \right) - \frac{R}{3\alpha} \frac{d\theta_a}{dz}. \quad (33)$$

Since the change in the atmospheric potential temperature over a height span of one initial plume radius is usually negligible compared to the plume's initial

FIG. 4. Conditions in plume initially on the verge of condensation.

excess temperature, we have

$$R_0 \frac{d\theta_a}{dz} \ll \Delta T_0. \quad (34)$$

Therefore, to a high degree of approximation

$$\Delta T = \frac{R_0^2}{R^2} \Delta T_0 - \frac{R}{3\alpha} \frac{d\theta_a}{dz}. \quad (35)$$

The second term on the right has a simple interpretation once we note, in the "initial phase" of plume rise [for which the first of Eqs. (11) and therefore (35) are valid], that $R/\alpha = z^*$, the height over the "virtual" source (the latter being located R_0/α below the chimney top); hence $(R/3\alpha) d\theta_a/dz = (z^* d\theta_a/dz)/3$, i.e., one-third of the environmental temperature difference between plume level and the level of the virtual origin. This term will only be appreciable at larger distances from the source, where the first term in Eq. (35) becomes comparatively small. On the other hand, this whole expression is only valid in the *initial* phase of plume rise, so that to first order we have from Eqs. (35) (without the gradient term) and (29) that

$$\frac{\Delta q}{\Delta q_0} = \frac{\Delta T}{\Delta T_0} = \frac{R_0^2}{R^2}, \quad (36)$$

exactly as in "well-mixed" surroundings, as may be expected. A somewhat closer estimate is

$$\frac{\Delta q}{\Delta q_0} = \frac{\Delta T}{\Delta T_0} - (1-\Lambda) \gamma \left[\left(\frac{\Delta T_0}{\Delta T} \right)^{\frac{1}{2}} - \frac{\Delta T}{\Delta T_0} \right], \quad (37)$$

where $\gamma = (R_0/3\alpha \Delta T_0) (d\theta_a/dz)$. The sign of the initial departure from the neutral environment case (a straight line in the Δq - ΔT diagram) depends on whether Λ is greater or less than 1. Without any approximations (other than those made in arriving at our basic equations) the Δq (ΔT) relationship may be expressed with the radius ratio as a parameter; thus

$$\left. \begin{aligned} \frac{\Delta T}{\Delta T_0} &= \frac{R_0^2}{R^2} (1+\gamma) - \gamma \frac{R}{R_0} \\ \frac{\Delta q}{\Delta q_0} &= \frac{R_0^2}{R^2} + \Lambda \gamma \left(\frac{R_0^2}{R^2} - \frac{R}{R_0} \right) \end{aligned} \right\}. \quad (38)$$

In the initial phase only, the radius ratio is proportional to height above the source, i.e.,

$$\frac{R}{R_0} = \frac{\alpha z}{R_0} + 1. \quad (39)$$

Although we have again arrived effectively at a Δq (ΔT) relationship, the relationship to saturation in a stratified atmosphere is no longer straightforward, because environmental conditions are no longer constant. The specific humidity and temperature of the plume become

$$\left. \begin{aligned} q_p &= q_a + \Delta q = q_1 + z \frac{dq_a}{dz} + \Delta q \\ T_p &= T_a + \Delta T = T_1 + z \frac{dT_a}{dz} + \Delta T \end{aligned} \right\} \quad (40)$$

For a given z these may be calculated from Eqs. (38)–(40). Although the method of calculation is a little more complicated, and the $q_p(T_p)$ curve may have a more complex shape, the result is basically similar to the case of a well-mixed environment illustrated in Fig. 2, i.e., the saturation line may be intersected at some point A at which condensation would occur. In a stable environment it is possible to have several “loops” where the two curves intersect, providing several clouds along the plume, although the conditions under which this should occur would seem to be rather special.

In the intermediate or final phases of plume growth, Eq. (29) remains valid but must be supplemented by a relationship describing the growth of the plume radius R , obtained from either (2b) or (2c). In the

final phase we find, for example, that

$$R^2 = \frac{2a_2 v_m L_t}{U} x + R_2^2, \quad (41)$$

where R_2 is an integration constant, an effective initial radius for the final phase. Using now the definition of the buoyancy flux $\bar{F}$ and Eq. (27), it is not difficult to show that

$$\frac{\Delta T}{\Delta T_0} = \frac{R_0^2}{R^2} \cos\left(\mu \frac{R^2}{R_0^2} - A\right), \quad (42)$$

where

$$\mu = \frac{NR_0^2}{2a_2 v_m L_t}, \quad A = \mu \frac{R_2^2}{R_0^2}.$$

Thus, in Eqs. (29) and (42) we again have parametric equations for Δq , ΔT in terms of the radius ratio, which can be related to distance downstream by Eq. (41).

Acknowledgments. This research work was supported by the U. S. Atomic Energy Commission under Contract NYO-3685. I am indebted to Dr. T. M. L. Wigley for some helpful comments.

REFERENCES

- Brunt, D., 1939: *Physical and Dynamical Meteorology*. Cambridge University Press, 428 pp.
- Morton, B. R., 1957: Buoyant plumes in a moist atmosphere. *J. Fluid Mech.*, **2**, 127–144.
- , G. I. Taylor and J. S. Turner, 1956: Turbulent gravitational convection from maintained and instantaneous sources. *Proc. Roy. Soc. London*, **A234**, 1–23.
- Slawson, P. R., and G. T. Csanady, 1967: On the mean path of buoyant, bent-over chimney plumes. *J. Fluid Mech.*, **28**, 311–322.
- and —, 1970: The effect of atmospheric conditions on plume rise. *J. Fluid Mech.* (in press).