A Square Equal-Area Map of the World

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ABSTRACT

The map is proposed for worldwide climatological statistics, to depict accurately the area covered by some specified meteorological condition or element. Since it is square, a grid overlay divides the map into small squares, each covering exactly the same amount of global area. The map is centered on the north pole where it is conformal. The parallels of latitude in each of the four quadrants of the square are represented by elliptical arcs that change from circular shape at either pole to a straight line at the equator.

Except for Antarctica no continent is split or divided in this projection. The Northern Hemisphere is presented without interruptions or discontinuities of direction. The map's four quadrants can be reassembled to place the south pole and the whole of Antarctica at the center of the representation, as an interim step in the drawing of isopleths in the Southern Hemisphere.

1. Introduction

An equal-area projection of a spherical surface, confined within a square, has the immediate purpose of providing for a grid overlay that will divide the area into equal fractional squares. If the two margins of the square grid are divided into 10 equal segments (Fig. 1) then there will be 100 small squares each mapping exactly 1% of the global surface. Ordinary Cartesian coordinates will locate any point on the map to correspond uniquely to a single point on the sphere, thus becoming an alternative to locating points by latitude and longitude.

The most apparent use for a square equal-area map is in an atlas of worldwide statistics related to such climatic elements as temperature, precipitation or cloud cover. The map of any one statistic can be presented on one page of a report or text, kept, if desired, to $8\frac{1}{2}$ inches by 11 inches, or book-shelf size. Each page of an atlas would be used to better advantage than with most projections, since the map would occupy 77% of the usable space of the page, allowing 23% of the page to be used for headings or description.

Raisz (1962) named at least 30 map projections that have been used or proposed for worldwide coverage throughout the years since ancient times, and reminds his readers that he did not exhaust the possibilities, even good ones. Among the equal-area projections there are several well-known maps, favored by climatologists. But all maps inevitably distort the shape of the global surface. The conformal maps conserve shapes of only small regions producing distortion of large shapes, like continents. Most maps will divide the global surface so that continents, or oceans, or both are split. In selecting a map, therefore, a decision must be made as to where, and how much, distortion can be tolerated, and where splits are acceptable.

The guiding purpose of a map may be to conserve angular measurements, as is the case of conformal projections, or to yield straight lines for all great circles as in the gnomonic projection, or to depict distances accurately from a focal point, or to represent all areas accurately with respect to one another, as in the cases of the sinusoidal, Mollweide, Eckert, Albers, Bonne, or azimuthal equal-area maps (Raisz, 1962).

In the equal-area maps it is possible to tell at a glance how dense the climatological coverage is in any part of the world, or where it is weak or non-existent. For most of the equal-area projections the representation is best along the geographic equator, and second best along some chosen meridian until a polar region is reached. It is the distortion at high latitudes that has prompted this effort to construct a new map with the north pole at the center, realizing that this could be accomplished with an acceptable distortion at the equator. The map's objective is specifically aimed at global climatology.

The map (Fig. 1) does not interrupt the global surface in the Northern Hemisphere. In the Southern Hemisphere there are four interruptions along four key meridians from the equator to the south pole, but the map can be cut into four parts and reassembled to give an uninterrupted representation of the Southern Hemisphere, if that should be desired. Actually the map is drawn so that the interruptions are over water, except for Antarctica. The three continents that reach into the Southern Hemisphere are represented in three of the four quadrants of the map. The map is closer to conformality the closer one gets to either pole.

Among the disadvantages of the square equal-area map, centered on the north pole, is that the lettering cannot be horizontal without favoring one continent. The distortion is greatest along the equator, along which are spread South America, Africa and Micronesia, with considerable distortion to Australia as well. But the distortion in no way interferes with the purpose of the map, to represent the relative sizes of all areas accurately.

2. Construction

The map is so constructed that the lines representing the parallels, in each quadrant, are sections of ellipses that become circular at the poles and a straight line at the equator. Thus, conformality will be achieved at each pole. Each parallel must inclose a map area directly proportional to the area of the globe from the pole to that parallel.

In Fig. 1 it is seen that, with respect to parallels and meridians, there are eight identical sections, each section being a right-angled isosceles triangle, with a pole represented at the apex and one-fourth of the equator represented along the base of the triangle (Fig. 2). The problem, therefore, becomes one of determining how, and where, to plot the curved lines to represent parallels of latitude and meridians of longitude within the triangle.

Let both the area of a global octant and the area of the triangle (Fig. 2) be made unity. Then the height of the triangle is a unit of length and the base is 2 units in length.

a. The parallels

The global area A from the pole down to latitude φ, in the octant, is

$$A=1-\sin\varphi. \tag{1}$$

Let the ellipse, corresponding to latitude φ, have long axis $2a$ and short axis $2b$, so that the equation of the ellipse, with respect to coordinates (ξ,η) from its center, becomes

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}=1. \tag{2}$$

As part of the problem the solution is required to make the map's parallels elliptical in shape intersecting the sides of the isosceles right-angled triangle perpendicularly so that the projection will show no discontinuity in mapping direction. Additionally, and for the same reason, the meridians should meet the hypotenuse representing the equator perpendicularly.

At the intersection (x,y) of the ellipse representing the parallel φ and the side of the triangle,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{3}$$

The ratio

$$r=b/a \tag{4}$$

will be chosen so that

$$\left.\begin{array}{ll} r=1 & \text{at } \varphi=\pi/2 \\ r=0 & \text{at } \varphi=0 \end{array}\right\},$$

thus making the elliptical arc become a circle at the pole and making it flatten to become identified with the base of the triangle. For all other φ

$$r=f(\varphi) \quad \text{and} \quad dr/d\varphi=f'(\varphi), \tag{5}$$

where $f(\varphi)$ is given below. If the height from the apex (or pole) to the long axis of the ellipse is h (Fig. 2), then

$$h+y=x. \tag{6}$$

The fractional area T_ξ of the ellipse is

$$T_\xi=\int_0^\xi \eta d\xi, \tag{7}$$

but from (2) and (4)

$$\eta=r(a^2-\xi^2)^{\frac{1}{2}}, \tag{8}$$

from which (7) gives

$$T_\xi=\frac{r}{2}[\xi(a^2-\xi^2)^{\frac{1}{2}}+a^2\sin^{-1}(\xi/a)], \tag{9}$$

and the area A enclosed by the elliptical curve becomes

$$A=h^2+r[x(a^2-x^2)^{\frac{1}{2}}+a^2\sin^{-1}(x/a)]-y^2. \tag{10}$$

To force the curve of the ellipse to meet the side of the triangle at right angles, the condition is imposed that

$$d\xi/d\eta=-1 \quad \text{at} \quad (x,y),$$

so that, from (2), by differentiation, and (4)

$$y=xr^2, \tag{11}$$

which together with (3) gives

$$a^2=x^2(1+r^2), \tag{12}$$

and with (6)

$$h=x(1-r^2). \tag{13}$$

If

$$z=\sin^{-1}[1/(1+r^2)^{\frac{1}{2}}], \tag{14}$$

whence

$$dz/dr=-1/(1+r^2), \tag{14}$$

and if

$$v=(1-r^2)+r(1+r^2)z, \tag{15}$$

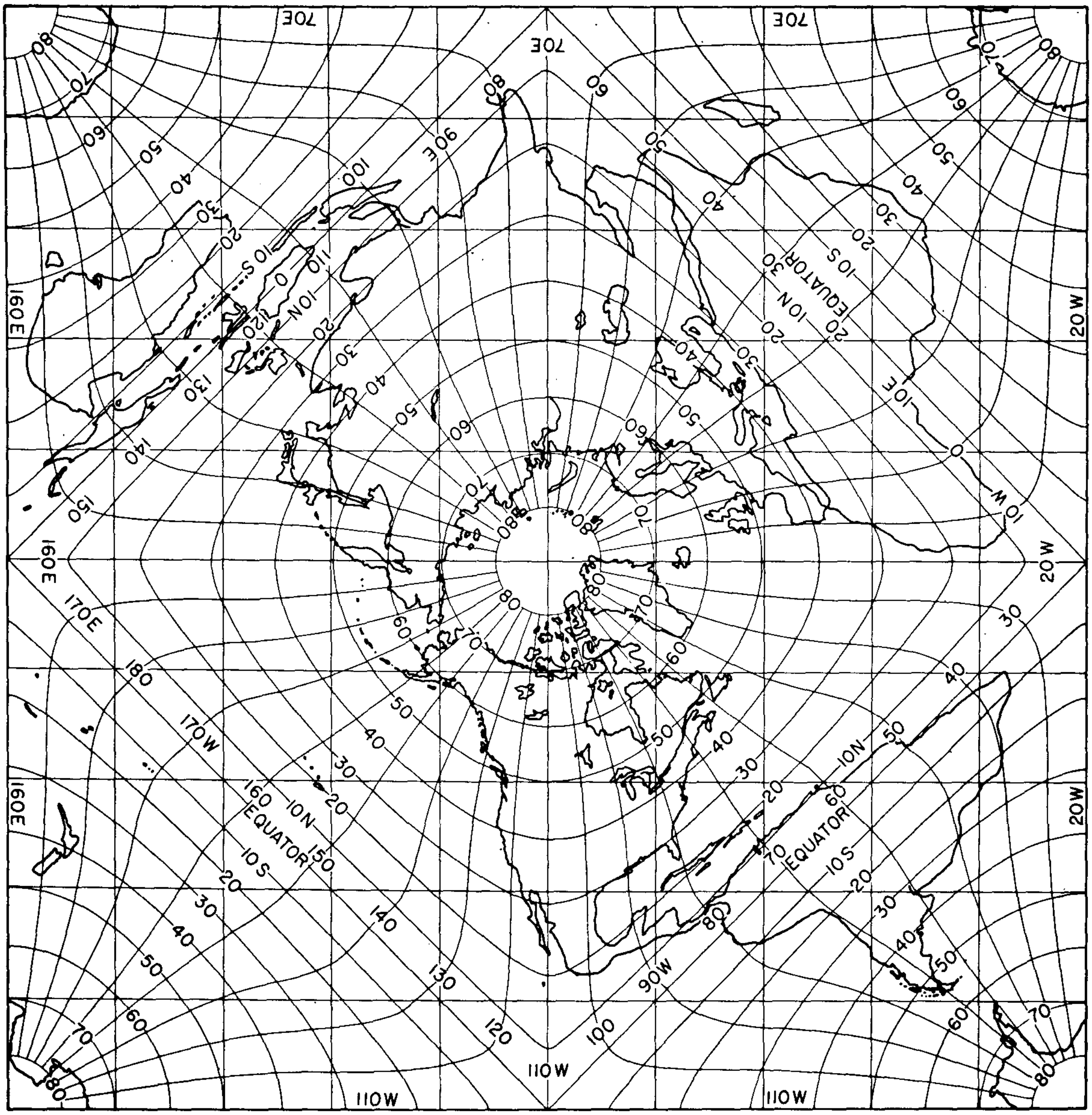

FIG. 1. A square equal-area map of the world. A square grid overlay divides the map into 100 equal small squares, each representing exactly 1% of the global surface.

then (10) becomes

$$A = vx^2. \tag{16}$$

Now, recognizing (1), i.e., the equivalence of the map area (A) and the global area (A),

$$x^2 = (1-\sin\varphi)/v. \tag{17}$$

Using (8), the elliptical curve representing the parallel (φ), in terms of coordinates (ξ,η') originating at the pole, is defined by

$$\eta' = h + r(a^2-\xi^2)^{\frac{1}{2}}, \tag{18}$$

which gives η' completely in terms of φ and ξ, given (5), (12), (13), (14), (15), (17).

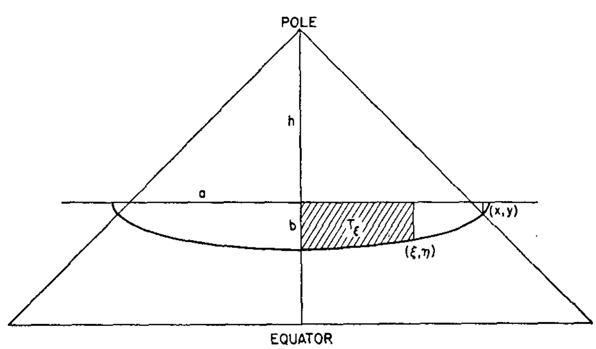

FIG. 2. Diagram illustrating the plotting of an elliptical curve that represents a given parallel of latitude (φ).

b. The meridians

Consider a meridian (λ) represented between the mid-line of the isosceles right-angled triangle and the right-hand side of the triangle (Fig. 3). From (1) the increment of area (δA) in a thin belt of latitudinal thickness ($\delta\varphi$) becomes

$$\delta A=(dA/d\varphi)\delta\varphi=-\cos\varphi\delta\varphi. \qquad (19)$$

For an equal-area projection the curve representing the meridian (λ) must intercept a fraction $\delta A'$ of the area δA such that

$$\lambda/(\pi/4)=(\delta A')/(\delta A/2). \qquad (20)$$

In the projection the areal increment ($\delta A'$) becomes

$$\delta A'=\int_0^\xi \delta\eta' d\xi=\delta\varphi\int_0^\xi (\partial\eta'/\partial\varphi)d\xi. \qquad (21)$$

From (18) by partial differentiation, at ξ

$$\partial\eta'/\partial\varphi=dh/d\varphi+r[(da^2/d\varphi)/2(a^2-\xi^2)^{\frac{1}{2}}]+(a^2-\xi^2)^{\frac{1}{2}}(dr/d\varphi),$$

which, when integrated with respect to ξ, gives for (21)

$$\delta A'/\delta\varphi=(dh/d\varphi)\xi+(dr/d\varphi)\xi(a^2-\xi^2)^{\frac{1}{2}}/2+[d(ra^2)/d\varphi]\sin^{-1}(\xi/a)/2, \qquad (21)$$

which, with (19) and (20), gives

$$\lambda/(\pi/4)=\zeta\xi+\mu\xi(a^2-\xi^2)^{\frac{1}{2}}+\nu\sin^{-1}(\xi/a), \qquad (22)$$

where

$$\zeta=-2(dh/d\varphi)/\cos\varphi, \qquad (23)$$

$$\mu=-(dr/d\varphi)/\cos\varphi, \qquad (24)$$

$$\nu=-[d(ra^2)/d\varphi]/\cos\varphi. \qquad (25)$$

From previous equations

$$(dh/d\varphi)=(1-r^2)(dx/d\varphi)-2rx(dr/d\varphi), \qquad (26)$$

$$d(ra^2)/d\varphi=r(1+r^2)(dx^2/d\varphi)+x^2(1+3r^2)(dr/d\varphi), \qquad (27)$$

$$dv/d\varphi=[-3r+z(1+3r^2)](dr/d\varphi), \qquad (28)$$

$$dx^2/d\varphi=[-v\cos\varphi-(1-\sin\varphi)(dv/d\varphi)]/v^2, \qquad (29)$$

$$dx/d\varphi=(dx^2/d\varphi)/2x. \qquad (30)$$

To impose the condition that each line representing a meridian of longitude will cross the base line representing the equator at right angles, we should make, at $\varphi=0$, for λ constant,

$$(d\xi/d\varphi)_0=0. \qquad (31)$$

By differentiating (22) with respect to φ, for λ constant,

$$0=(d\zeta/d\varphi)\xi+(d\mu/d\varphi)\xi(a^2-\xi^2)^{\frac{1}{2}}+(d\nu/d\varphi)\sin^{-1}(\xi/a)+(da^2/d\varphi)(\mu-\nu)\xi/2(a^2-\xi^2)^{\frac{1}{2}}+g(\varphi)(d\xi/d\varphi), \qquad (32)$$

where $g(\varphi)$ is a function of φ.

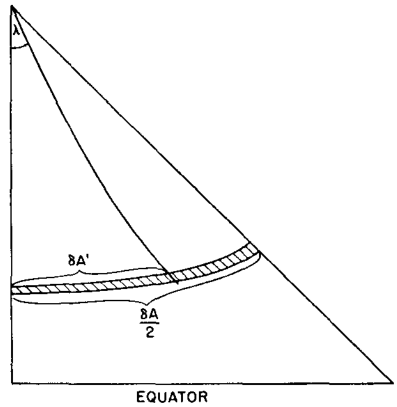

FIG. 3. Diagram illustrating the method of locating the intersection of a curve of meridian (λ) with a curve of latitude (φ).

At $\varphi=0$, where $a=1$, $r=0$ and using (31), the last two terms of (32) vanish, to give

$$0=(d\zeta/d\varphi)_0\xi+(d\mu/d\varphi)_0\xi(1-\xi^2)^{\frac{1}{2}}+(d\nu/d\varphi)_0\sin^{-1}\xi. \qquad (33)$$

For all $0\leqslant\xi\leqslant 1$, (33) is accurate only if $(d\zeta/d\varphi)_0=(d\mu/d\varphi)_0=(d\nu/d\varphi)_0=0$ which conditions, because of (23)–(25), are the same as

$$(d^2h/d\varphi^2)_0=(d^2r/d\varphi^2)_0=[d^2(ra^2)/d\varphi^2]_0=0. \qquad (34)$$

As it happens, the three conditions of (34) cannot be accurately satisfied simultaneously, and hence it is not possible to construct the map with the meridional lines crossing the equatorial line exactly at right angles. But the condition can be nearly satisfied. Beginning with the Taylor expansion of the terms in (33), and dropping terms of order higher than ξ^2, Eq. (33) yields the ξ-free relation

$$2(d^2h/d\varphi^2)_0+(d^2r/d\varphi^2)_0+[d^2(ra^2)/d\varphi^2]_0=0,$$

a condition that is nearly satisfied by

$$r=f(\varphi)=\sin^2\varphi. \qquad (35)$$

This relation, when used to define $f(\varphi)$ in (5), leads finally to the plot of (φ,λ) in terms of (ξ,η').

c. Plotting

For the computations of ξ using successive values of

$$\varphi=0°(1°)5°(5°)80°,$$

the equations (35), (5), (14), (15), (28), (17), (29), (30), (26), (27), (23), (24), (25), (22) were used in this order to obtain λ vs ξ. The value of ξ was found by trial and error (Table 1) to correspond to values of

$$\lambda=0°, 5°(10°)45°.$$

Finally, the corresponding values of η' were obtained using (18).

TABLE 1. The Cartesian coordinates (ξ,η') for the plotting of points given by latitude (φ) and longitude (λ) in each of sixteen congruent triangles in a square equal-area map of the world.

Latitude (φ)	0° ξ	0° η'	5° ξ	5° η'	15° ξ	15° η'	25° ξ	25° η'	35° ξ	35° η'	45° $\xi=\eta'$
	Longitude (λ)										
80°	0	0.139	0.012	0.138	0.036	0.134	0.059	0.126	0.080	0.114	0.098
75°	0	0.208	0.018	0.207	0.054	0.201	0.089	0.188	0.120	0.171	0.148
70°	0	0.276	0.025	0.274	0.073	0.266	0.119	0.250	0.161	0.227	0.197
65°	0	0.342	0.031	0.341	0.092	0.331	0.149	0.312	0.201	0.284	0.247
60°	0	0.408	0.038	0.406	0.111	0.395	0.181	0.372	0.243	0.340	0.296
55°	0	0.471	0.044	0.469	0.131	0.457	0.212	0.432	0.285	0.396	0.346
50°	0	0.532	0.052	0.530	0.154	0.516	0.248	0.490	0.330	0.450	0.398
45°	0	0.590	0.061	0.588	0.179	0.574	0.286	0.545	0.378	0.505	0.451
40°	0	0.645	0.070	0.643	0.204	0.628	0.324	0.607	0.425	0.561	0.505
35°	0	0.696	0.080	0.694	0.232	0.681	0.366	0.654	0.477	0.617	0.562
30°	0	0.744	0.090	0.743	0.261	0.730	0.411	0.707	0.533	0.673	0.623
25°	0	0.789	0.100	0.788	0.290	0.778	0.455	0.759	0.590	0.731	0.686
20°	0	0.832	0.108	0.831	0.314	0.824	0.496	0.810	0.645	0.790	0.754
15°	0	0.874	0.113	0.873	0.331	0.869	0.528	0.861	0.695	0.848	0.822
10°	0	0.915	0.115	0.915	0.338	0.913	0.548	0.909	0.736	0.903	0.889
5°	0	0.957	0.114	0.960	0.339	0.956	0.557	0.956	0.764	0.954	0.950
4°	0	0.966	0.114	0.966	0.338	0.965	0.557	0.965	0.768	0.964	0.961
3°	0	0.974	0.112	0.974	0.337	0.974	0.558	0.974	0.771	0.973	0.971
2°	0	0.983	0.112	0.983	0.336	0.983	0.557	0.982	0.774	0.982	0.982
1°	0	0.991	0.111	0.991	0.334	0.991	0.556	0.991	0.776	0.991	0.991
0°	0	1.000	0.111	1.000	0.333	1.000	0.556	1.000	0.778	1.000	1.000

Table 1 gives enough information to plot lines representing parallels and meridians in $\frac{1}{16}$ of the world map. But the latitudinal and meridional lines in all other parts of the map are readily seen to be exactly the same, or mirror images. In Fig. 1 the key meridians were chosen to be 20W, 70E, 160E, 110W.

3. Concluding remarks

Once the map has been constructed, the grid overlay can be made arbitrarily as a fine or coarse mesh. In Fig. 1 the rectangular coordinates of Boston, Mass. (44°22′N, 71°1′W) from the north pole as center are (0.1346, −0.1738).

The one principal body of land that is split in the map is Antarctica. But the four squares in the four corners of the map can be reassembled to make Antarctica appear without splits. Depending on page space, this could be done in one corner of the map, making Antarctica appear as an appendage to the basic large square of the whole map.

If this map should become popular as a practical and useful background for the drawing of isopleths of weather elements, it might be desirable to have an auxiliary map that shows the Southern Hemisphere reassembled. The auxiliary map would be used for ease in the initial drawing of the isopleths for the Southern Hemisphere, before their transfer to the original map.

In the construction of this map there were several arbitrary decisions that could be replaced with one or more alternatives. For example, assumption (35), which was adopted because it causes the map meridians to cross the equator almost perpendicularly, could be replaced by

$$r=(2/\pi)\varphi,$$

which will yield map parallels that do not straighten out as rapidly, while nearing the equator, as they do in Fig. 1. One modification that might be considered seriously is to make the key meridians 0°, 180°, and 90 E and 90 W. This was not done herein primarily because of the resulting unattractive distortion of Africa. But New Zealand and Australia would fall into the same quadrant and would be well centered. It is also conceivable that the map might be centered on another point instead of the north pole. But in any modification the main purpose should be kept in mind, to obtain a square equal-area map of the world.

REFERENCE

Raisz, Erwin, 1962. *Principles of Cartography.* New York, McGraw–Hill, 315 pp.