A Height-Dependent Model of Eddy Viscosity in the Planetary Boundary Layer

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ABSTRACT

A height-dependent model of the vertical eddy diffusivity for momentum, $K(Z)$, has been formulated for purposes of studying numerically the momentum (and heat) transfer arising from turbulent motions within the planetary boundary layer (PBL). The model possesses those features believed to be characteristic of the PBL: 1) an approximate linear increase in K through the surface boundary layer, 2) a local maximum value of K in the lower portion of the Ekman layer, 3) an exponential decrease in the profile above the level of maximum K, and 4) a $K(Z)$ function that is continuously differentiable. Specifically the model is given as

$$K(Z)=a[\exp(-bZ/Z_T)-\exp(-bcZ/Z_T)]$$

where Z represents height, Z_T the top of the PBL and the scaling factor, and a, b and c are arbitrarily chosen parameters that specify a unique K profile. The applicability of the model to atmospheric data is shown by fitting curves to the K distributions derived by Pandolfo's study using BOMEX data.

1. Introduction

In many studies concerned with the vertical transport of momentum, heat, water vapor, pollutants, and other such quantities within the atmosphere's planetary boundary layer (PBL), it is necessary to know the vertical distribution of the transport coefficient for the quantity being considered. Due to the turbulent nature of the atmosphere the molecular transport is dominated by the effects of eddy motion, and consequently the exchange coefficient is a property of the flow and not of the fluid. Of particular interest in this paper is the vertical eddy diffusivity for momentum, K, which is related to the stress induced by the vertical shear in the horizontal wind field. Specification of the vertical distribution of K is quite difficult, however, and efforts continue to provide a more realistic physical description of the structure of the PBL. An example of this continuing refinement is the variangular theory given in a recent paper by Lettau and Dabberdt (1970). There have been several important studies on the structure of the PBL and the reader is referred to a report by Hanna (1969) that reviews many of these investigations.

In principle, the vertical distribution of the horizontal wind (and therefore K) in the PBL is determined by the combined effect of surface roughness, thermal stratification, horizontal pressure gradients, and the earth's rotation. In the surface boundary layer, shown in Fig. 1, the shearing stress is nearly constant and the structure of the wind field is primarily determined by surface roughness and the vertical gradient of temperature. Immediately above the surface boundary layer is the Ekman layer, also shown in Fig. 1, where the shearing stress varies with height and the wind field is influenced primarily by surface roughness, horizontal pressure gradients, and the earth's rotation. At the top of the PBL, motion approximates that of an inviscid rotating fluid that is usually geostrophic or balanced gradient flow. A study by Lettau (1962) concluded that the structure of the PBL is a unique function of the surface Rossby number, $\mathrm{Ro}=V_g/(Z_0 f)$, where V_g is the geostrophic wind speed, f the Coriolis parameter, and Z_0 the roughness parameter of the ground. He has also shown that the maximum value K_M of vertical eddy diffusivity for momentum is directly proportional to V_g^2/f. Therefore, K_M is expected to increase as the speed of the wind increases and as the latitude decreases. Also, the height level of K_M, given as Z_M, should be higher for increasing surface roughness. O'Brien (1970) has pointed out that Z_M should be greater than the thickness of the surface boundary layer by taking into account the physical requirement that the K distribution and its first derivative be continuous with height. He further concludes that Z_M is located about one-third of the distance from the top of the surface boundary layer to the top of the PBL at Z_T. According to

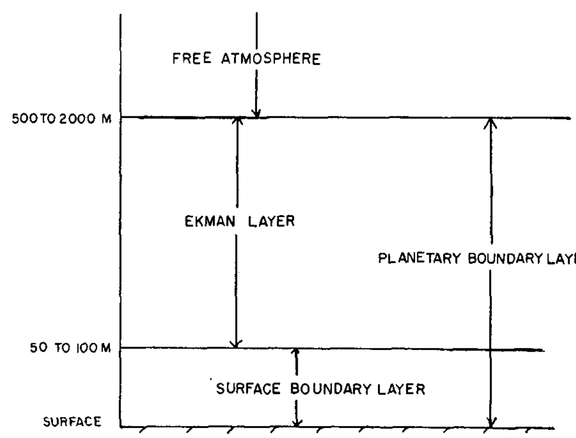

FIG. 1. The geometry of the planetary boundary layer.

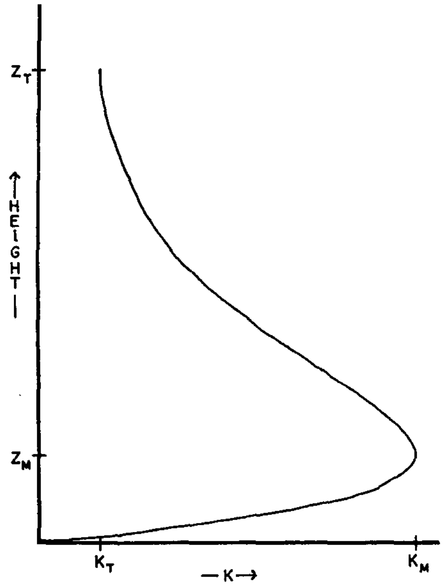

FIG. 2. The general shape of the K profile in the PBL: Z_M is the height level of maximum eddy diffusivity, K_M, and K_T the value of eddy diffusivity at the top of the PBL, Z_T,

Blackadar and Tennekes (1968), the thickness of the Ekman layer (or for all practical purposes the PBL) is equal to U_*/f for barotropic flow under neutral steady-state conditions, where U_* is the friction velocity. Finally, the likely general shape of the vertical distribution of K given in Fig. 2 shows a near-linear increase in K with height close to the ground, a local maximum at Z_M, and then an exponential type of decrease to the top of the PBL.

There are two approaches to modeling the vertical profile of K where each is taken for different research objectives. First is the more scientific approach which is based on a physical model of the PBL relating the K distribution to physical controls such as wind shear, thermal stability, and surface roughness. Although somewhat empirical, the models proposed by Pandolfo (1971) relating K to vertical shear, height, Richardson number, and state of the sea surface represents examples of this approach. The second approach to modeling K is through the mathematical formulation of a height-dependent function $K(Z)$ that possesses the known shape characteristics. This method, although somewhat less scientific, requires a certain physical understanding of the problem enabling one to make a better mathematical formulation of $K(Z)$, as evident in O'Brien's model. For practical studies in which one wants to compute estimates of the vertical flux of momentum (or other quantities) it is necessary to use the physical model, as Pandolfo (1971) has done with BOMEX data. On the other hand, a mathematical formulation of K, based on sound physical reasoning, is particularly useful in numerical experimentation and the study of PBL motions responsible for momentum and heat transfer. An excellent example of this latter approach is the numerical study of sea breeze circulation conducted by McPherson (1970). In this convection study, near-neutral (adiabatic) conditions were assumed thus allowing the eddy diffusivity for heat to be set equal to K. It has been demonstrated in a recent study by Businger *et al.* (1971), however, that heat transfer might exceed momentum transfer at neutral conditions. McPherson also pointed out the physical importance of allowing for curvature in the K profile, an improvement over the linear model proposed earlier by Estoque (1963). However, in McPherson's study the usage of a K profile with a first-order discontinuity and maximum value of K at the top of the surface boundary layer represents a profile with objectionable features. O'Brien's model of K, a cubic polynomial, has eliminated these undesirable features, but has enforced the new restriction previously mentioned that K_M can occur no closer to the ground than about one-third of the distance from the top of the surface boundary layer to the top of the PBL. It is the writers' opinion that in many instances the Z_M level suggested by O'Brien's model would be higher than the actual level of K_M. It is interesting to note that equiangular spiral solutions of PBL motions obtained by several investigators indicate a second-order polynomial K profile, while Lettau and Dabberdt's (1970) variangular theory produces a height distribution of K which corresponds to a fourth-order polynomial.

The approach taken here was to devise a mathematical formulation of K, based upon physical reasoning, appropriate for numerical experimentation in atmospheric modeling. The specific interest of the writers was to generate a height-dependent profile of K to incorporate in a frictional model of mesoscale cellular convection. The important role played by a K profile with curvature is evident by examining such terms as $(\partial/\partial z)(K\partial u/\partial z)$ and $(\partial/\partial z)(K\partial\theta/\partial z)$ in the momentum and heat equations, respectively. The $K(Z)$ formulation obtained is given below, and its applicability to real cases is demonstrated by fitting curves to the K distributions derived by Pandolfo (1971) using BOMEX data.

2. The model

The objective of this study was to develop a height-dependent model of vertical eddy diffusivity for momentum in the PBL with features similar to those of the profile shown in Fig. 2. Specifically, it was recognized that $K(Z)$ should have an approximate linear increase with height through the surface boundary layer, a local maximum value of K in the lower portion of the Ekman layer, and an exponential decrease in the profile above the level of K_M. Further, it was desirable to choose a $K(Z)$ that was also continuously differentiable with respect to height. Through a trial-and-error procedure the following height-dependent model was developed:

$$K(Z)=a[\exp(-bZ/Z_T)-\exp(-bcZ/Z_T)], \quad (1)$$

where a, b and c are arbitrarily chosen parameters that primarily affect the magnitude of K, the height scale, and the ratio of K_M to K_T, respectively. By having the flexibility of combining many different values of these three parameters, one can generate many different profiles of K with varying K_M, K_T, Z_M and Z_T, each possessing the desired general shape shown in Fig. 2. Conversely, if one has specific values in mind for K_M, K_T, Z_M and Z_T, then by determining the corresponding a, b and c for the model given in (1) a unique $K(Z)$ profile can be specified. As pointed out in the Introduction the values of K_M, K_T, Z_M and Z_T are related to physical effects such as wind shear, thermal stability, and surface roughness. By knowing the general nature of these relationships many different K profiles can be generated and their effects studied in modeling experiments. Based upon the reports and studies by several investigators it is typically considered that K_M ranges between 10^4 and 10^6 cm² sec⁻¹, K_T between 10^3 and 10^5 cm² sec⁻¹, Z_M between 50 and 500 m, and Z_T between 1 and 2 km. Of course, the values of K could be substantially less if one allow the winds to become nearly calm.

In order to solve for the model parameters a, b and c it is necessary to specify the three conditions that generate a solvable system. The conditions enforced on our model are

$$K=K_T \quad \text{at} \quad Z=Z_T, \tag{2}$$

$$K=K_M \quad \text{at} \quad Z=Z_M, \tag{3}$$

$$\left.\frac{dk}{dz}\right)_{Z_M}=0 \quad \text{at} \quad Z=Z_M. \tag{4}$$

These conditions are straightforward and should raise no objections. Using (1)–(4) and certain algebraic manipulations, the following results were obtained:

$$a=\frac{K_T c^{[(Z_T/Z_M)/(c-1)]}}{1-c^{(-Z_T/Z_M)}}, \tag{5}$$

$$b=\frac{(\ln c)Z_T}{(c-1)Z_M}, \tag{6}$$

$$\frac{K_M}{K_T}=[1-\exp[-(Z_T/Z_M)\ln c]^{-1} \times[\exp\{(Z_T-Z_M)\ln c/[Z_M(c-1)]\} -\exp\{(Z_T-cZ_M)\ln c/[Z_M(c-1)]\}]. \tag{7}$$

Eqs. (5)–(7) constitute a solvable system for determining a, b and c, if values are given for K_M, K_T, Z_M and Z_T. An iteration scheme was used to numerically solve for c in (7) by converging to within $\epsilon=10^{-4}$ distance of the chosen K_M/K_T. In turn, (5) and (6) were solved, respectively, for a and b.

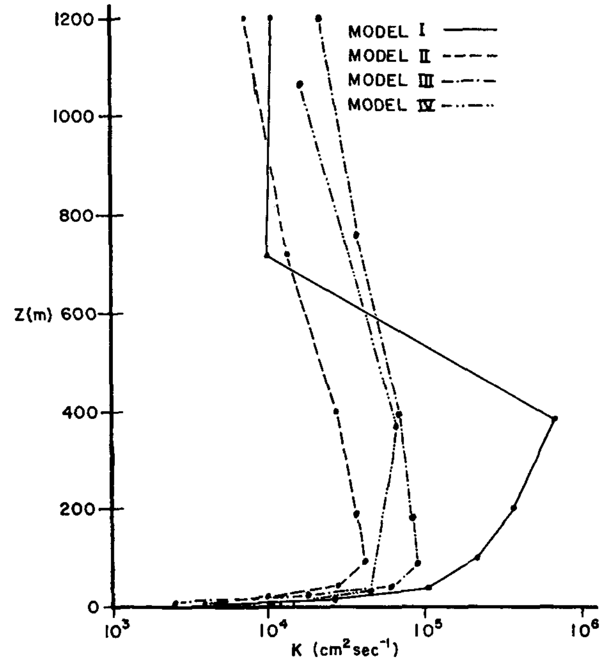

FIG. 3. Semi-log vertical distributions of eddy diffusivity K determined by Pandolfo (1971) from his four models of the air-sea boundary layer.

Before proceeding to specific test cases, certain limitations of the model should be discussed. From (1) it can be seen that a (a coefficient factor) only affects the magnitude of K. However, b and c appear as factors in the exponents and it is therefore desirable to examine more closely the limiting values of these parameters. From the iterative solutions for c obtained from (7) it was determined that the ratio K_M/K_T became larger as c became smaller, i.e., as it approaches 1. From (1) it is seen that $K=0$ if $c=1$; however, since b is a function of c, the appropriate limiting value for b and K_M/K_T must be obtained by approaching $c=1$ from the right. Therefore, applying L'Hospital's rule to (6), we have

$$\lim_{c\to 1} b=\frac{Z_T}{Z_M}. \tag{8}$$

Similarly from (7),

$$\lim_{c\to 1}\left(\frac{K_M}{K_T}\right)=(Z_T/Z_M)\exp[(Z_T-Z_M)/Z_M]. \tag{9}$$

3. Application of model to Pandolfo's BOMEX cases

To test the applicability of the model to realistic cases, the K profiles obtained by Pandolfo (1971) were used. Pandolfo's profiles, given in Fig. 3 with the author's permission, were derived using BOMEX data in four different models of the air-sea boundary layer. These profiles are given in a semi-log plot due to the

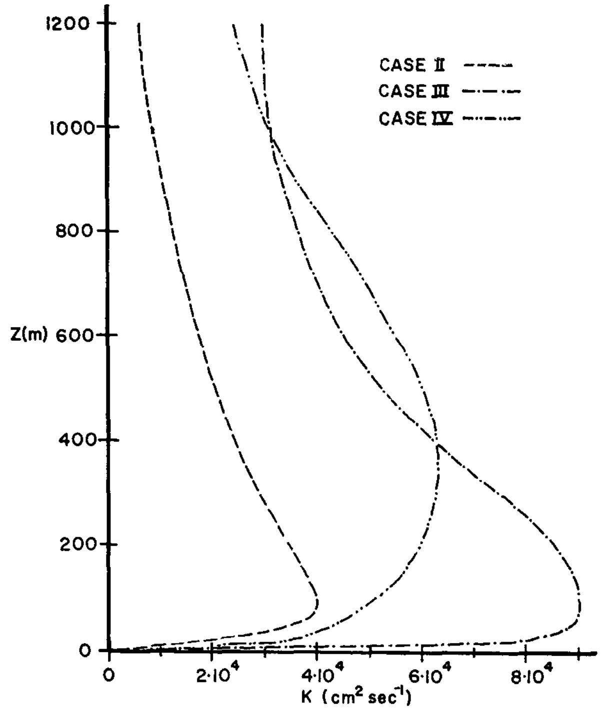

FIG. 4. Vertical profiles of linear eddy diffusivity K from the height-dependent model, estimating Model II, III and IV profiles obtained by Pandolfo (1971). For case II: $a=48994$ cm²sec⁻¹, $b=2.046$, $c=20.76$; case III: $a=96509$ cm²sec⁻¹, $b=1.168$, $c=51.60$; case IV: $a=172984$ cm²sec⁻¹, $b=1.902$, $c=2.812$.

extreme maximum value for Model I. Of the four profiles shown the procedure outlined in the previous section was used successfully to fit the height-dependent model given by (1) to the profiles of Models II, III and IV. These fitted curves and the corresponding values for a, b and c are given in Fig. 4. A fit of the height-dependent model could not be made to Pandolfo's Model I profile, as seen from (9), since K_M/K_T is too large for the given Z_M (360 m) and Z_T(1200 m). It is the writers' opinion that the approximately two orders of magnitude decrease in K for the layer 400 to 700 m is probably unrealistic. However, this case does demonstrate the limitations of the height-dependent model.

4. Conclusions

As intended, a height-dependent model of the vertical eddy diffusivity for momentum has been formulated for the purpose of studying frictional effects in modeling experiments. It has further been shown that the model is capable of assuming those features believed to be most characteristic of the distribution of K through the PBL.

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