# NOTES AND CORRESPONDENCE

## On the Graphical Computation of the Montgomery Streamfunction

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13 April 1973 and 29 July 1973

ABSTRACT

A thermodynamic diagram useful in the graphical computation of the Montgomery streamfunction is developed. Examples of the use of the diagram are shown together with an analysis of error for the graphical technique.

### 1. Introduction

Outlined in this note are the theory and application of a new thermodynamic diagram for the plotting of atmospheric soundings. The diagram, developed as a graphical means of computing the Montgomery streamfunction, is called the "$\theta$-$\Delta\psi$ diagram." It is a true thermodynamic diagram on which the isobars and the dry adiabats are straight lines and the angle between isotherms and adiabats is large. Thus, in addition to the computation of the Montgomery streamfunction, the diagram is suitable for all of the usual stability and energy calculations. The theory upon which the diagram rests is developed below, followed by a sample diagram, an error analysis, and a case study illustrative of the use of the diagram in the computation of the Montgomery streamfunction.

### 2. Theory

The virtual potential temperature $\theta^*$ is defined as

$$\theta^* = T^*\left(\frac{1000}{P}\right)^\kappa, \tag{1}$$

where $T^*$ is the virtual temperature, $P$ the pressure, $\kappa = R/c_p$, $R$ the specific gas constant for dry air, and $c_p$ the specific heat at constant pressure for dry air. $\theta^*$ is neither more nor less conservative in dry adiabatic processes than is the ordinary potential temperature; both suffer from the minor fault that a very small correction should be applied to the exponent $\kappa$ to account for the presence of water vapor (see the discussion in Godske *et al.*, 1957). It is, however, necessary to introduce the virtual temperature in the derivation of the Montgomery streamfunction when the equation of state for an ideal gas is used and this, in turn, requires the introduction of the virtual potential temperature. The Montgomery streamfunction is then defined as

$$\psi = c_p T^* + gz, \tag{2}$$

where $\psi$ is the Montgomery streamfunction, $g$ the acceleration of gravity, and $z$ height. These definitions together with the hydrostatic assumption lead to

$$\frac{\partial\psi}{\partial\theta^*} = c_p\left(\frac{P}{1000}\right)^\kappa. \tag{3}$$

Integration of (3) with respect to $\theta^*$ from one isentropic surface to another results in

$$\psi(\theta_2^*) = \psi(\theta_1^*) + c_p(\theta_2^* - \theta_1^*)\overline{\left(\frac{P}{1000}\right)^\kappa}, \tag{4}$$

where the mean of the pressure term is defined as

$$\overline{\left(\frac{P}{1000}\right)^\kappa} = \frac{1}{\theta_2^* - \theta_1^*}\int_{\theta_1^*}^{\theta_2^*}\left(\frac{P}{1000}\right)^\kappa d\theta^*.$$

Thus, the Montgomery streamfunction at a given level may be found by determining the mean value of the pressure term from that level upward or downward to a level at which the streamfunction is known. At the surface of the earth $\psi$ may be determined directly from (2) and the upper level streamfunction values then built up through repeated application of (4). This is analogous to the usual method of determining the geo-

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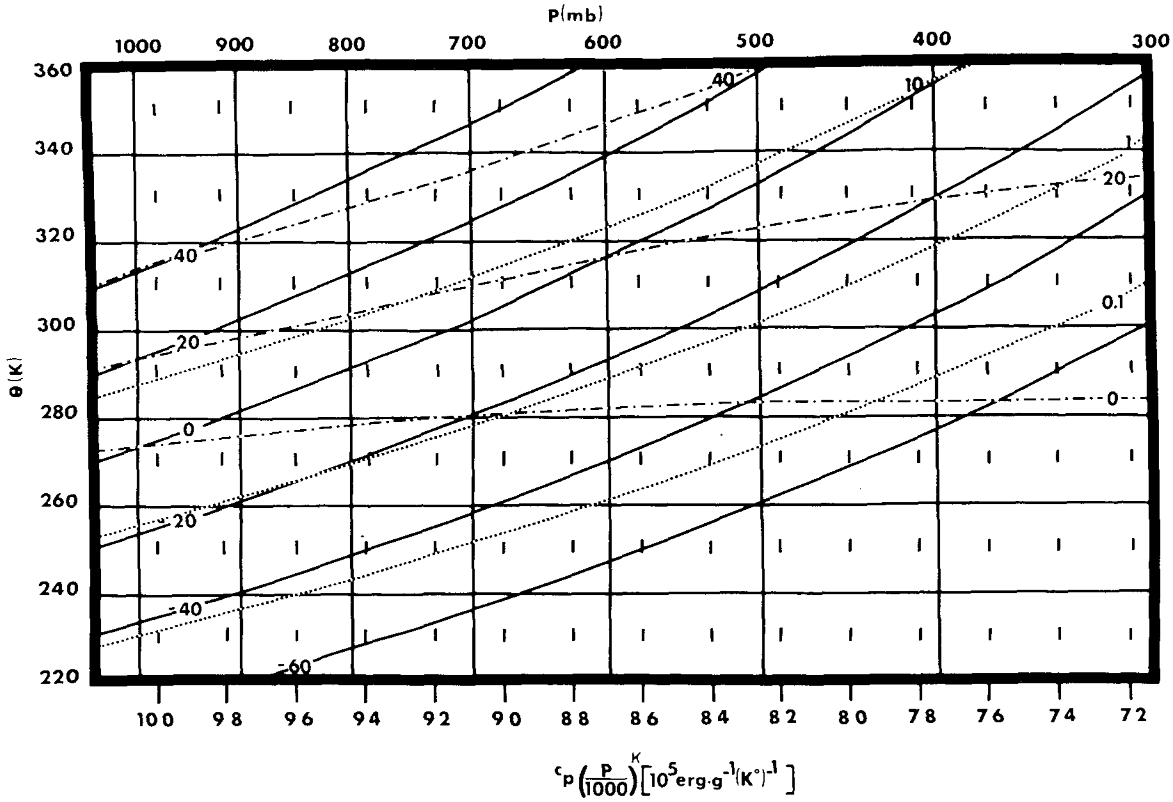


FIG. 1. Simplified $\theta$-$\Delta\psi$ diagram (see text for description).

potential on a constant pressure surface (see, for example, Saucier, 1955).

## 3. The diagram

A simplified version of the $\theta$-$\Delta\psi$ diagram is illustrated in Fig. 1. Along the vertical axis the virtual potential temperature increases linearly upward while the pressure term, $c_p(P/1000)^\kappa = \partial\psi/\partial\theta^*$, increases *leftward* along a linear horizontal scale. The horizontal axis is labeled in terms of pressure along the top and $c_p(P/1000)^\kappa$ along the bottom of the diagram. The scale for the pressure term is indicated in the body of the diagram by short vertical line segments. The isotherms (°C) are shown as curved, solid lines labeled at the left. The dash-dotted lines are moist adiabats labeled at the right in terms of wet bulb potential temperature (°C) while the saturation mixing ratios (gm kg$^{-1}$) are indicated by dotted lines labeled at the right. An increase in the size of the diagram combined with the use of color would allow all of the lines to be at the smaller interval necessary for accurate computations.

That the diagram is an equal-area transformation of an $(\alpha,P)$ diagram may be shown by the following derivation for which I am indebted to one of the reviewers:

$$\text{AREA}\,(\theta,\Delta\psi) = \oint \theta^* d\left[c_p\left(\frac{P}{1000}\right)^\kappa\right] = \oint \frac{\theta^* c_p \kappa P^{\kappa-1}}{(1000)^\kappa} dP.$$

Substitution of the definitions of $\theta^*$ and $\kappa$ yields

$$\text{AREA}\,(\theta,\Delta\psi) = \oint \frac{T^*}{P} R dP$$

$$= -\oint \alpha dP$$

$$= -\text{AREA}\,(\alpha,P) = -(\text{WORK DONE}).$$

## 4. The evaluation of a sample sounding

Fig. 2 illustrates a temperature sounding plotted on the diagram. The Montgomery streamfunction values have been calculated by building up from the surface in 5K increments of virtual potential temperature. The mean of the pressure term is indicated by a short vertical line segment for each layer, labeled with its numerical value in units of $10^5$ ergs gm$^{-1}$ (°K)$^{-1}$. Multiplication of this value by the virtual potential temperature increment for the layer (5K for all layers except 3.7K for the layer nearest the ground) yields $\Delta\psi$ for the layer. Beginning with the values of $\psi$ at the surface ($\psi_0 = 28{,}276 \times 10^5$ ergs gm$^{-1}$), the $\Delta\psi$ values are added sequentially to find $\psi$ at the top of each layer.

## 5. Error analysis

The sensitivity of the Montgomery streamfunction computation to measurement errors and to improper

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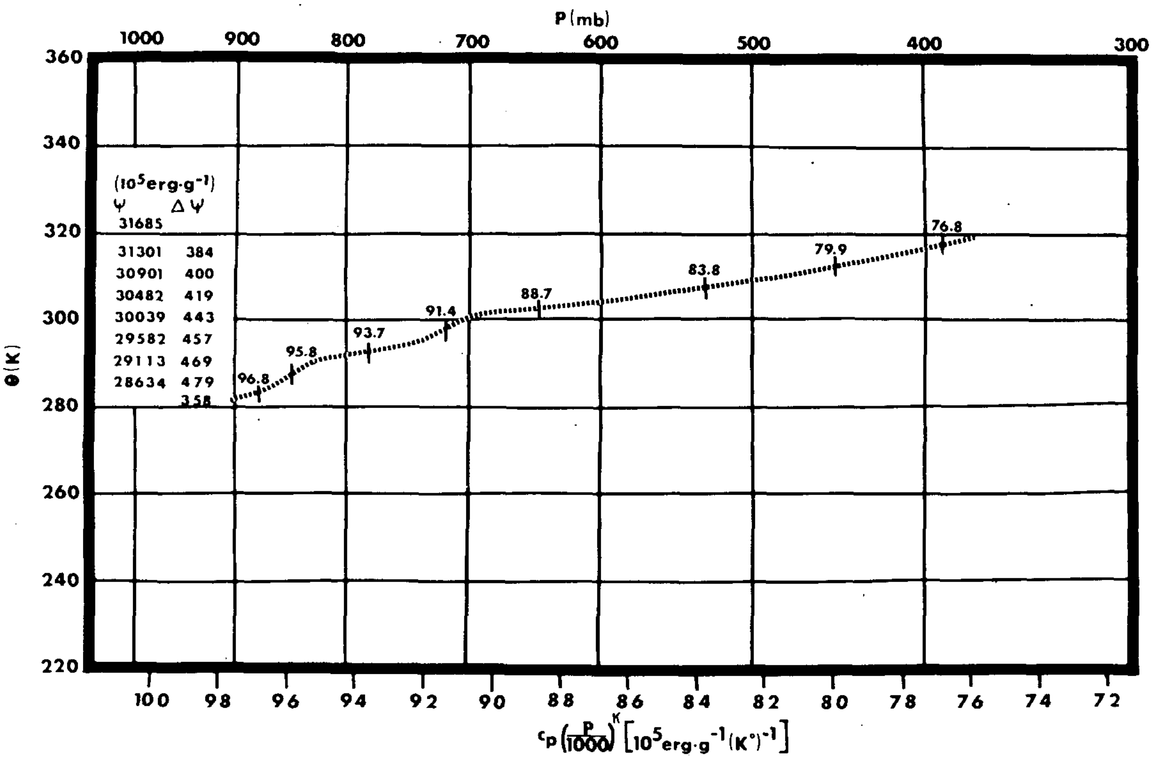


FIG. 2. Sounding from Dodge City, Kansas, 1200 GMT 10 December 1971, plotted on a $\theta$-$\Delta\psi$ diagram.

data handling has been emphasized by Danielson (1959) and by Reiter (1963). Their work is not, however, directly applicable to the technique here outlined, because they use pressure as the independent variable. It is therefore necessary that the error inherent in this technique, in which $\theta^*$ is the independent variable, be examined. Integration of (3) from the surface of the earth to a selected isentropic surface aloft leads to

$$\psi_1=\psi_0+\int_{\theta_0^*}^{\theta_1^*} c_p\left(\frac{P}{1000}\right)^{\kappa} d\theta^*, \qquad (5)$$

where subscript 1 indicates the selected isentropic surface aloft and 0 indicates the surface of the earth. Eq. (5) may be differentiated through the application of Leibnitz' rule to obtain

$$\delta\psi_1=\delta\psi_0+\int_{\theta_0^*}^{\theta_1^*}\frac{R}{P}\left(\frac{P}{1000}\right)^{\kappa}\delta P_{\theta^*}d\theta^* +c_p\left(\frac{P_1}{1000}\right)^{\kappa}\delta\theta_1^*-c_p\left(\frac{P_0}{1000}\right)^{\kappa}\delta\theta_0^*, \qquad (6)$$

where $\delta$ indicates an error. Noting that $\delta\theta_1^*=0$, that $\delta P_{\theta}{}^*=-(\partial P/\partial\theta^*)\delta\theta^*$, and differentiating (1) and (2), we obtain

$$\delta\psi_1=g\delta z_0-\int_{\theta_0^*}^{\theta_1^*}\frac{R}{P}\frac{\partial P}{\partial\theta^*}\left[\delta T^*-\frac{\kappa T^*}{P}\delta P\right]d\theta^*+\frac{RT_0^*}{P}\delta P_0. \qquad (7)$$

If $\delta T^*$, $\delta P$ and $\partial P/\partial\theta^*$ are assumed constant with respect to $\theta^*$, $|\partial P/\partial\theta^*|=100$ mb $(5\text{K})^{-1}$ [approximately the moist adiabatic lapse rate], and typical values for the other variables are inserted in (5), then

$$\delta\psi_1=(9.8\times10^4)\delta z_0+(7.2\times10^5)\delta T^* -(7.0\times10^4)\delta P+(8.7\times10^5)\delta P_0, \qquad (8)$$

with $\delta\psi_1$ in ergs gm$^{-1}$, $\delta z_0$ in m, $\delta T^*$ in °K, and $\delta P$ and $\delta P_0$ in mb. For example, if $\delta z_0=5$ m, $\delta T^*=1$K, $\delta P=5$ mb, and $\delta P_0=1$ mb, then each error term in (6) is roughly $5\times10^5$ ergs gm$^{-1}$. As shown by Danielson (1959), this is comparable with the error in determining the geopotential on a constant pressure surface [e.g., $\delta T^*=1$K results in $\delta(gz)=7.2\times10^5$ ergs gm$^{-1}$ under similar circumstances]. It should be acknowledged, however, that as the lapse rate approaches the dry adiabatic, $|\partial P/\partial\theta^*|$ increases without limit and the consequences of errors in temperature and pressure become greatly magnified.

The only other source of error which must be considered is that which results from reading values from the diagram. Using a diagram approximately 20 cm by 30 cm it is possible to read the values of $c_p(P/1000)^{\kappa}$ to the nearest $0.2\times10^5$ ergs gm$^{-1}$ (°K)$^{-1}$ which is well inside the limits imposed by instrument errors.

## 6. Case study

The development of a major storm with strong low-level winds prompted the selection of 10 December 1971

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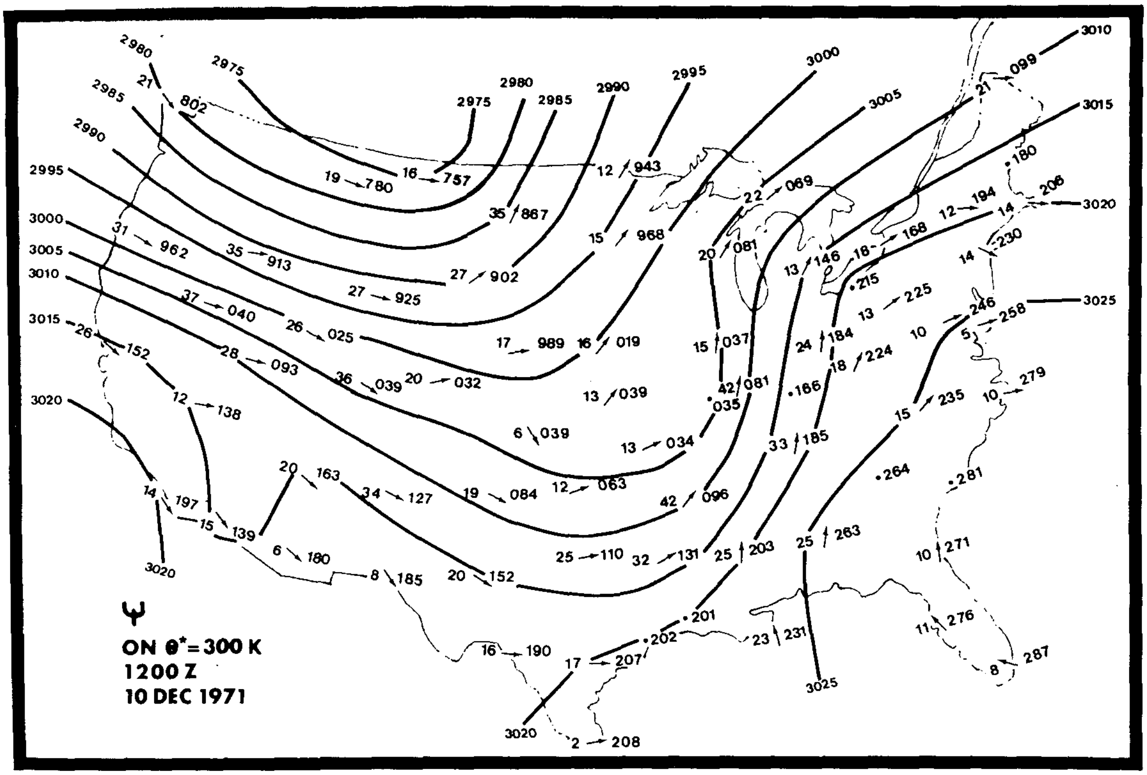


FIG. 3. The Montgomery streamfunction on the $\theta^*=300$K surface. The last three digits of $\psi$ (in units of $10^5$ ergs gm)$^{-1}$ are plotted to the right of each station. Wind directions are indicated by arrows with wind speeds (m sec$^{-1}$) plotted to the left of the station. Contours labeled for $\psi$ are in units of $10^6$ ergs gm$^{-1}$.

as a test of the graphical Montgomery streamfunction computation. The soundings were plotted on $\theta$-$\Delta\psi$ diagrams directly from the teletype transmissions using a Hewlett-Packard 9100A desk-top calculator and plotter. The mean value of the pressure term for each 5K layer was found manually by the equal-area method and the $\psi$ values were determined as in the example. The results are shown in Fig. 3 where the Montgomery streamfunction analysis may be compared with the observed winds on the 300K surface of constant virtual potential temperature. The analysis was done without the aid of the wind observations. Clearly, the graphical technique produces $\psi$ values which are spatially consistent and closely related to the winds.

## 7. Summary

The $\theta$-$\Delta\psi$ diagram is a true thermodynamic diagram designed for use in the graphical computation of the Montgomery streamfunction. It produces accurate, consistent results with a minimum of effort. The diagram is not intended to replace the computation of the streamfunction by computer methods where large volumes of data must be processed. It may be used where small amounts of data make it impractical or inefficient to use a large computer or where a computer is simply not available. It should also be noted that the theory of the $\theta$-$\Delta\psi$ diagram has been used at the University of Wisconsin to develop an operational computer program for the computation of the Montgomery streamfunction (Whittaker, 1973).[1]

*Acknowledgments.* The work reported here was sponsored by the National Science Foundation under Grant NSF GA-30978X. The author wishes to thank Mr. Thomas Whittaker for computer programming which was helpful in the preparation of the sample diagram and Mr. Scott Singer for his help with the computations and drafting. Special thanks are due Dr. Lyle Horn for his critical reading of the manuscript.

[1] Personal communication.

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