Equations for Calculating the Terminal Velocities of Water Drops

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ABSTRACT

The experimental data of Gunn and Kinzer, Beard and Pruppacher, and Davies are used to curve-fit a polynomial for Re in terms of $C_d\mathrm{Re}^2$. The resulting equations predict very accurately all known experimental data for drop fall velocities at sea level and at higher altitudes. Iteration of the equations is not necessary.

1. Introduction

The most definitive measurements of drop fall velocities have commonly been considered to be those of Gunn and Kinzer (1949), which span the radius range from 0.005 to 0.29 cm or in Reynolds numbers (Re) from 1.8 to 3549. More recent experiments by Beard and Pruppacher (1969), in the radius range from 0.001 to 0.0475 cm (Re from 0.2 to 200) tend to corroborate the data of Gunn and Kinzer except for finding slightly smaller velocities near 0.005 cm radius, the lower limit used by Gunn and Kinzer. Careful work by Eaton (1971) demonstrates that the values of Beard and Pruppacher are highly accurate for $20<\mathrm{Re}<80$.

It is of interest to note that the data of Beard and Pruppacher almost exactly duplicate the data for rigid spheres gathered from several investigators and listed by Davies (1945). Fig. 1 shows the Gunn and Kinzer data well above the rest in Re and therefore in terminal velocity. The data of Gunn and Kinzer approach the Stokes law at higher radii than do the others. This difference was attributed by Beard and Pruppacher to the possibility that the drop size determined by Gunn and Kinzer at the end of the fall through air with a relative humidity of 50% had been reduced by evaporation and was therefore slightly smaller than when its terminal velocity was measured. Davies, in fact, shows the deviation of these measured terminal velocity data and Stokes law for values of Re from 0.037 to 0.82. The deviation he gives is 0.5% for the smaller to 10% for the larger Re. Fig. 2 shows the data in good agreement at higher Re.

The fall velocity of water drops is approximately that of rigid spheres up to a certain size. Beyond that size, however, the deviations in velocity V and drag coefficient C_d are significant as indicated in Fig. 3. The data in Fig. 3 for water drops is from Gunn and Kinzer and for spheres from Davies. Mason (1957) gives a formula for $C_d\mathrm{Re}$ as a function of Re for the rigid spheres of Goldstein (1929a,b).

Of the several methods for calculating the terminal velocities of water drops, as discussed by Beard and Pruppacher, the most useful seems to be that of establishing the relationship between Re and $C_d\mathrm{Re}^2$. The origin of this method is attributed by Beard and Pruppacher to Langmuir (1948), Mason (1957) and McDonald (1960). Heymsfield (1972) attributes this method to Best (1950) in his decision to call $C_d\mathrm{Re}^2$ the "Best number." An earlier use of this method, however, is to be found in Davies (1945), who derived a least-squares fit to the terminal velocity data for rigid spheres.

In this paper, we apply the method of Davies to the data of Gunn and Kinzer and also of Beard and Pruppacher for water drops. The improved slip correction from Ranz and Wong (1952) is added. The result is an accurate and versatile method for computing the Reynolds number and thence the terminal velocity for water drops under a wide variety of atmospheric conditions from sea level to high altitudes.

2. Curve-fits

Since

$$\mathrm{Re}=2rV\rho/\eta, \tag{1}$$

$$C_d=2mg/(\rho V^2A), \tag{2}$$

where r is the radius of a spherical drop of mass m, ρ the density of the air, η the dynamic viscosity, m the mass of the drop, g the acceleration of gravity, and

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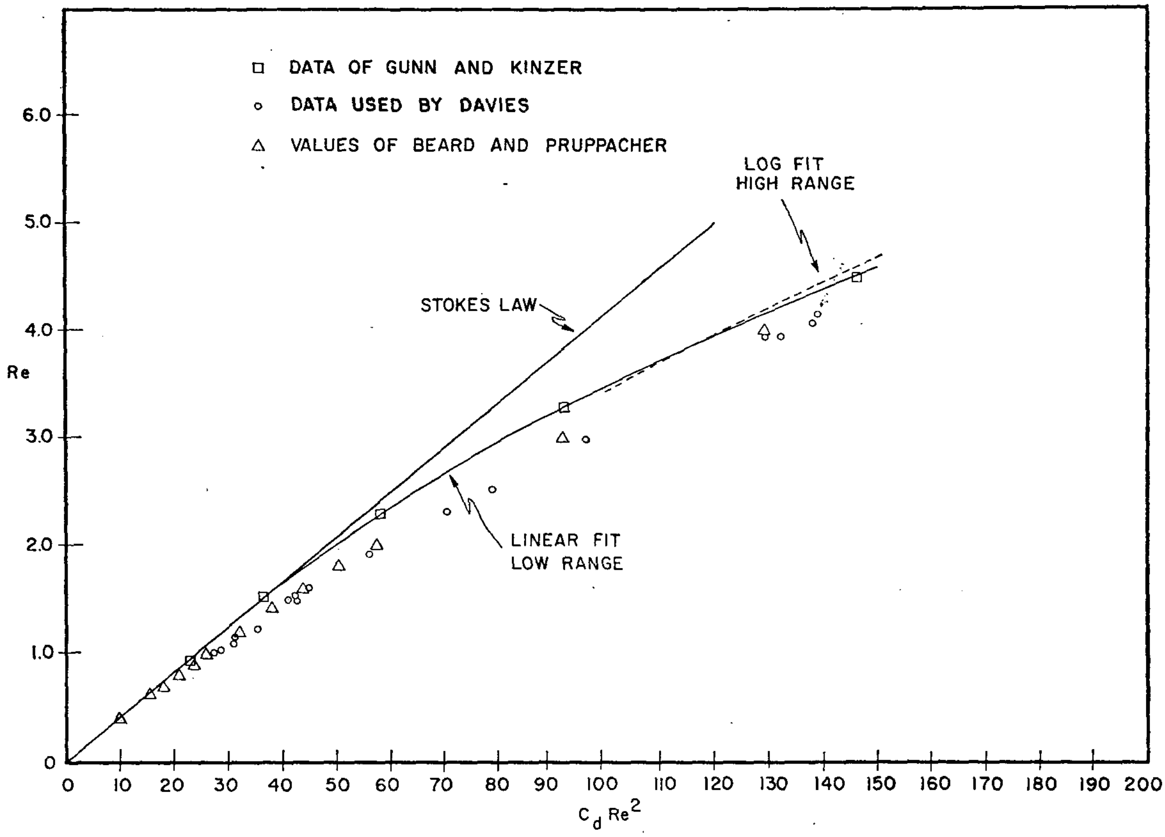

FIG. 1. Re as a function of $C_d\mathrm{Re}^2$, in region of lower Re, where the data of Beard and Pruppacher for water drops agree quite well with those given Davies for rigid spheres. Both of these, however, show measurably lower Re than do the data of Gunn and Kinzer for water drops. The solid, curved line shows the curve-fit of Eq. (6) to the data of Gunn and Kinzer. The curve-fits of Eqs. (8), (12), (13) and (14) would show a similarly accurate fit of the Beard and Pruppacher data.

A the cross-sectional area for a spherical drop $[=\pi r^2]$, then the product

$$C_d\mathrm{Re}^2=(8/\pi)(mg\rho/\eta^2) \qquad (3)$$

is independent of the fall velocity. If the drop shape deviates from that of a sphere, the computed C_d in (2) will apply to the deformed drop.

Our initial, least-squares curve-fits, which were made in 1967 as a basis for studies in stochastic collection, used the data of Gunn and Kinzer because we then believed them to be more accurate than those presented by Davies. In retrospect, we might better have used the data of Davies since they match so well with the newer data of Beard and Pruppacher. Here we show curve-fits for the newer data as well.

Following Davies (1945), we use the following polynomials, where $X=C_d\mathrm{Re}^2$:

$$\mathrm{Re}=a_{11}X_1+a_{12}X_1^2+a_{13}X_1^3+a_{14}X_1^4, \quad \text{for smaller } X_1, \qquad (4)$$

$$\ln\mathrm{Re}=a_{20}+a_{21}\ln X_2+a_{22}(\ln X_2)^2+a_{23}(\ln X_2)^3, \quad \text{for larger } X_2. \qquad (5)$$

Specifically, we find for the best fit to the Gunn and Kinzer data:

$$\left.\begin{array}{l} a_{11}=+0.428259\times10^{-1}\\ a_{12}=-0.656156\times10^{-5}\\ a_{13}=-0.119872\times10^{-5}\\ a_{14}=+0.464525\times10^{-8}\end{array}\right\}, \quad \text{for } 0<X_1\leqslant 115.20, \qquad (6)$$

$$\left.\begin{array}{l} a_{20}=-0.227924\times10^{+1}\\ a_{21}=+0.744612\times10^{0}\\ a_{22}=+0.735351\times10^{-2}\\ a_{23}=-0.817884\times10^{-3}\end{array}\right\}, \quad \text{for } 115.20\leqslant X_2\leqslant 10^7. \qquad (7)$$

For the data of Beard and Pruppacher up to Re=200 and Gunn and Kinzer from there to Re=3549, as listed in Table 1, we find:

$$\left.\begin{array}{l} a_{11}=+0.412657\times10^{-1}\\ a_{12}=-0.150074\times10^{-3}\\ a_{13}=+0.758804\times10^{-6}\\ a_{14}=-0.168841\times10^{-8}\end{array}\right\}, \quad \text{for } 0<X_1\leqslant 175.270, \qquad (8)$$

$$\left.\begin{array}{l} a_{20}=-0.236534\times10^{+1}\\ a_{21}=+0.767787\times10^{0}\\ a_{22}=+0.535826\times10^{-2}\\ a_{23}=-0.763554\times10^{-3}\end{array}\right\}, \quad \text{for } 175.270\leqslant X_2\leqslant 10^7. \qquad (9)$$

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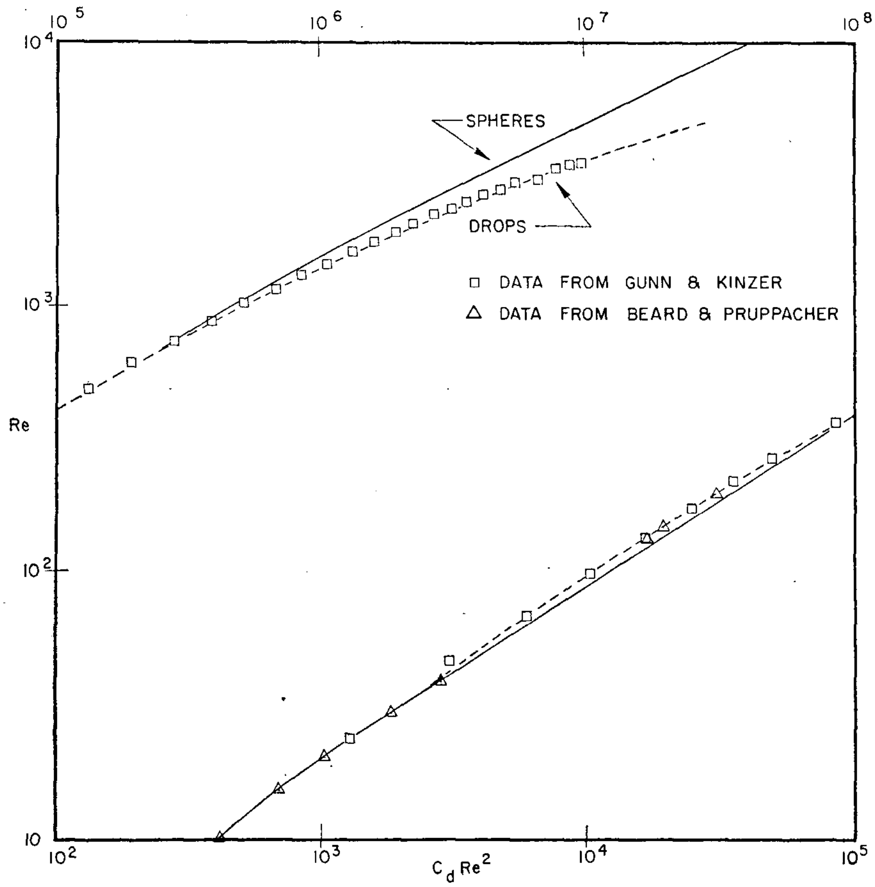

FIG. 2. Re as a function of $C_d\mathrm{Re}^2$ in the region of higher Re, where the data of Beard and Pruppacher agree quite well with the data of Gunn and Kinzer. The dashed line shows the curve-fit of Eq. (7) to the Gunn and Kinzer data. Curve fits (9), (13) and (14) would be similarly accurate. The solid line shows the Davies' curve fit of (11) for rigid spheres.

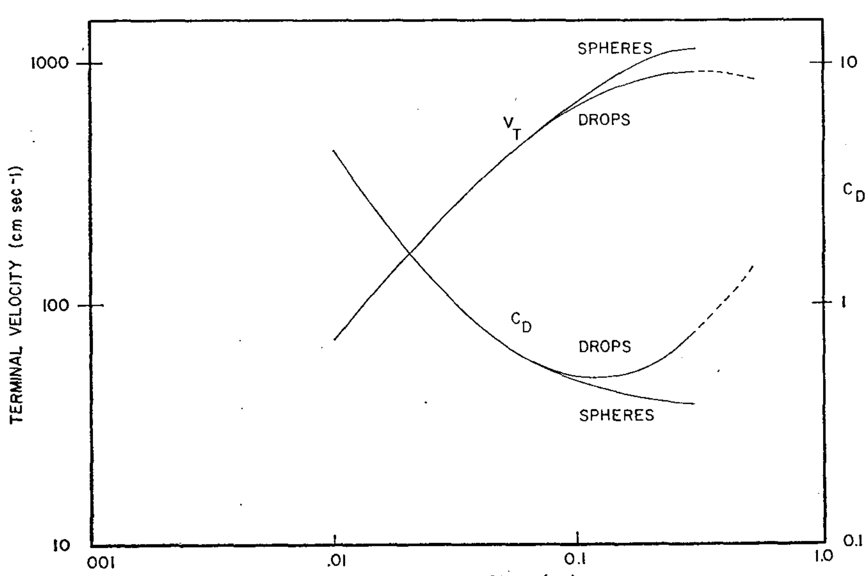

FIG. 3. Terminal velocities and drag coefficients for water drops (from data of Gunn and Kinzer) and for rigid spheres (from data of Davies) at 760 mm, 20C, 50% relative humidity.

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These two equations [(8) and (9)] give the most accurate values of the terminal velocities of water drops known to these authors.

The coefficients given by Davies (here transposed to natural logarithms) for data on *rigid spheres*, which approximate the data of Beard and Pruppacher in the low Re range, are:

$$\left.\begin{aligned} a_{11}&=+0.416666\times10^{-1}\\ a_{12}&=-0.23363\times10^{-3}\\ a_{13}&=+0.20154\times10^{-5}\\ a_{14}&=-0.69105\times10^{-8}\end{aligned}\right\}, \quad \text{for } 0<X_1<140, \tag{10}$$

$$\left.\begin{aligned} a_{20}&=-0.298268\times10^{0}\\ a_{21}&=+0.986\times10^{0}\\ a_{22}&=-0.202716\times10^{-1}\\ a_{23}&=+0.211905\times10^{-3}\end{aligned}\right\}, \quad \text{for } 100<X_2<4.5\times10^{7}. \tag{12}$$

In addition, because of possible convenience to the user, we have made the following curve-fits. The low Re range data of Beard and Pruppacher may be approximated by (5) rather than (4) by use of the coefficients:

$$\left.\begin{aligned} a_{20}&=-0.31944\times10^{+1}\\ a_{21}&=+0.100773\times10^{+1}\\ a_{22}&=-0.271842\times10^{-2}\\ a_{23}&=-0.223127\times10^{-2}\end{aligned}\right\}, \quad \text{for } 2.4<X_2\leqslant 221.385. \tag{12}$$

This may be used to replace (8) for the lower range where (9) is still used for the higher range. The advantage is that only one formula, namely (5), is now needed. The disadvantage is that (12) cannot be used for Re $\lesssim 0.1$. The upper limit on X for (12) becomes the cross-over point that matches the values of (12) with those from (9). Within its range, (12) is just as accurate as (8).

With the same restriction on low Re, two curve-fits have been achieved over the whole range of listed data using (5). Their accuracy is sufficient for many applications. These are:

$$\left.\begin{aligned} a_{20}&=-0.312611\times10^{1}\\ a_{21}&=+0.101338\times10^{1}\\ a_{22}&=-0.191182\times10^{-1}\end{aligned}\right\} \begin{bmatrix} 2.4<X_2<10^7\\ 0.1<\mathrm{Re}<3550\end{bmatrix}, \tag{13}$$

$$\left.\begin{aligned} a_{20}&=-0.327486\times10^{1}\\ a_{21}&=+0.112394\times10^{1}\\ a_{22}&=-0.440777\times10^{-1}\\ a_{23}&=+0.214874\times10^{-2}\\ a_{24}&=-0.619713\times10^{-4}\end{aligned}\right\} \begin{bmatrix} 2.4<X_2<10^7\\ 0.1<\mathrm{Re}<3550\end{bmatrix}. \tag{14}$$

The use of two extra terms in (14) decreases the standard deviation over (13) by a factor of 2, as is seen in the following paragraph. Eq. (13) provides a reasonably accurate (within 3% of Re), simple means of determining the fall velocity of a liquid drop. As is indicated in the next section, it should retain this accuracy over a large range of altitudes.

The standard deviation of the above curve-fits from their respective data is as follows:

$\sigma(\mathrm{Re})=0.01$	for (6)
$\sigma(\ln\mathrm{Re})=0.01$	for (7)
$\sigma(\mathrm{Re})=0.0013544$	for (8)
$\sigma(\ln\mathrm{Re})=0.007293$	for (9)
$\sigma(\mathrm{Re})=$ (from Davies)	for (10)
$\sigma(\ln\mathrm{Re})=$ (from Davies)	for (11)
$\sigma(\ln\mathrm{Re})=0.0013551$	for (12)
$\sigma(\ln\mathrm{Re})=0.023069$	for (13)
$\sigma(\ln\mathrm{Re})=0.011471$	for (14).

Table 1 lists the data used for curve fits (8), (9), (12), (13) and (14), and compares the values of Re calculated by (8), (9), (12) and (13).

3. Calculation of fall velocities

From known values of m of the drop and g, ρ and η at the position in the atmosphere, $X=C_d\mathrm{Re}^2$ is calculated from (3). Using (4) or (5) for the appropriate range of X, and the coefficients of (6)–(14) corresponding to the desired curve fit, Re is calculated.

Finally, when the size of the drop is comparable to or smaller than the mean free path of the air molecules, the hydrodynamic theory of a continuous medium no longer applies, and a correction factor S is needed.

The basic equation for the correction factor, called the "slip correction," was deduced by Knudsen and Weber (1911). Davies (1945) used data from several investigators, most notably Millikan (1920), and the definition of mean free path from Chapman and Enskog, to evaluate the coefficients of the equation. These coefficients were only slightly modified by Ranz and Wong (1952), whose resulting equation for the slip correction factor S is

$$S=1.0+(l/r)[1.23+0.41\exp(-0.88r/l)], \tag{15}$$

where l is the mean free path which may be determined from

$$l=0.812\times10^{-8}/\rho. \tag{16}$$

Using (1) and (15), the fall velocity is now calculated from,

$$V=S\,\mathrm{Re}\,\eta/(2r\rho). \tag{17}$$

4. Evaluation of the results

The calculated fall velocities at altitudes agree very closely with the experimental data of Beard and Pruppacher for radii up to 0.045 cm and pressures to 400 mb. They also agree very closely with the calculated curves of Foote and DuToit up to 0.30 cm radius at sea level but begin to deviate significantly at higher levels for radii >0.20 cm. At 500 mb, for instance, the equations developed here predict a fall velocity for 0.30 cm radius drops some 4% greater than that calculated by Foote and DuToit.

In analyzing this discrepancy, we must note that the equation of Foote and DuToit for fall velocities at

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TABLE 1. Data of Beard and Pruppacher to Re=200 and Gunn and Kinzer to higher values, and the associated values of Re produced by the equations as designated in the text.

Low range

$C_d\mathrm{Re}^2$	Re	Eq. (8) Linear	Eq. (12) Log-log	Eq. (13) Log-log
2.427	0.1	0.0993	0.0998	0.106
4.905	0.2	0.1989	0.2003	0.209
7.433	0.3	0.2987	0.3006	0.310
10.008	0.4	0.3987	0.4006	0.409
12.631	0.5	0.4988	0.5003	0.507
15.302	0.6	0.5989	0.6000	0.604
18.019	0.7	0.6991	0.6996	0.701
20.783	0.8	0.7993	0.7992	0.797
23.592	0.9	0.8995	0.8988	0.893
26.448	1	0.9996	0.9985	0.988
29.349	1.1	1.100	1.098	1.08
32.296	1.2	1.200	1.198	1.18
35.288	1.3	1.300	1.298	1.28
38.326	1.4	1.400	1.399	1.37
41.408	1.5	1.500	1.499	1.47
44.536	1.6	1.601	1.600	1.56
47.708	1.7	1.701	1.701	1.66
50.924	1.8	1.801	1.802	1.76
54.186	1.9	1.902	1.903	1.85
57.491	2	2.002	2.005	1.95
91.984	3	2.996	3.005	2.91
129.56	4	4.002	4.000	3.87
170.17	5	5.000	4.993	4.84

High range

$C_d\mathrm{Re}^2$	Re	Eq. (9) Log-log	Eq. (13) Log-log
213.69	6	5.99	5.82
260	7	6.95	6.81
310	8	7.94	7.83
360.7	9	8.89	8.83
414.95	10	9.88	9.86
471.73	11	10.87	10.89
530.99	12	11.88	11.94
592.68	13	12.89	13.00
656.8	14	13.91	14.06
723.27	15	14.94	15.14
792.07	16	15.98	16.22
863.17	17	17.02	17.31
936.56	18	18.08	18.40
1012.2	19	19.13	19.51
1090	20	20.19	20.62
1887.7	30	30.06	30.88
2827.4	40	40.11	41.26
3887.7	50	50.19	51.62
5059.2	60	60.29	61.95
6334.5	70	70.39	72.22
7707.8	80	80.49	82.45
9174.5	90	90.57	92.62
10731	100	100.6	102.7
12373	110	110.7	112.8
14098	120	120.7	122.8
15903	130	130.7	132.7
17786	140	140.6	142.6
19746	150	150.5	152.4
21779	160	160.4	162.2
23885	170	170.2	171.9
26061	180	180.0	181.5
28306	190	189.7	191.1
30620	200	199.5	200.7
35400	220	218.6	219.4
48500	269	265.9	265.6
83600	372	369.9	366.3
133000	483	485.4	476.6
198000	603	608.2	595.5
282000	731	738.7	720.9
388000	866	875.9	853.0
517000	1013	1017	989.1
670000	1164	1159	1125
850000	1313	1304	1267
1.05E+6	1461	1443	1404
1.31E+6	1613	1601	1564
1.59E+6	1764	1750	1713
1.91E+6	1915	1900	1865
2.22E+6	2066	2030	1998
2.65E+6	2211	2191	2164
3.10E+6	2357	2340	2321
3.58E+6	2500	2483	2472
4.13E+6	2636	2631	2630
4.72E+6	2772	2774	2785
5.35E+6	2905	2912	2937
6.06E+6	3033	3053	3094
6.80E+6	3164	3187	3245
7.61E+6	3293	3321	3398
8.50E+6	3423	3455	3554
9.45E+6	3549	3587	3708

higher latitudes is a curve fit to unpublished data of Davies, as quoted by Sutton (1942), which corresponds to radii from 0.0894 to 0.168 cm, and has no strong basis for being correct for radii much larger than the last data point. Wobus *et al.* (1971) incorrectly attribute the data of Davies as covering the range from 0.169 to 0.297 cm. As quoted by Sutton, Davies' unpublished data is valid for $0.4<X_0<1.4$ where

$$X_0=4r^2\rho_0/g\sigma,$$

where ρ_0 is the density of water and σ the surface tension of water. From this the limits mentioned above directly follow. The curve-fits in the present paper use data to radii of 0.29 cm, and seem to have in the use of Re a more physical basis for extrapolation to higher altitudes.

The evaluation of these equations, or any others, for high altitudes and larger drops must await new experimental data. Aside from the use of $C_d\mathrm{Re}^2$ and the Reynolds number, the physical basis for extrapolations of these equations far outside the limits of the data is weak. Internal circulations, mechanical pulsations, and irregular vortex shedding combine to make the fall velocity of large drops somewhat irregular (Dingle and Lee, 1972) and certainly difficult to calculate.

5. Conclusion

The curve-fits given here provide a very close fit to experimental data at sea level. The use of Re as a function of $C_d\mathrm{Re}^2$ for the curve-fits provides a physical basis for extrapolation to other atmospheric conditions. Without the benefit of any other curve-fits or equations, the described equations accurately predict the fall velocities for all available data for higher altitudes. The

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solution for the fall velocities is direct and requires no iteration.

The data of Beard and Pruppacher are very close to those gathered by Davies, which for Reynolds numbers from 1 to 5 give smaller fall velocities than those of Gunn and Kinzer.

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