On the Correct Use of the Wet Adiabatic Lapse Rate in Stability Criteria of a Saturated Atmosphere

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(Manuscript received 4 October 1973, in revised form 15 January 1974)

ABSTRACT

A rigorous stability analysis of a saturated atmosphere is carried out and is compared with the parcel method. It is shown that the stability parameter $\tilde{n}_w^2$ (the Brunt-Väisälä frequency) one obtains by the two methods is identical. It is further shown that the replacement of the dry adiabatic by the wet adiabatic lapse rate in studying the stability of a saturated atmosphere is inadequate in certain circumstances. In particular, for an atmosphere at rest with negative temperature gradients, such a replacement may lead to erroneous prediction of instability. Similarly, for an atmosphere with a background wind, the same replacement will lead to underestimation of stability for sufficiently negative temperature gradients.

1. Introduction

The stability of a dry atmosphere is usually analyzed in terms of the parameter

$$n_d^2=\frac{g}{T_0}\left(\frac{dT_0}{dz}-\Gamma_d\right), \tag{1}$$

where g is the gravitational acceleration acting in the negative z direction; T_0 is the actual temperature of the atmosphere; dT_0/dz is the observed rate of change of the temperature with height, i.e., the temperature lapse rate; Γ_d is the dry adiabatic lapse rate; and n_d, the Brunt-Väisälä frequency of the dry air, is the frequency of oscillation of a parcel of dry air when perturbed from its initial position. For the oscillation to be stable, n_d^2 must be positive; therefore, the stability criterion can be stated as follows: if the observed lapse rate is less than the dry adiabatic lapse rate, the air is stable; if greater, the air is unstable; and if equal to the dry adiabatic lapse rate, neutral.

Similarly, the stability of a saturated atmosphere relative to adiabatic perturbation is usually analyzed in terms of the parameter

$$n_w^2=\frac{g}{T_0}\left(\frac{dT_0}{dz}-\Gamma_w\right), \tag{2}$$

where n_w is the Brunt-Väisälä frequency of oscillation of a parcel of saturated air and Γ_w the moist adiabatic lapse rate whose analytical expression will be given shortly. In the limit of no water vapor, n_w and Γ_w reduce to n_d and Γ_d, respectively. Since the stability of the system is now analyzed in terms of the sign of n_w^2, it appears that the criterion for the stability or instability of saturated air is that of the dry air but with Γ_d, the dry adiabatic lapse rate, replaced by Γ_w, the wet one.

This is not always so, however. A number of approximations have been incorporated in the derivation of the expression for n_w^2. Some are related to the actual analytical expression for Γ_w, others to the actual form of the relation between n_w^2, dT_0/dz and Γ_w. In the process of studying wave-induced instabilities in an atmosphere near saturation, it was found that circumstances arise when Eq. (2) is not an adequate approximation for the Brunt-Väisälä frequency for the system even though the correct expression for Γ_w is used.

Thus, to clearly specify the correct stability parameter and to assist parameterization studies like that of Betts (1973), a rigorous stability analysis is undertaken in Section 2, which results in a new stability parameter $\tilde{n}_w^2$, in general different from n_w^2. In Section 3, it is shown that careful application of the parcel method yields the same stability parameter $\tilde{n}_w^2$ and not n_w^2. Finally, in Section 4, the regions where $\tilde{n}_w^2$ and n_w^2 lead to substantially different results for a saturated atmosphere with and without a background wind are indicated and illustrative numerical examples are given.

2. Equations of motion and a dynamic stability analysis

The equations governing the behavior of perturbations of a gravitationally stratified compressible saturated atmosphere were obtained by Einaudi and

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Lalas (1973), hereafter referred to as I. They considered the system as a mixture of three fluids, dry air, water vapor and water droplets. The droplets were assumed very small compared to characteristic scales of the system and numerous enough so as to be treated as a continuum. The linearized form of these equations, in the presence of an horizontal, height-dependent background wind $u_0(z)$, are as follows:

Conservation of mass

$$\frac{d}{dt}\overset{(i)}{\rho_1}+\overset{(i)}{\rho_0}\nabla\cdot\mathbf{v}_1+v_{1z}\frac{d}{dz}\overset{(i)}{\rho_0}=\overset{(i)}{\Gamma_1},\quad i=1,2,3 \tag{3}$$

$$\frac{d}{dt}\rho_{M1}+\rho_{M0}\nabla\cdot\mathbf{v}_1+v_{1z}\frac{d}{dz}\rho_{M0}=0 \tag{4}$$

Conservation of momentum

$$\rho_{M0}\frac{d}{dt}v_{1x}+\frac{\partial}{\partial x}p_{M1}+\rho_{M0}v_{1z}\frac{d}{dz}u_0=0 \tag{5}$$

$$\rho_{M0}\frac{d}{dt}v_{1z}+\frac{\partial}{\partial z}p_{M1}+\rho_{M1}g=0 \tag{6}$$

Conservation of energy

$$[\overset{(1)}{\rho_0}\overset{(1)}{c_p}+\overset{(2)}{\rho_0}\overset{(2)}{c_p}+\overset{(3)}{\rho_0}\overset{(3)}{c_p}]\left[\frac{d}{dt}T_1+v_{1z}\frac{d}{dz}T_0\right]-\left[\frac{d}{dt}p_{M1}+v_{1z}\frac{d}{dz}p_{M0}\right]=-L_v\Gamma_1 \tag{7}$$

Equations of state

$$\overset{(i)}{p_1}=R^{(i)}[T_0\overset{(i)}{\rho_1}+\overset{(i)}{\rho_0}T_1],\quad i=1,2 \tag{8}$$

Clausius-Clapeyron equation

$$\rho_{\omega1}=\overset{(2)}{\rho_0}\left[\frac{L_v}{R^{(2)}T_0}-1\right]\frac{T_1}{T_0} \tag{9}$$

Equation for mass exchange

$$\Gamma_1=[\rho_{\omega1}-\overset{(2)}{\rho_1}]/\tau_m,\qquad 1/\tau_m=4\pi n_p a_0 D. \tag{10}$$

The background quantities are denoted by the suffix "0" and depend on z only; suffix "1" refers to perturbation quantities; superscripts $i=1, 2, 3$ refer to dry air, water vapor and water droplets continuum, respectively; the symbols $p^{(i)}$, $\rho^{(i)}$, T, v_{1x} and v_{1z} denote the partial pressures, partial densities, temperature, horizontal and vertical velocity perturbations, respectively; $c_p^{(i)}$ is the specific heat at constant pressure and $R^{(i)}$ the gas constant; L_v is the latent heat of evaporation; n_p is the number of droplets, all assumed of radius a_0, per unit volume of the mixture and is an externally specified quantity depending on the number of nuclei around which each droplet can grow; D is the diffusivity of water vapor; $\Gamma_1^{(1)}=0$; $\Gamma_1^{(2)}=-\Gamma_1^{(3)}=\Gamma_1$ is the production of water vapor per unit volume and unit time; $\rho_{\omega1}$ is the partial density of vapor at the droplet surface; and the total pressure and density, and the operator d/dt are defined as

$$\left.\begin{aligned}\rho_{M0}&=\sum_{i=1}^{3}\overset{(i)}{\rho_0}, & \rho_{M1}&=\sum_{i=1}^{3}\overset{(i)}{\rho_1}\\ p_{M0}&=\sum_{i=1}^{2}\overset{(i)}{p_0}, & p_{M1}&=\sum_{i=1}^{2}\overset{(i)}{p_1}\\ \frac{d}{dt}&=\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\end{aligned}\right\}. \tag{11}$$

The background quantities have to satisfy the zero-order equations

$$\frac{dp_{M0}}{dz}=-\rho_{M0}g, \tag{12a}$$

$$\overset{(i)}{p_0}=R^{(i)}\overset{(i)}{\rho_0}T_0, \tag{12b}$$

$$\overset{(2)}{\rho_0}=\rho_{0\omega}, \tag{12c}$$

$$\frac{d\overset{(2)}{p_0}}{dz}=\frac{dp_{0\omega}}{dz}=\frac{L_v\overset{(2)}{\rho_0}}{T_0}\frac{dT_0}{dz}. \tag{12d}$$

Eqs. (4)–(7) are the same as those governing the propagation of acoustic-gravity waves in a dry atmosphere (see Hines, 1960), except for a source term in the energy equation equal to minus the product of the latent heat times the rate of production of water vapor; Eq. (9) is obtained by linearizing the Clausius-Clapeyron equation and neglecting the weak temperature dependence of the latent heat of evaporation; and Eq. (10) stems from the assumption that the controlling mechanism of mass exchange between droplet and ambient vapor is diffusion and τ_m is therefore the characteristic time for mass transfer (see Byers, 1965).

A more detailed justification and discussion of the above equations can be found in I. We mention here only the fact that these equations are limited to periods which are large compared to both the characteristic time scale with which the droplet velocity approaches that of the gas and the time scale for heat transfer via conduction. Since in atmospheric problems, this condition for the validity of the equations demands periods much greater than 10^{-3} sec, we focus our attention to infrasonic and internal gravity waves. Consistent with this, we have neglected the terminal velocity u_{0z} of the droplets since it is much smaller than a typical gravity wave speed of a few meters per second. Finally, the atmosphere is assumed saturated at all heights.

In order to study the stability of the above system of equations, we first assume that all the perturbation quantities are of the form

$$A(x,z,t)=A(z)\exp[i(\omega t-k_x x)], \tag{13}$$

where k_x is real and $\omega=\omega_r+i\omega_i$ is complex. The set of equations can then be reduced to the following, second-order, linear, homogeneous differential equation for the variable q related to $v_{1z}(z)$:

$$\frac{d}{dz}\left(\frac{\Omega^2 r dq/dz}{k_x^2-\Omega^2/c_w^2}\right)+r(\tilde{n}_w^2-\Omega^2)q=0, \tag{14}$$

where:

$$q=\frac{v_{1z}(z)}{i\Omega}\exp\left(-\int\frac{g}{c_w^2}dz\right) \tag{15}$$

$$\Omega=\omega-k_x u_0(z),\quad r=\rho_{M0}\exp\left(2\int\frac{g}{c_w^2}dz\right) \tag{16}$$

$$\tilde{n}_w^2=-g\left(\frac{1}{\rho_{M0}}\frac{d\rho_{M0}}{dz}+\frac{g}{c_w^2}\right) \tag{17}$$

$$c_w^2=\frac{\overset{(1)}{\rho_0}T_0R^{(1)}}{\rho_{M0}}\frac{1}{\{1-[R^{(1)}\rho_0^{(1)}\Theta^2/\Psi]\}} \tag{18}$$

$$\Theta=1+L_v\overset{(2)}{\rho_0}/[R^{(1)}\overset{(1)}{\rho_0}T_0]$$
$$\Psi=\overset{(1)}{\rho_0}\overset{(1)}{c_p}+\overset{(2)}{\rho_0}\overset{(2)}{c_p}+\overset{(3)}{\rho_0}\overset{(3)}{c_p} \tag{19}$$

$$+\left[\frac{L_v^2\overset{(2)}{\rho_0}}{R^{(2)}T_0}\right]\left[\frac{R^{(1)}\overset{(1)}{\rho_0}+R^{(2)}\overset{(2)}{\rho_0}}{R^{(1)}\rho_0^{(1)}T_0}\right]. \tag{20}$$

For typical atmospheric conditions, $a_0\approx10^{-3}$ cm, $D=2.58\ 10^{-1}$ cm^2 sec^{-1} and $n_p\approx300$ particles cm^{-3}; hence by Eq. (10), $\tau_m\approx1$ sec. Thus, since the period of the disturbance was assumed much larger than τ_m, Eq. (14) is valid for Ω such that

$$|\Omega|\tau_m=|\omega-k_x u_0|\tau_m\ll1, \tag{21}$$

which implies that we restrict ourselves to frequencies in the internal gravity wave range and, for $u_0\approx10-100$ m sec^{-1}, to horizontal wavelengths λ_x of at least 1 km. For smaller values of u_0, correspondingly smaller values of λ_x can be considered.

The sufficient condition for the stability of any system governed by Eq. (14), with any given real $\tilde{n}_w$, c_w, k_x and r, is that throughout the flow

$$\tilde{n}_w^2\geqslant\frac{(du_0/dz)^2}{4\Delta_1},\quad \Delta_1=1+|\Omega|^2/(k_x^2c_w^2). \tag{22}$$

The proof of this result is given in Lalas and Einaudi (1973), hereafter referred to as II, who extended the method of Howard (1961) and Chimonas (1970) to a saturated atmosphere. They derived Eq. (14), with terms of order $L_v\rho_0^{(2)}/[R^{(1)}\rho_0^{(1)}T_0]$ neglected. In Section 4 we will show that the above simplification, although widely used, may not be justified under conditions of near-neutral static stability. In the limit of no background wind, Eq. (22) reduces to

$$\tilde{n}_w^2>0. \tag{23}$$

Thus $\tilde{n}_w^2$, given by Eq. (17), is indeed the parameter one should use to study the stability of an atmosphere saturated at all heights, with or without a background wind.

The physical significance of $\tilde{n}_w^2$ is given in the next section, in connection with the application of the parcel method to this problem. We emphasize here that inequalities (22) and (23) have been obtained by a rigorous dynamic analysis of the equations governing the propagation of low-frequency disturbances in an atmosphere saturated at all heights.

3. Stability analysis using the parcel method

The parcel method is limited by a number of simplifying assumptions which have been discussed in the literature (see Sutton, 1953). In particular, the displacement of the parcel of air should be small and its movement slow so that for a vertical displacement ξ taking place in the time interval Δt we can assume that

$$\frac{\partial}{\partial t}+v_z\frac{\partial}{\partial z}\approx\frac{\xi}{\Delta t}\frac{\Delta}{\Delta z} \tag{24}$$

holds when no background wind is present.

Following Eckart (1960), we consider a parcel of the air, vapor, water mixture enclosed in a flaccid balloon-like membrane which allows changes of density so that pressures inside and outside are always the same. Let us assume that the initial temperature of the parcel is the same as that of the surrounding air and let us displace the balloon from the position $z=z_0$ to $z=z_0+\xi$ in the time interval Δt. The balloon will then experience a buoyancy force given by

$$\mathbf{f}=-\mathbf{g}[\rho_{Me}(z_0+\xi)-\rho_{Mi}(z_0+\xi)], \tag{25}$$

where e and i refer to values outside and inside the membrane, respectively. In the context of the parcel method, it is not meaningful to divide each quantity into the sum of a background part and a perturbation part. For this reason, in this section quantities without a suffix "0" or "1" may appear and they then indicate total actual values of those quantities.

For convenience, let us rewrite here the nonlinear energy equation for the system [i.e., Eq. (4.20) of I]:

$$[\rho^{(1)}\overset{(1)}{c_p}+\rho^{(2)}\overset{(2)}{c_p}+\rho^{(3)}\overset{(3)}{c_p}]\left[\frac{\partial}{\partial t}T+v_i\frac{\partial}{\partial x_i}T\right]-\left[\frac{\partial}{\partial t}p_M+v_i\frac{\partial}{\partial x_i}p_M\right]=-L_v\Gamma, \quad (26)$$

where we have adopted the notation

$$v_i\frac{\partial}{\partial x_i}=v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}.$$

Eq. (26) is quite general and, as a matter of fact, Eq. (7) is nothing but the linearized form of Eq. (26). Let us also write, for convenience, the nonlinear form of the continuity equations for dry air, water vapor and water [i.e., Eqs. (3.20) minus (3.21), (3.21), and (3.22) in I, respectively]:

$$\frac{\partial}{\partial t}\rho^{(1)}+v_i\frac{\partial}{\partial x_i}\rho^{(1)}+\rho^{(1)}\nabla\cdot\mathbf{v}=0 \quad (27)$$

$$\frac{\partial}{\partial t}\rho^{(2)}+v_i\frac{\partial}{\partial x_i}\rho^{(2)}+\rho^{(2)}\nabla\cdot\mathbf{v}=\Gamma \quad (28)$$

$$\frac{\partial}{\partial t}\rho^{(3)}+v_i\frac{\partial}{\partial x_i}\rho^{(3)}+\rho^{(3)}\nabla\cdot\mathbf{v}=-\Gamma. \quad (29)$$

By adding Eqs. (28) and (29), eliminating the divergence term by use of Eq. (27), and using the equation of state for dry air, we obtain

$$\frac{\partial}{\partial t}[\rho^{(2)}+\rho^{(3)}]+v_i\frac{\partial}{\partial x_i}[\rho^{(2)}+\rho^{(3)}]=[\rho^{(2)}+\rho^{(3)}]\times\left\{\frac{1}{R^{(1)}\rho^{(1)}T}\left[\frac{\partial}{\partial t}p^{(1)}+v_i\frac{\partial}{\partial x_i}p^{(1)}\right]-\frac{1}{T}\left(\frac{\partial}{\partial t}T+v_i\frac{\partial}{\partial x_i}T\right)\right\}. \quad (30)$$

By eliminating the divergence term in Eq. (28) through the use of Eq. (27), and then using Eq. (11) and the equation of state for the mixture of dry air and water vapor, we obtain

$$\Gamma=-\frac{\rho^{(2)}}{R^{(1)}\rho^{(1)}T}\left(\frac{\partial}{\partial t}p_M+v_i\frac{\partial}{\partial x_i}p_M\right)+\left[\frac{R^{(1)}\rho^{(1)}+R^{(2)}\rho^{(2)}}{R^{(1)}\rho^{(1)}}\right]\left[\frac{\partial}{\partial t}\rho^{(2)}+v_i\frac{\partial}{\partial x_i}\rho^{(2)}+\frac{\rho^{(2)}}{T}\left(\frac{\partial}{\partial t}T+v_i\frac{\partial}{\partial x_i}T\right)\right]. \quad (31)$$

Having derived these auxiliary equations, let us calculate the change of density which takes place inside the balloon:

$$[\rho_M(z_0+\xi)]_i=(\rho_M)_i+\left(\frac{\Delta}{\Delta z}\rho_M\right)_i\xi. \quad (32)$$

The changes of density inside the membrane depend on the pressure and temperature changes, which in turn are related by the energy equation. Using (24), (31), and the Clausius-Clapeyron equation, Eq. (26) can be written, to first order:

$$\Psi\left(\frac{\Delta T_0}{\Delta z}\right)_i=\Theta\left(\frac{\Delta p_M}{\Delta z}\right)_i, \quad (33)$$

where Θ and Ψ are given by Eqs. (19) and (20). Also, to first order:

$$\left(\frac{\Delta p_M}{\Delta z}\right)_i=\left(\frac{\Delta p_M}{\Delta z}\right)_e=-g\rho_{M0}. \quad (34)$$

The quantities Θ and Ψ in Eq. (33) and ρ_{M0} in Eq. (34) are evaluated at $z=z_0$. Using (34), Eq. (33) gives the expression for the rate of change of temperature inside the membrane, i.e., the moist adiabatic lapse rate

$$\Gamma_w=-g\rho_{M0}\Theta/\Psi. \quad (35)$$

This expression for Γ_w reduces, in the limit $\rho_0^{(3)}\rightarrow 0$, to that first derived by Brunt (1933) and also reported in Brunt's book (1952). If one, in addition to $\rho_0^{(3)}\rightarrow 0$, neglects terms of order $\rho_0^{(2)}/\rho_0^{(1)}$, unless they are multiplied by $L_v/[R^{(2)}T_0]$ which is of order 20, Eq. (35) reduces to

$$\Gamma_w=-g\frac{\Theta}{c_p^{(1)}+\dfrac{L_v^2\rho_0^{(2)}}{R^{(2)}T_0^2}}, \quad (36)$$

which is the standard expression for the wet adiabatic lapse rate, given in the meteorological books (see, for example, Haurwitz, 1941, and Tverskoi, 1965).

From (30), using (24), we obtain

$$\left\{\frac{\Delta}{\Delta z}[\rho^{(2)}+\rho^{(3)}]\right\}_i=\left\{[\rho^{(2)}+\rho^{(3)}]\times\left[\frac{1}{R^{(1)}\rho^{(1)}T}\frac{\Delta p^{(1)}}{\Delta z}-\frac{1}{T}\frac{\Delta T}{\Delta z}\right]\right\}_i. \quad (37)$$

From Eqs. (12d), (34), (37) and the equation of state for dry air, we can finally write, to first order:

$$\left\{\frac{\Delta}{\Delta z}[\rho^{(1)}+\rho^{(2)}+\rho^{(3)}]\right\}_i=-\rho_{M0}\left[g\frac{\rho_{M0}}{R^{(1)}\rho_0^{(1)}T_0}+\frac{1}{T_0}\Gamma_w\Theta\right]_i, \quad (38)$$

where the right-hand side is evaluated at the initial position of the parcel.

The change of density which takes place outside the parcel can be readily evaluated:

$$[\rho_M(z_0+\xi)]_e=[\rho_M(z_0)]_e+\left[\frac{\Delta\rho_M}{\Delta z}\right]_e\xi. \tag{39}$$

The quantity $(\Delta\rho_M/\Delta z)_e$ is, to first order, simply $d\rho_{M0}/dz$ at $z=z_0$.

Since we have assumed $[\rho_M(z_0)]_i=[\rho_M(z_0)]_e$, Eq. (25) can now be written as

$$f=g\left\{\frac{d\rho_{M0}}{dz}+\rho_{M0}\left[g\frac{\rho_{M0}}{R^{(1)}\rho_0^{(1)}T_0}+\frac{\Gamma_w}{T_0}\Theta\right]\right\}\xi=-\rho_{M0}\tilde{n}_w^2\xi,$$

where

$$\tilde{n}_w^2=-g\left\{\frac{1}{\rho_{M0}}\frac{d\rho_{M0}}{dz}+\left[g\frac{\rho_{M0}}{R^{(1)}\rho_0^{(1)}}+\Gamma_w\Theta\right]\frac{1}{T_0}\right\}. \tag{40}$$

The equation of motion of the parcel can finally be written as

$$\frac{d^2\xi}{dt^2}+\tilde{n}_w^2\xi=0. \tag{41}$$

From Eq. (41), the physical meaning of $\tilde{n}_w$ becomes apparent: it represents the frequency of oscillation of the parcel of saturated air. For the oscillations to be stable, $\tilde{n}_w$ must satisfy the inequality

$$\tilde{n}_w^2>0. \tag{42}$$

Since it can be proven easily that

$$g\frac{\rho_{M0}}{R^{(1)}\rho^{(1)}T_0}+\frac{\Gamma_w\Theta}{T_0}=\frac{g}{c_w^2},$$

it follows that $\tilde{n}_w^2$ given by (40) is the same as that given by (17) and therefore the stability condition (42) derived by this method is the same as (23) derived by a rigorous stability analysis.

Contrary to widespread belief then, the parcel method gives the correct stability parameter. It is the authors' opinion that this is due to two factors. On one hand the parcel method has been applied here with careful consideration for the role played by the divergence of the velocity term; and on the other, the stability analysis of Section 2 has been carried out for periods τ sufficiently long, i.e., $\tau\gg 2\pi$ sec (see condition 21).

4. Comparison of $\tilde{n}_w^2$ and n_w^2 and some numerical examples

We have shown that, by either a rigorous stability analysis or by careful application of the parcel method, the correct criterion for the stability of the moist saturated atmosphere at rest is

$$\tilde{n}_w^2>0,$$

with $\tilde{n}_w^2$ given by either (17) or (40). As far as the present authors could ascertain, however, the stability analysis of a saturated atmosphere is always carried out in terms of the parameter n_w^2 defined by (2). In addition to the references already mentioned, see, for example, the books by Hewson and Longley (1944), Hess (1959), Matveev (1967) and Karalis (1972). At times, the stability parameter is expressed in terms of the equivalent potential temperature and such a parameter is related to n_w^2, within terms of order $\rho_0^{(2)}/\rho_0^{(1)}$ (see Holton, 1972). In II also, the stability parameter used was effectively n_w^2 since terms of order $\rho_0^{(2)}L_v/[R^{(1)}\rho_0^{(1)}T_0]$ were neglected so that $\Theta\rightarrow 1$.

In order to compare $\tilde{n}_w^2$ and n_w^2, we express $\tilde{n}_w^2$ in terms of n_w^2 by the use of (17) and (40) as

$$\tilde{n}_w^2=n_w^2+\left\{\left[\frac{L_v}{R^{(2)}T_0}\right]\left[\frac{R^{(2)}}{R^{(1)}}\right]\frac{\rho_0^{(2)}}{\rho_0^{(1)}}n_w^2 -g\left[\frac{1}{\rho_{M0}}\frac{d\rho_{M0}}{dz}-\frac{1}{\rho_0^{(1)}}\frac{d\rho_0^{(1)}}{dz}\right]\right\}. \tag{43}$$

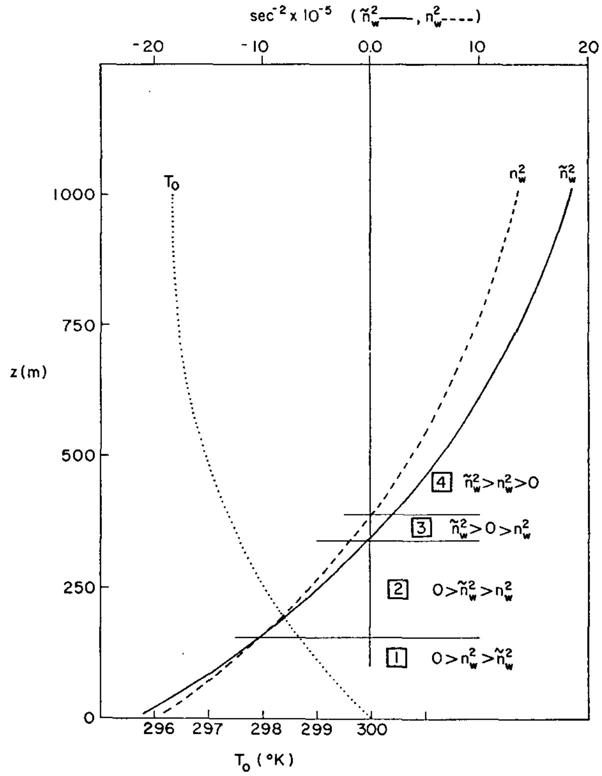

FIG. 1. Plot of n_w^2, $\tilde{n}_w^2$ and T_0 vs height. The temperature distribution (°K) is given by $T_0=300-0.01z\exp(-z/1000)$, with z in meters. The temperature scale is on the bottom and the frequency scale on top. Water droplet density $\rho^{(3)}$ was assumed negligible.

A simplified formula can be obtained if one uses the fact that $[\rho_0^{(2)}+\rho_0^{(3)}]/\rho_0^{(1)}$ is always less than 0.04. If we write

$$\frac{1}{\rho_{M0}}\frac{d\rho_{M0}}{dz} \approx \frac{1}{\rho_0^{(1)}}\frac{d}{dz}[\rho_0^{(1)}+\rho_0^{(2)}+\rho_0^{(3)}] - \frac{\rho_0^{(2)}+\rho_0^{(3)}}{\rho_0^{(1)}}\frac{1}{\rho_0^{(1)}}\frac{d\rho_0^{(1)}}{dz},$$

then Eq. (43) can be written as

$$\tilde{n}_w^2 = n_w^2(\Theta) - g\left[\frac{1}{\rho_0^{(1)}}\frac{d\rho_0^{(3)}}{dz} - \frac{\rho_0^{(3)}}{\rho_0^{(1)}}\frac{1}{\rho_0^{(1)}}\frac{d\rho_0^{(1)}}{dz}\right] - g\frac{\rho_0^{(2)}}{\rho_0^{(1)}}\left[\frac{g}{R^{(1)}T_0}+\frac{L_v}{R^{(2)}T_0}\frac{1}{T_0}\frac{dT_0}{dz}\right]. \qquad (44)$$

In trying to analyze the general differences between n_w^2 and $\tilde{n}_w^2$, let us neglect the effect of liquid water and let us first discuss the case of an atmosphere at rest. The case of greatest interest is, perhaps, when $n_w^2 \approx 0$, i.e., when there are negative temperature lapse rates sufficiently close to the wet adiabatic lapse rate. In this case $\tilde{n}_w^2 > n_w^2$ since, typically,

$$\left[\frac{L_v}{R^{(2)}T_0}\right]\frac{1}{T_0}\left|\frac{dT_0}{dz}\right| \gg \frac{g}{[R^{(1)}T_0]};$$

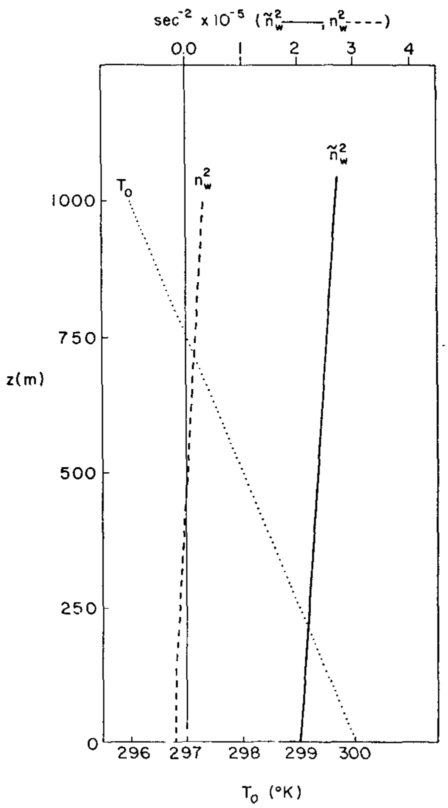

FIG. 2. As in Fig. 1 except for $T_0=300-0.004\ z$.

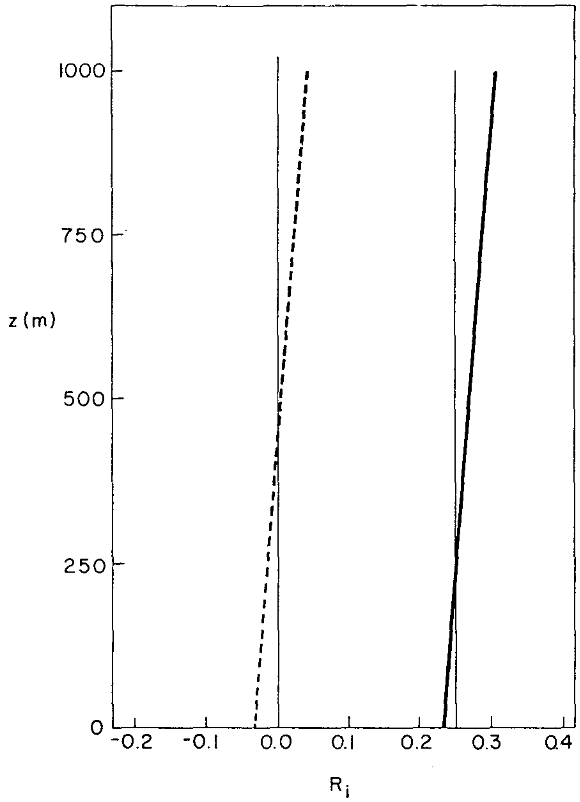

FIG. 3. Plot of the Richardson numbers using $\tilde{n}_w^2$ (solid line) and n_w^2 (dashed line) for the previous temperature profile $T_0=300-0.004\ z$ and for constant shear equal to 9.3×10^{-3} sec^{-1}.

therefore, a range of negative temperature gradients will exist for which a saturated atmosphere seems unstable if (2) is used while it is, indeed, stable. In general, a numerical analysis of (44) is required: in Fig. 1, plots of $\tilde{n}_w^2$ and n_w^2 are shown for a temperature profile with variable temperature gradient to illustrate all the possible kinds of relation between $\tilde{n}_w^2$ and n_w^2.

The difference between n_w^2 and $\tilde{n}_w^2$, in addition, becomes significant in discussing the stability of a saturated atmosphere in the presence of a background wind. The criterion for stability is then

$$\mathrm{Ri} = \tilde{n}_w^2 \Big/ \left(\frac{dU_0}{dz}\right)^2 > \frac{1}{4}, \qquad (45)$$

which is essentially (22). Since the percentage difference between $\tilde{n}_w^2$ and n_w^2 is sometimes large, situations will arise in which, for negative temperature gradients, the use of n_w^2 may indicate instability, whereas in fact the system is stable. This situation is shown in Figs. 2 and 3 where for wind shears below 8.9×10^{-3} sec^{-1} the saturated atmosphere is statistically and dynamically stable while the use of n_w^2 would indicate static instability.

5. Conclusions

In this paper, it is shown that the replacement of the dry adiabatic by the wet adiabatic lapse rate in studying the stability of a saturated atmosphere is inadequate in certain circumstances, and the new form of the stability parameter $\tilde{n}_w^2$ is given. The new Brunt-Väisälä frequency to be used in the stability criterion was derived by both a careful application of the parcel method and a perturbation fluid dynamic stability analysis. For an atmosphere at rest, the usual stability criterion $n_w^2>0$ is especially inadequate when the temperature and moist adiabatic lapse rates are nearly equal. In the presence of a background wind, the same inadequacy is evident for gradient Richardson number near $\frac{1}{4}$.

Acknowledgments. This research was supported in part by Grant GA-40243 from the National Science Foundation.

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