A Controlled Experiment with One-Dimensional Interpolation

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ABSTRACT

The performances of various one-dimensional interpolation schemes are evaluated using mean square errors of the point-wise estimates of the function values and component-wise estimates of the power spectrum. The schemes of interpolation considered include Gandin's optimum interpolation, cubic spline, and two-point linear interpolations. The test function is a random analytic function defined in a closed domain with a prescribed power spectrum, satisfying the conditions of homogeneity and ergodicity.

The experiment is carried out under the circumstance of interpolation which includes a conservative simulation of the large-scale state of the atmosphere and reflects the levels of data acquisition and resolution as practiced currently in meteorology.

The results of the experiment suggest that under all practical circumstances the error of interpolation, which is defined to be the ratio of the sample variance of estimation errors to the sample variance of true values, may not be reduced too much below the value of 0.05. The study also shows that the cubic spline interpolation performs very much like Gandin's optimum interpolation.

1. Introduction

Interpolation occupies a special role in numerical weather prediction. It enters into the preparation of the initial data, the process referred to as objective analysis, as the data gathered by individual meteorological stations are processed to estimate the initial values at the grid points on which the time integration is performed to produce a forecast. It is also needed in the verification of the forecast to furnish predicted values at points where observations are made. It thus takes part both in initiating a prediction and in evaluating its success.

Interpolation is necessary in both of these processes because the network of grid points for numerical computation is different from the network of meteorological stations. The current state of the art of solving a system of partial differential equations, such as the one believed to describe the time evolution of the global weather, whether using the finite-difference method or the spectral method, requires at one stage or another, or throughout the entire numerical process, a network of grid points that possesses some sort of regularity or repetitiveness. On the other hand, the location of a meteorological station is in almost every case decided by a compromise of many conflicting lines of thought and can rarely be dictated by mathematical convenience alone.

The fact that numerical weather prediction cannot do without interpolation and that the success of the former depends heavily on that of the latter makes it imperative that we have a clear understanding of interpolation both in its intended role and in its quality of performance. Thus, for example, when the height of an isobaric surface at a grid point is estimated by interpolation using the observations of the quantity at weather stations, the closeness of the estimated value of the height to the true value is desired. On the other hand, if the wind is estimated at a point using the geopotential or streamfunction, it is then the trueness of the estimated values of the first-order derivatives, rather than the value of the function itself, that is desired. The two different interests may coincide as in the case of the finite-difference method, or share little common ground as in the case of the spectral method.

The uniqueness of the circumstance under which interpolation enters into numerical weather prediction can be readily appreciated by noting that meteorological observations are taken mostly over the parts of land that are accessible to scientific technology, and that the number of available observations at any one time is far less than the number of grid points required for the numerical computation. In an area abundant with weather stations, a straightforward interpolation using contemporaneous observations may be adequate; in another area with few or no observations, however, the climatological normals, often obtained circum-
stantially, may be the only data that can be used to construct the necessary estimates.

The challenge of this reality has been met by the meteorologist with the device of objective analysis. A typical objective analysis constructs the scheme of interpolation upon the statistical norms of the relationships among the deviations of a quantity from its normal values at various points. In other words, in order to complete the required estimation of an individual situation it imposes upon the individual a pattern or a rule which is supposed to be valid only on the statistical basis. This contrivance toward statistical normalcy can be recognized as not only a necessary but also a best means of estimation under the circumstances, as long as the statistical norms employed in the analysis are sufficiently accurate. A lucid exposition of objective analysis is available in L. S. Gandin’s (1963, 1965) work Objective Analysis of Meteorological Fields.

However, the wisdom of introducing a bias toward normalcy could be questioned, if, as in all practice, the statistical norms themselves have to be assumed and/or estimated, or if, as has been done in a number of diagnostic studies of the atmosphere (e.g., Julian et al., 1970), the estimates are used for a purpose other than that of preparing for prediction. In the former case, propriety of the assumptions and accuracy of the estimation concerning the statistical norms most likely affect the accuracy of the forecast prepared. In the latter case, even if the assumptions are correct and the estimation accurate, truth in individuals is modulated by the collective characteristics of an ensemble from which the statistical norms have been inferred.

These observations on interpolation make obvious the desirability of a test by which performances of various schemes of interpolation may be evaluated. Both the level of performance of any particular scheme and the relative merits of different schemes are affected by a number of elements that define the circumstance of interpolation. These elements are, to name the important few, the nature of a quantity being interpolated, the number and distribution of available observations, the desired degree of resolution in the estimates, and the purpose of interpolation.

Such a test was designed to compare the effects of interpolation specifically on the spectral profiles of the function that had a prescribed power spectrum [Yang and Shapiro, 1973; hereafter referred to as (1)]. In it, the mean value of the mean square errors of the point-wise estimates of the value of the function and the mean values of the estimates of the spectral components were used to measure the performance of interpolation. The present study adopts the same approach by the use of similar circumstances of interpolation and by the use of similar parameters in measuring the performance of interpolation. An extension of the test is achieved through a judicious definition of the test function, by which a stereotype of the optimum interpolation proposed by Gandin could be examined and compared with other schemes of interpolation.

2. Experimental design

The important features of the experiment are described in the following subsections.

a. Test function \( f(\lambda) \)

The function is a real and single-valued analytic function of distance \( \lambda \) defined in the closed domain (0,2\( \pi \)) with a prescribed power spectrum. It is represented in a Fourier series as

\[
f(\lambda) = \sum_{k=1}^{N_w} (A_k \cos k\lambda + B_k \sin k\lambda),
\]

in which

\[ C_k^2 = A_k^2 + B_k^2, \quad k = 1, 2, \ldots, N_w, \]

are prescribed.

The randomness of the function is embodied in the phase-angle vector \( \varphi = \{ \varphi_k, k = 1, \ldots, N_w \} \), the component of which is defined by the relations

\[
A_k = C_k \sin \varphi_k, \\
B_k = C_k \cos \varphi_k,
\]

where each \( \varphi_k \) is rectangularly distributed in (0,2\( \pi \)) and statistically independent of any other component. The test function is thus defined to be a random analytic function,

\[
f(\lambda, \varphi) = \sum_{k=1}^{N_w} C_k \sin(k\lambda + \varphi_k) = \sum_{k=1}^{N_w} f_k(\lambda, \varphi_k),
\]

with the power spectrum

\[
P_k = \frac{C_k^2}{2}, \quad k = 1, 2, \ldots, N_w.
\]

It follows from the above that the test function \( f(\lambda, \varphi) \) is a sum of statistically independent random variables \( f_k, k = 1, \ldots, N_w \). The statistical characteristics of the individual components \( f_k \) are most conveniently specified in terms of their moments of distribution:

\[
E(f_k^n) = \begin{cases}
0, & \text{if } n = 2m - 1 \\
\frac{(2m-1)(2m-3)\cdots3\cdot1}{m(m-1)\cdots2\cdot1} \left(\frac{C_k^2}{2}\right)^m, & \text{if } n = 2m
\end{cases}, \\
m = 1, 2, \ldots,
\]
where \( E(f) \) denotes the statistical expectation of \( f \), i.e.,

\[
E(f) = \frac{1}{(2\pi)^{N_w}} \times \int_0^{2\pi} \cdots \int_0^{2\pi} f(\varphi_1, \ldots, \varphi_{N_w}) d\varphi_1 \cdots d\varphi_{N_w}.
\]

The formulas in (6) may be compared with the moments of the normal variate \( t(0, \sigma) \):

\[
E(t^n) = \begin{cases} 0 & \text{if } n = 2m - 1 \\ (2m-1)(2m-3) \cdots 3 \cdot 1(\sigma^2)^m & \text{if } n = 2m \\ m = 1, 2, \ldots. \end{cases}
\]

The moments of distribution of \( f(\lambda, \varphi) \) may be computed by combining (4) and (6). It can thus be readily verified that

\[
E(f(\lambda)) = \frac{N_w C_k^2}{2} \sum_{k=1}^{N_w} P_k,
\]

with skewness and kurtosis, respectively, of

\[
\gamma_3 = \frac{E(f^3(\lambda))}{[E(f(\lambda))]}^3 = 0
\]

\[
\gamma_4 = \frac{E(f^4(\lambda))}{[E(f^2(\lambda))]}^2 = 3 - 3 \sum_{k=1}^{N_w} \left( \frac{C_k^2}{2} \right) \left( \sum_{k=1}^{N_w} \frac{C_k^2}{2} \right)^2.
\]

Finally, we observe the homogeneity and ergodicity of the test function as revealed in the following identities:

\[
E(f(\lambda, \varphi)) = E(f(\lambda)) = 0;
\]

\[
f(\lambda, \varphi) = E(f(\lambda)) = \sum_{k=1}^{N_w} P_k;
\]

\[
E(f(\lambda, \varphi)f(\lambda+s, \varphi)) = E(f(\lambda)f(\lambda+s)) = \sum_{k=1}^{N_w} P_k \cos ks,
\]

for all \( \varphi \). Here

\[
F(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} F(\lambda) d\lambda
\]

denotes the spatial average. We also note that the autocorrelation function \( \rho(s) \) may be given by

\[
\rho(s) = \frac{E(f(\lambda)f(\lambda+s))}{E(f^2(\lambda))) = \frac{f(\lambda, \varphi)f(\lambda+s, \varphi)}{f^2(\lambda, \varphi)} = \sum_{k=1}^{N_w} P_k \cos ks \sum_{k=1}^{N_w} P_k.
\]

\section{b. Observations and estimations}

The test function \( f(\lambda, \varphi) \) is observed (or measured) for each \( \varphi \) at a prescribed network of observation stations \( \{\lambda_j, j = 1, 2, \ldots, N_0\} \). The network is constructed randomly except for a restraint concerning the distances between stations. It is that no neighboring stations are separated by a distance which is less than \( d_m \) or greater than \( d_M \). In addition, the distribution of the stations is made to give these distances of separation a bell-shaped distribution around the mean value \( d = 2\pi/N_0 \).

Each set of observations \( \{f(\lambda_j, \varphi), j = 1, 2, \ldots, N_0\} \) is then used to estimate the value of \( f(\lambda, \varphi) \) at each of the equally spaced grid points \( \lambda_i, i = 1, 2, \ldots, N_\sigma \), with the following schemes of interpolation, based on the formula:

\[
f(\lambda_i, \varphi) = \sum_{j=1}^{N_0} w_{ij} f(\lambda_j, \varphi), \quad i = 1, 2, \ldots, N_\sigma.
\]

\section{1) Ultimate optimum interpolation}

In this the autocovariance function as prescribed in Eq. (12) is employed to determine the weight of interpolation \( w_{ij} \) contributed by observation at \( \lambda_i \) toward the estimate at \( \lambda_i \) according to the least-squares principle. The system of equations for \( \{w_{ij}, j = 1, 2, \ldots, N_0\} \) for each of \( i = 1, 2, \ldots, N_\sigma \) is given by

\[
\sum_{j=1}^{N_0} w_{ij} \rho_{il} = \rho_{il},
\]

where \( \rho_{il} \) denotes the correlation coefficient between observations at \( \lambda_i \) and \( \lambda_l \) and \( \rho_{il} \) the correlation coefficient between observations at \( \lambda_i \) and \( \lambda_i \). It is ultimate because the values of the autocovariance function used are the true values for the given test function.

\section{2) Real optimum interpolation}

Except for the autocovariance function that is estimated with the use of available data, the scheme is the same as that of the ultimate optimum interpolation. The details of the estimation are given in Section 4.

\section{3) Cubic spline fitting}

This finds for each set of \( \{f(\lambda_i, \varphi)\} \) a spline \( f(\lambda, \varphi) \) which is continuous with continuous first- and second-order derivatives in the entire domain and a piecewise linear second-order derivative in each segment bounded by two neighboring observation stations.

\section{4) Two-point linear interpolation}

This seeks for each set of \( \{f(\lambda_i, \varphi)\} \) an estimate \( f(\lambda_i, \varphi) \) which is continuous in the entire domain and
piecwise linear in each segment bounded by two consecutive observations.

We note here that while the aim of the optimum interpolation is to reduce the mean square errors of the estimates at a given set of grid points, there is no such expressed objective in the other two. Moreover, in the view of the optimum interpolation a set of observational data \( \{f(A_i, \varphi), j = 1, 2, \ldots, N_0\} \) constitutes a sample in an ensemble whose stochastic properties are either known or to be estimated. On the contrary, in the application of the other two methods the concept of ensemble is rather an after-thought, and no concern is given to relations that might or might not exist among the individual samples that are being considered for the purpose of performance evaluation.

c. Evaluation

Two measures are employed to evaluate the performances of different interpolation schemes. The first one is the ratio of the mean square errors of the estimates to the mean squares of the true values of the function calculated on the grid points \( A_i, i = 1, 2, \ldots, N_0 \), and is called the error of interpolation \( \epsilon \). It may be defined by

\[
ed = \frac{\sum_{i=1}^{N_0} [f(A_i, \varphi_i) - f(A_i, \varphi)]^2}{\sum_{i=1}^{N_0} f^2(A_i, \varphi_i)}. \tag{17}
\]

The second measures are the deviations from the expected values of the normalized spectral intensities that are calculated from the estimates \( \{f(A_i, \varphi_i), i = 1, 2, \ldots, N_0\} \). These depict the spectral distribution of the mean square error of interpolation and measure the alteration of the power spectrum caused by the interpolation scheme.

The averages of these measures over an ensemble of \( N_S \) random and independent samples \( \{f(A, \varphi_j), j = 1, 2, \ldots, N_S\} \), that are obtained by random sampling of the phase-angle vector \( \varphi \), are used to characterize the effects of interpolation.

We observe at this point that the mean error of interpolation of the ultimate optimum interpolation is a ratio of the sample estimate of the least mean square error to the sample estimate of the total variance of the test function for the given set of grid points and the given number of samples in the ensemble. It may therefore be regarded as a landmark with which the errors of interpolation of other interpolation schemes may be compared.

d. Specification

The experiments reported here have the following specifications. The test function has the spectral amplitude \( C_k \) for wavenumber \( k \) given by

\[
C_k = \begin{cases} 
C_1, & \text{if } 1 \leq k \leq 8 \\
C_1(k/8)^{-1}, & \text{if } 9 \leq k \leq N_w 
\end{cases}
\]

with

\[ C_1 = 0.4658 , \quad N_w = 72. \]

The observational network contains 50 stations, i.e., \( N_0 = 50 \). The average (\( d \)), maximum (\( d_m \)), and minimum (\( d_n \)) of the distances between two adjacent stations are, respectively, 7.2°, 11.0°, and 3.4°.

The grid-point network is specified by \( N_0 = 72 \), and the number of samples \( N_S \) in an ensemble is 72.

3. Comparison

Comparisons among three interpolation schemes, ultimate optimum, cubic spline and two-point linear interpolations, are made on six different sets of conditions formed by two phase-angle vectors \( \Phi_0 \) and \( \Phi_2 \), and three observational networks designated by vectors \( \Lambda_0, \Lambda_1, \Lambda_2 \). \( \Phi_0 \) and \( \Phi_2 \) are vectors of 72 components in which each component represents a phase angle assigned to a wave component of the test function. The ensemble of seventy-two samples is then generated with each \( \Phi \) by rotating its components. \( \Lambda_0, \Lambda_1, \) and \( \Lambda_2 \) are vectors of 50 components in which each component designates the longitude of an observation station. The distribution of the components of these vectors except \( \Lambda_0 \) in the domain of definition (0°, 360°) are displayed in Fig. 1. Other pertinent statistical characteristics of the observational networks are presented in Table 1, which also makes obvious the reason why vector \( \Lambda_0 \) has been omitted from the display in Fig. 1.

The matter of concern here is to insure the validity of the inferences we might draw on the experiments. We desire the conclusions of these experiments to be as general, and, at the same time, as definitive as possible even after allowance is made for unavoidable sampling fluctuations. It was in this spirit that the specifications mentioned at the end of the last section were drawn and it was hoped that the number of the grid points and the size of the ensembles would be large enough to keep deviations due to sampling fluctuations within such a bound that the inferences drawn would remain true for most of the "reasonable" conditions. Thus, we see no characteristic difference either between \( \Phi_1 \) and \( \Phi_2 \) nor between \( \Lambda_1 \) and \( \Lambda_2 \), and regard any difference among the four sets of conditions involving these to be entirely due to sampling fluctuations. On the other hand, we recognize the distinction of \( \Lambda_0 \) from \( \Lambda_1 \) and \( \Lambda_2 \) in its uniformity of station distribution.

Table 2 summarizes the values of sample means and standard deviations of the errors of interpolation under these six conditions.
The mean value of the error of interpolation is consistently smallest in the ultimate optimum interpolation and largest in the two-point linear interpolation. The closeness in the values of both mean and standard deviation between the cubic spline and the ultimate optimum interpolation is quite remarkable. If we venture to assume that the sample means of the errors of interpolation are normally distributed, these figures indicate that statistical significance of the difference between the ultimate optimum and cubic spline interpolation is rather marginal, while those between the two-point linear interpolation and the other two are definite and unequivocal in all six cases.

The other noticeable feature in Table 2 is the smaller mean error of interpolation obtained in observation system A9 than in A1 and A9, observed consistently in all the interpolation schemes. The assumption of normal distribution of the sample means would lead us to believe that the difference between the uniform network of observations A9 and the irregular networks A1 and A2 is statistically significant in all the interpolation schemes.

It may be worthwhile to remark here that the observations as employed in these experiments contain no error of observation, so that the errors of interpolation arise only from application of the interpolation schemes. It can be readily demonstrated that the error of interpolation would be greater if there were errors in the observations, insofar as the errors may be assumed to be random with null mean and statistically independent of each other as well as of the measured quantity. Consequently, the mean values in Table 2 may also be regarded as sample values of the minima of the errors of interpolation, associated with the given interpolation scheme and for each observational network.

Table 2 furthermore suggests that for a given observational network the ultimate optimum interpolation furnishes the least value of the error of interpolation among all interpolation schemes, and that for each interpolation scheme the observational network with uniform distribution yields the smallest error of interpolation among all systems with the same number of stations.

Fig. 2 presents the power spectra calculated from the estimates on the 72 grid points in the sample associated with phase-angle vector Φ1 by the interpolation schemes. It can be readily shown that from the test function defined in Section 2, the true spectral estimates \( \hat{P}_k \) obtained with the true values of the function at the grid-points have the following properties:

\[
\begin{align*}
E(\hat{P}_k) & = \hat{P}_k + \sum_{m=1}^{N} \left( P_{mN_{G-k}} + P_{mN_{G+k}} \right), \\
V(\hat{P}_k) & = E(\hat{P}_k - E(\hat{P}_k))^2 = 2 \sum_{N \neq M} \sum_{M} P_{M} P_{N}, \\
E(\hat{P}_k - E(\hat{P}_k))^3 & = 0
\end{align*}
\]

where

\[
m = 1, 2, \ldots; \quad M, N = k, \quad mN_{G} \pm k,
\]

for all \( k = 0, 1, \ldots, N_{G}/2 \). Here it is assumed without much offense that \( N_{G} \), the number of grid points in the domain \((0,2\pi)\), is even so that \( N_{G}/2 \) defines the Nyquist wavenumber. If, in addition, we assume that the sample means of \( \hat{P}_k \) are normally distributed, we can state that 95% of the sample means of the true spectral estimates based on ensembles of 72 random samples are expected to fall within the region defined by

\[
E(\hat{P}_k) \pm 1.96 \sqrt{\frac{V(\hat{P}_k)}{N}},
\]

which is represented by the area bounded by the two curves in each of Fig. 2a; the same region is shaded in each of Fig. 2b. The shaded region is a measure of unavoidable sampling fluctuation of the true estimates.

The mean spectral intensities obtained from the harmonic analyses of the values of the function at the grid points are the sample values of the true estimates for each given set of samples, \( \Phi_1 \) or \( \Phi_3 \), and are represented by the symbol \( \times \) in Fig. 2a. We infer from these

| Table 1. Statistics of the distances (deg) between two adjacent stations in observation systems, \( A_9, A_1, A_2 \). |
|---|---|---|---|
| Statistics | \( A_9 \) | \( A_1 \) | \( A_2 \) |
| Mean | 7.2 | 7.2 | 7.2 |
| Standard deviation | 0 | 1.4 | 1.2 |
| Maximum | 7.2 | 9.8 | 9.7 |
| Minimum | 7.2 | 3.8 | 4.7 |
that both samples are typical or representative of the population with very little apparent bias.

The estimates of the spectral intensities based on the estimates of various schemes of interpolation are presented in Fig. 2b, in which (and also in all other spectral profiles) any normalized intensity less than $10^{-2}$ is marked as $10^{-2}$. One obvious feature is the apparent lack of fidelity in the estimates obtained with the two-point linear interpolation, a point much belabored in (I). It is also easily observed that the ultimate optimum and the cubic spline interpolations give estimates that are close to each other. It may also be noted, however, that the cubic spline estimates seem to have a slight edge in fidelity over the ultimate optimum interpolation. Both remain reasonably accurate up to about $k=15$ and then deviate by an ever greater amount as wavenumber increases.

The closeness of the levels of performance by the ultimate optimum and cubic spline interpolations does not end here. As shown in Fig. 3, we note even in the standard deviations and the rms errors of the spectral estimates the almost identical profiles in these two schemes, both being clearly set apart from those in the two-point linear interpolation. The profiles in only two conditions are shown here, because those for the remaining four conditions are indistinguishably different.

It may also be pointed out in Fig. 3 that the values of the standard deviations in all three schemes at higher wavenumbers are smaller than those of the true estimate, contrary to those in the lower wavenumbers. This means that the underestimation of the spectral intensities in this wavenumber range, as shown in Fig. 2b, is inherently associated with the degree of resolution that is attained by the observational network under study.

### 4. Ultimate and real optimum interpolations

The ultimate optimum interpolation discussed in the last section assumes and uses the complete knowledge of the autocovariance function or, equivalently, the true power spectrum at every point in the domain of the function that is subject to interpolation. It has been referred to as ultimate because every value of the autocovariance function that enters into determining the weights of interpolation is true of the function and, therefore, the resulting error of interpolation is the true value of expectation; namely, the value that would be obtained as the result of infinitely many trials. Such perfect knowledge is not available in practice when one encounters situations in which interpolation is required. In most cases, the intrinsic statistical character of the function is surmised and deduced from the physical rather than statistical evidence of the function, and it is often difficult to prove or disprove their veracity in any conclusive manner. Moreover, in most of the atmospheric variables it is extremely hard to insure
homogeneity or randomness of the samples from which the autocovariance function is estimated.

The experiment described in this section eliminates these uncertainties by drawing random samples from the test function, the statistical properties of which are known, but retains sampling fluctuations that affect the estimation of the autocovariance function. It is presented here to illustrate by way of an example a number of issues that may arise in the course of the application of a real optimum interpolation.

We imagine that we have an ensemble of errorless observations $f(\Phi_i,\Lambda_i)$ of the test function $f(\lambda)$ that is assumed a priori to be ergodic and homogeneous in the sense described in Section 2. We shall first estimate the autocovariance function of $f(\lambda)$ from this ensemble in the manner that will be explained below. We next use the estimate in the determination of the values of the weights of interpolation that define the scheme of the optimum interpolation from the given observational network to the equally-spaced 72 grid points. We then apply this scheme to the ensembles pertaining to the observational network $\Lambda_1$ and evaluate its performances.

The autocovariance function $R_f(s)$ may be defined by

$$R_f(s) = V(f)\rho_f(s),$$

where $V(f) = R_f(0)$ is the variance of $f(\lambda)$ and $\rho_f(s)$ is the autocorrelation coefficient at distance $s$. In view of the assumed ergodicity and homogeneity of $f(\lambda)$
we first computed the following sample values:

\[
\bar{E}(f) = \frac{1}{N_s} \sum_{i=1}^{N_s} f_i(A_i), \quad \bar{E}(f) = \frac{1}{N_0} \sum_{j=1}^{N_0} \bar{E}(f_j)
\]

\[
\bar{V}(f_j) = \frac{1}{N_s} \sum_{i=1}^{N_s} f_i^2(A_i) - [\bar{E}(f_i)]^2, \quad \bar{V}(f) = \frac{1}{N_0} \sum_{j=1}^{N_0} \bar{V}(f_j)
\]

\[
\hat{\rho}(d_{jk}) = \left[ \frac{1}{N_s} \sum_{i=1}^{N_s} f_i(A_i) f(A_k) - \bar{E}(f_i) \bar{E}(f_k) \right] / \left[ \bar{V}(f_i) \bar{V}(f_k) \right]^{1/2}
\]

for \( j, k = 1, 2, \ldots, N_0 \), in which \( \rho(d_{jk}) \) denotes the correlation coefficient of \( f(A_j) \) and \( f(A_k) \) observed, respectively, at stations \( A_j \) and \( A_k \) that are separated by the distance \( d_{jk} = |A_j - A_k| \). Table 3 shows the true and sample values of the mean and variance of \( f(\lambda) \).

Our next task was to obtain an estimate of the autocorrelation function \( \rho_j(s), 0 \leq s \leq 2\pi \), from the collection of \( \hat{\rho}(d_{jk}) \), the scatter diagram of which for condition \( (\Phi_0, \Lambda_0) \) is shown in Fig. 4. We began this by first subdividing the entire domain into 144 equal sub-intervals and grouping \( \hat{\rho}(d_{jk}) \) into these sub-intervals according to the value of \( d_{jk} \). The group mean evaluated for each sub-interval was then considered to be the sample value of \( \hat{\rho}_0(s) \) at the mid-point of each sub-interval. We then took account of the fact that the autocorrelation function is an even function of the distance of separation by putting

\[
\hat{\rho}_1(s) = \frac{1}{2} [\hat{\rho}_0(s) + \hat{\rho}_0(2\pi - s)], \quad 0 \leq s \leq \pi.
\]  

(21)

The set \( \{\rho_i(s_i), s_i = 2\pi i/72, i = 0, 1, \ldots, 72\} \) was then subjected to a cosine transform. We obtained the estimated power spectrum \( \{P_k, k = 0, 1, \ldots, 36\} \) by truncating the cosine transform at wavenumber 36 and nullifying any wave component which had a negative power. The last step was taken to comply with the intrinsic property of a power spectrum and the corresponding autocorrelation function. The final estimate of the autocorrelation function was then defined as the cosine transform of this power spectrum. These estimates are compared with the true values and also with the estimates that would be obtained if the function were measured at the grid points. They are shown in Figs. 5 and 6.

The area bounded by the two curves in Fig. 5 is defined by

\[
E(\rho) \pm \frac{1.96}{\sqrt{N_s}} [1 - E^2(\rho)] \leq \hat{\rho} \leq E(\rho) \pm \frac{1.96}{N_s} [1 - E^2(\rho)]
\]

and represents the 95% confidence region of the correlation coefficient between the two components of a binormal variate when the size of the sample is large enough. The true value, \( E(\rho) \), at any given distance is obviously represented by the midpoint of the shaded interval. Except for the immediate neighborhood of
the origin, where the value of the autocorrelation function is large and both estimates tend to overestimate, the estimate based on the observations seems quite satisfactory, even though the discrepancy in the power spectra is made obvious by Fig. 6.

The schemes of the real optimum interpolation constructed using the autocorrelation function \( \hat{\rho}(s) \) and the variance \( \hat{V}(f) \) estimated from ensemble \((\Phi_1, \Lambda_1)\) were then applied to both \((\Phi_1, \Lambda_1)\) and \((\Phi_2, \Lambda_1)\). The performances of these and others for observation system \( \Lambda_2 \) as measured by the mean errors of interpolation are summarized in Table 4 and the estimated power spectra are presented in Fig. 6. The larger values of the error of interpolation in the real than in the ultimate optimum interpolation are quite obvious in Table 4. Indeed, these values place the real optimum interpolation at the level of the two-point linear interpolation in performance (see also Table 2). This observation, however, should not exclude a possibility that another way of estimating the autocorrelation

![Graphs showing correlation coefficients vs. distance for two types of waves.](image)

**Fig. 4.** Scatter diagrams of the estimated correlation coefficients under conditions \((\Phi, \Gamma)\) and \((\Phi_2, \Lambda_1)\), where \( \Gamma \) designates the 72-point, equally-spaced grid-point network.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Ultimate</th>
<th>Real 1*</th>
<th>Real 2*</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\Phi_1, \Lambda_1))</td>
<td>0.0567 m 0.0089 s</td>
<td>0.0693 m 0.0130 s</td>
<td>0.0910 m 0.0185 s</td>
</tr>
<tr>
<td>((\Phi_2, \Lambda_1))</td>
<td>0.0560 m 0.0087 s</td>
<td>0.0636 m 0.0105 s</td>
<td>0.0704 m 0.0101 s</td>
</tr>
<tr>
<td>((\Phi_3, \Lambda_1))</td>
<td>0.0639 m 0.0122 s</td>
<td>0.0943 m 0.0134 s</td>
<td>0.0756 m 0.0120 s</td>
</tr>
<tr>
<td>((\Phi_4, \Lambda_2))</td>
<td>0.0531 m 0.0098 s</td>
<td>0.0688 m 0.0137 s</td>
<td>0.0612 m 0.0155 s</td>
</tr>
</tbody>
</table>

* Real 1 uses the weights obtained from the ensemble itself. Real 2 uses the weights obtained from the other ensemble.
function might produce a more accurate estimate and thereby improve the performance of the real optimum interpolation.

5. Optimum interpolation and region of influence

The profiles of both the expected and the estimated autocorrelation functions presented in Fig. 4 show a long interval away from the origin in which the values of the autocorrelation functions remain small. In both the ultimate and real optimum interpolations considered in the preceding section, all available observations were taken into consideration in determining the weights contributed by individual observations toward the estimate at each grid point. In Eq. (16), therefore, there were many components with small coefficients and many an equation with a small inhomogeneous term. This meant that the system of equations for \( \{w_{ij}, j=1, 2, \ldots, N_0\} \), for each value of \( i \), were large \((N_0=50)\) and tended to be ill-conditioned. Also, it resulted in many weights of insignificant magnitude mostly imposed by those observations taken at far distances from the grid point of concern.

Thus, there was more than one reason that made us wonder what would happen to the performance of an optimum interpolation if the region of influence, which is defined as the region outside of which the autocorrelation function is assumed zero, is reduced. In an attempt to answer this question, we took an autocorrelation function, imposed the restriction that \( \rho(s)=0 \) if \( |s| \geq D \) for a given value of \( D \), and obtained the scheme of optimum interpolation by solving Eq. (16) with the use of this truncated autocorrelation function.

When the expected autocorrelation function was used, the results on \( D=\pi \) (which covers the entire domain), \( \pi/2 \), \( \pi/4 \) and \( \pi/8 \) revealed no discernible changes in the performances as measured by both the mean error of interpolation and the mean spectral estimates among the different values of \( D \) for each of the six different conditions studied. Table 5a summarizes the statistics of these errors of interpolation.

When the estimated autocorrelation functions were used in the real optimum interpolation, however, the error of interpolation, as shown in Table 5b, decreased
steadily as the size of region of influence was reduced, although no appreciable difference could be observed among the corresponding mean spectral profiles. The only obvious change was noted in the total energy of the estimated power spectrum which showed a decrease with the reduction of the size of region of influence. The largest reduction in all the four cases considered amounted to less than 2.5%.

These results strongly suggest that, with the test function employed here, it is advantageous to limit the region of influence to as little as one-eighth of the entire domain, which contains barely the first lobe of the profile of the autocorrelation function.

6. Summary

A test function in a bounded one-dimensional domain is proposed that enables us to compare easily and fairly the performance of various schemes of interpolation. The stochastic nature of the test function is so defined that it can be used in those interpolation schemes that are dependent on the stochastic properties of the quantity under study. In view of the usage commonly made of the results of interpolation in meteorology, two kinds of parameters are proposed to measure the performance of interpolation schemes.

These are then applied to a comparative study of Gandin’s optimum interpolation and two other schemes.
Table 5. Means (m) and standard deviations (s) of the errors of interpolation.

<table>
<thead>
<tr>
<th>Condition</th>
<th>( m )</th>
<th>( \pi )</th>
<th>( s )</th>
<th>( m/2 )</th>
<th>( \pi/4 )</th>
<th>( s )</th>
<th>( m/8 )</th>
<th>( \pi/8 )</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\Phi_1, A_0))</td>
<td>0.0505</td>
<td>0.0085</td>
<td>0.0509</td>
<td>0.0084</td>
<td>0.0510</td>
<td>0.0084</td>
<td>0.0510</td>
<td>0.0085</td>
<td></td>
</tr>
<tr>
<td>((\Phi_1, A_1))</td>
<td>0.0567</td>
<td>0.0089</td>
<td>0.0578</td>
<td>0.0086</td>
<td>0.0578</td>
<td>0.0086</td>
<td>0.0578</td>
<td>0.0086</td>
<td></td>
</tr>
<tr>
<td>((\Phi_2, A_0))</td>
<td>0.0560</td>
<td>0.0087</td>
<td>0.0567</td>
<td>0.0087</td>
<td>0.0567</td>
<td>0.0087</td>
<td>0.0567</td>
<td>0.0087</td>
<td></td>
</tr>
<tr>
<td>((\Phi_2, A_1))</td>
<td>0.0462</td>
<td>0.0056</td>
<td>0.0466</td>
<td>0.0054</td>
<td>0.0466</td>
<td>0.0054</td>
<td>0.0466</td>
<td>0.0054</td>
<td></td>
</tr>
<tr>
<td>((\Phi_3, A_0))</td>
<td>0.0653</td>
<td>0.0122</td>
<td>0.0653</td>
<td>0.0120</td>
<td>0.0653</td>
<td>0.0119</td>
<td>0.0653</td>
<td>0.0120</td>
<td></td>
</tr>
<tr>
<td>((\Phi_3, A_1))</td>
<td>0.0551</td>
<td>0.0098</td>
<td>0.0539</td>
<td>0.0098</td>
<td>0.0539</td>
<td>0.0098</td>
<td>0.0539</td>
<td>0.0098</td>
<td></td>
</tr>
</tbody>
</table>

\(a.\) Ultimate optimum interpolation with expected autocorrelation functions

\(b.\) Real optimum interpolation with estimated autocorrelation functions*

* The weights of interpolation are obtained with the ensemble of observations associated with the alternate phase angle vector.

of interpolation. While the optimum interpolation
presumes the stochastic nature of the quantity under
investigation, the other two, namely the two-point linear and cubic spline interpolations, do not require
any hypothesis regarding the statistical nature of the
function.

The circumstance of interpolation and the structure
of ensembles, with which the performances of these
interpolation schemes are evaluated, are selected in
such a way to strike a reasonable compromise between
generality and practicality of the experiments, and
that goal, in our subjective view, is believed to have
been attained.

Among the observational networks with a given
number of stations the one with equally-spaced stations
is found to furnish the point-wise estimates of the
observed quantity with the least error of interpolation
in each of the interpolation schemes considered in
this study.

The optimality of the ultimate optimum inter-
polation among different schemes employed on a
statistically stationary quantity, as measured by the
error of interpolation, has been borne out by these
experiments taken under various conditions.

Based on these findings, and armed with the belief
that a resolution of a function in the wavenumber
range 0–36 on data collected by the total of 50 stations
around a circle is a good type-casting of the current
state of art of meteorological data analysis, we suggest
that 5% error of interpolation is the best we can
achieve in accuracy under all practical circumstances.

The harmonic analyses on the point estimates
obtained through interpolation show consistent and
significant underestimation in the greater half of the
wavenumber range. In the ultimate optimum and cubic
spline interpolations, this divergence starts at about
wavenumber 18; in the two-point linear interpolation
at about wavenumber 10. The exact extent of the sig-
nificance of this fact cannot be known without more
knowledge than is currently available on the corre-
spondence between the finite-difference method and
the spectral method of solving, and on the relative
roles played by the initial conditions and the boundary
conditions on the numerical solutions of the initial-
boundary value problems. It is believed, however, that
attention should be called to the fact that the estimates
are consistently less than the true values in the short-
wave end.

The performance of the cubic spline interpolation
was found very close to that of the ultimate optimum
interpolation in many aspects. These include, in
addition to mean error of interpolation and mean
spectral estimates, mean square errors and standard
deviations of the individual spectral estimates. The
high level of performance of the cubic spline inter-
polation makes it a highly preferable tool of inter-
polation, especially when the stochastic nature of the
quantity under investigation is either not sufficiently
well known or is not stationary.

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