# Bayesian and Classical Statistical Methods Applied to Randomized Weather Modification Experiments[1]

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ABSTRACT

Statistical procedures for analyzing the results of randomized weather modification experiments are presented in a format designed to emphasize their underlying assumptions. A parallel development of Bayesian and classical statistical techniques is given to demonstrate that both methodologies can be used under the assumed experimental conditions and that the difficulties in applying either are comparable.

## 1. Introduction

One of the many perplexing problems in weather modification is the statistical analysis of a randomized experiment. Problems are introduced by the nonnormality of the precipitation distribution, the postulated multiplicative seeding effect, and the time and expense involved in obtaining experimental data. Statistical procedures have been proposed that overcome some of these problems. Some require distributional assumptions (Neyman and Scott, 1967), while others are nonparametric in nature (Duran and Mielke, 1968). Recently, Simpson *et al.* (1973b) utilized an analysis based on the foundations of the Bayesian statistical methodology to analyze the Experimental Meteorology Laboratory (EML) series of dynamic cloud seeding experiments conducted in south Florida.

The Bayesian analysis of the EML experiments adds another method of analyzing a cloud seeding experiment. However, it is more than just another alternative under the usual classical statistical methodology. When evaluating a cloud seeding experiment analyzed using this methodology, this difference must be recognized and an understanding of the Bayesian statistical approach should be obtained.

The following sections are designed to present some of the assumptions possible and the resulting analysis procedures as they relate to the EML analyses. At the same time classical statistical procedures will be given based on the same distributional assumptions. Hopefully, this parallel development will help show some of the differences in the methodologies and at the same time give an alternative classical analysis procedure based on the same assumptions used in the EML Bayesian analysis.

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## 2. Basic model

In a randomized treatment-control cloud seeding experiment two independent data sets are obtained, one set giving independent replications of the measurement of a variable (rainfall volume) associated with the control experimental units and the other set similar measurements on the treated (seeded) experimental units. The purpose of the analysis is to determine if there is a seeding effect and to obtain an estimate of its magnitude.

Two basic assumptions used in the analysis of EML's experiments are 1) that a highly skewed gamma distribution describes the daily rain volume and 2) that the effect of seeding, if any, is multiplicative. Although a justification for the assumptions specifically used in the EML experiments is not given here [see Simpson *et al.* (1973b) for such a discussion], this aspect of the analysis must be carefully considered when applying the methodology to a specific data set. With these assumptions a random variable $X$ representing the rainfall on a control experimental unit has the probability density function

$$p(x|\alpha,\beta_c)=\frac{\beta_c^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta_c x},\quad x>0,$$

where $\alpha>0$ and $\beta_c>0$, and a random variable $Y$, associated with the treatment, has the same distributional form, but with a possibly different scale parameter $\beta_t$.

If the expected values of $X$ and $Y$ are given by $\mu_c=\alpha/\beta_c$ and $\mu_t=\alpha/\beta_t$, then the multiplicative seeding effect parameter of interest to the meteorologist is defined by $\theta=\mu_t/\mu_c$. Under these conditions $\theta$ is the factor by which, hypothetically, the treatment multiplies the average response that would occur if the treatment had not been applied to the experimental unit. Note that $\beta_t=\beta_c/\theta$.

In summary, at the completion of the randomized treatment-control cloud seeding experiment, it is assumed there are $m$ independent observations on $X$ with density $p(x|\alpha,\beta_c)$ and $n$ independent observations on $Y$ with density $p(y|\alpha,\beta_c/\theta)$ upon which the investigation of the seeding effect $\theta$ is to be based.

## 3. Control distribution assumed known

It is usually unrealistic to assume that the control, i.e., natural rainfall, distribution is completely known. However, the analysis procedures are presented for this case since this assumption was implicitly made in some of the preliminary development of the Bayesian procedures used at EML (Simpson *et al.*, 1973a). The control distribution being known implies that both $\alpha$ and $\beta_c$ are known so that the only unknown parameter remaining is the seeding effect $\theta$, and that only the $n$ treated observations on $Y$ enter into the analysis. The probability density function (pdf) for $Y$ can then be rewritten as

$$p(y|\theta)=\left(\frac{\alpha}{\mu_c\theta}\right)^{\alpha}\frac{y^{\alpha-1}}{\Gamma(\alpha)}\exp\left(-\frac{\alpha y}{\mu_c\theta}\right),\quad y>0.$$

### a. Bayesian analysis

The Bayesian analysis is based on Bayes equation which takes the form

$$p(\theta|\mathbf{y})\propto p(\theta)p(\mathbf{y}|\theta),$$

where $p(\theta|\mathbf{y})$ is the posterior pdf for the seeding effect $\theta$, given the sample $\mathbf{y}$, $p(\theta)$ is the prior pdf assigned to the parameter $\theta$, and $p(\mathbf{y}|\theta)$, viewed as a function of $\theta$, is the well-known likelihood function. The posterior pdf presents all the evidence upon which inferences can be made about the parameter $\theta$. In this case

$$p(\mathbf{y}|\theta)=\prod_{i=1}^{n}p(y_i|\theta)=\left(\frac{\alpha}{\mu_c\theta}\right)^{n\alpha}[(\prod_{i=1}^{n}y_i)^{\alpha-1}/\Gamma(\alpha)^n]\times\exp\left(-\frac{n\alpha\bar{y}}{\mu_c\theta}\right),$$

where $\bar{y}$ is the arithmetic mean. A family of prior distributions that is mathematically tractable and allows the inclusion of a wide range of prior information is the inverse gamma priors given by

$$p(\theta)=\frac{K_2^{K_1+1}}{\Gamma(K_1+1)}\theta^{-K_1-2}\exp(-K_2/\theta),\quad \theta>0.$$

The resulting posterior distribution for $\theta$ is also an inverse gamma,

$$p(\theta|\mathbf{y})=\frac{(K_2+\Delta)^{n\alpha+K_1+1}\theta^{-(n\alpha+K_1)-2}}{\Gamma(n\alpha+K_1+1)}\exp[-(K_2+\Delta)/\theta],$$

where $\Delta=n\alpha\bar{y}/\mu_c$. The posterior mean, mode and variance are given respectively by

$$\mu=\frac{K_2+\Delta}{K_1+n\alpha},\ \eta=\frac{K_2+\Delta}{K_1+n\alpha+2},\ \sigma^2=\frac{(K_2+\Delta)^2}{(K_1+n\alpha)^2(K_1+n\alpha-1)}.$$

Bayes equal-tail and shortest posterior intervals on the parameter $\theta$ with a prescribed content can also be obtained (Olsen, 1973), but require the use of tables or numerical integration.

### b. A classical analysis

Under the present assumptions, test and estimation procedures concerning the seeding effect $\theta$ are readily determined. A point estimate $\theta$ is given by $\hat{\theta}=\bar{y}/\mu_c$ which is unbiased and a maximum likelihood estimate. Moreover, $\hat{\theta}$ is the minimum variance unbiased estimator for $\theta$.

The hypothesis test of interest is to test the null hypothesis $H_0$: $\theta=1$, i.e., no seeding effect, versus the alternative $H_A$: $\theta\neq1$. An equal-tail test procedure is based on the test statistic

$$Z=\frac{2n\alpha\bar{y}}{\mu_c},$$

which under the null hypothesis has a chi-square distribution with $2n\alpha$ degrees of freedom. The rejection region is $Z<Z_L$ or $Z>Z_U$ where $Z_L$ and $Z_U$ are determined from chi-square tables so that $\Pr\{Z<Z_L|\theta=1\}=\Pr\{Z>Z_U|\theta=1\}=\xi/2$ where $\xi$ is the significance level.

An equal-tail confidence interval is given by

$$\frac{2n\alpha\bar{y}}{\mu_cZ_U}\leqslant\theta\leqslant\frac{2n\alpha\bar{y}}{\mu_cZ_L}$$

with confidence level $1-\xi$.

It is interesting to note that the same numerical results concerning interval estimates of $\theta$ can be obtained from the classical and Bayesian approaches. This agreement is shown by Olsen (1973) to occur when the improper prior $1/\theta(K_1=-1,\ K_2=0)$ is used in the Bayesian analysis.

## 4. Shape parameter $\alpha$ assumed known

The easiest relaxation of the assumption utilized in Section 3 is to allow the scale parameter $\beta_c$, or equivalently the mean $\mu_c$, to be unknown while still assuming the shape parameter $\alpha$ is known. This is a more realistic situation than the previous one but is still probably an overspecified analysis. The analysis now requires both the control $\mathbf{x}$ and seeded $\mathbf{y}$ samples and there are the two unknown parameters $\mu_c$ and $\theta$.

### a. Bayesian analysis

The Bayesian analysis follows the same procedures given in Section 3a except that now a joint prior distribution on $\theta$ and $\mu_c$ must be specified. One possible family of priors assumes that $\theta$ and $\mu_c$ are independent and that both $\theta$ and $\mu_c$ have inverse gamma priors. That is

$$p(\theta,\mu_c)=p(\theta)p(\mu_c)$$
$$=\frac{K_2^{K_1+1}}{\Gamma(K_1+1)}\theta^{-(K_1+2)}e^{-K_2/\theta}\frac{k_2^{k_1+1}}{\Gamma(k_1+1)}\mu_c^{-(k_1+2)}e^{-k_2/\mu_c}.$$

The resulting joint posterior pdf is proportional to

$$\theta^{-(n\alpha+K_1+2)}e^{-K_2/\theta}\mu_c^{-(n\alpha+m\alpha+k_1+2)}\times\exp\{-[k_2+m\bar{x}\alpha+(n\bar{y}\alpha/\theta)]\mu_c\},$$

where $\theta$, $\mu_c>0$. The marginal posterior pdf for $\theta$ is obtained by integrating on $\mu_c$ and results in

$$p(\theta|\mathbf{x},\mathbf{y})\propto\frac{\theta^{-(n\alpha+K_1+2)}e^{-K_2/\theta}}{[k_2+m\bar{x}\alpha+(n\bar{y}\alpha/\theta)]^{n\alpha+m\alpha+k_1+1}},\quad\theta>0.$$

The required normalizing constant as well as the posterior mean, mode and variance can be obtained numerically. Similarly, Bayes shortest and equal-tail posterior intervals of a prescribed content can be determined numerically using the procedures given in Olsen (1973).

### *b. A classical analysis*

When the shape parameter $\alpha$ is the only known parameter, point and interval estimation as well as hypothesis testing for $\theta$ must be completed in the presence of the nuisance parameter $\mu_c$. It is assumed that there is no direct interest in estimating $\mu_c$ and, additionally, that the classical procedures used in making inferences concerning $\theta$ are to be invariant under changes in $\mu_c$.

A point estimator for $\theta$ can be obtained by forming the log-likelihood function and deriving the maximum likelihood estimators. That is,

$$\mathrm{LN}=-(m\alpha+n\alpha)\log\mu_c-[m\bar{x}\alpha+(n\bar{y}\alpha/\theta)]/\mu_c-n\alpha\log\theta,$$

and the maximum likelihood estimators are

$$\tilde{\mu}_c=\bar{x},\quad\tilde{\theta}=\bar{y}/\bar{x}.$$

In this case $\tilde{\theta}$ is invariant under scale changes as desired, but in general it is a positively biased estimator for $\theta$, i.e.,

$$E(\tilde{\theta})=\frac{m\alpha}{m\alpha-1}\theta.$$

Since $\alpha$ is assumed known an unbiased estimator is easily determined to be $\hat{\theta}=[(m\alpha-1)/m\alpha]\tilde{\theta}$. It can also be shown that $\hat{\theta}$ is the minimum variance unbiased estimator for $\theta$.

Confidence intervals and hypothesis tests ($H_0$: $\theta=1$ vs $H_A:\theta\neq1$) are based upon

$$T(\theta)=\frac{\bar{y}}{\theta\bar{x}},$$

which has a Snedecor's $F$-distribution with $2n\alpha$ and $2m\alpha$ degrees of freedom. The two-tailed critical values for an equal-tail test with significance level $\xi$ are determined from the appropriate $F$-tables such that

$$\Pr\{T(\theta)\leqslant T_1|\theta=1\}=\xi/2=\Pr\{T(\theta)\geqslant T_2|\theta=1\}.$$

A $1-\xi$ equal-tail confidence interval is given by

$$\tilde{\theta}/T_2\leqslant\theta\leqslant\tilde{\theta}/T_1.$$

Further details as well as the procedures necessary for determining an unbiased test are given by Olsen (1973).

By allowing $\mu_c$ to be an unknown parameter the maximum likelihood estimator for $\theta$ becomes biased, but easily adjusted however, and the length of the confidence intervals is increased compared to those in Section 3b. Furthermore, as expected, the same numerical results for the confidence intervals can be obtained in the Bayesian framework when the improper priors $1/\theta$ and $1/\mu_c$ are used.

## 5. Procedures when all parameters are unknown

The situation that would more realistically be encountered in a cloud seeding experiment is to know only that the underlying rainfall distributions were gamma and that the seeding effect is multiplicative. That is, no parameters in the model presented in Section 2 are known. In some cases, or for some experimenters, even these assumptions will not seem to be justified.

### *a. A Bayesian Analysis*

The analysis procedure for this case is essentially the same as in section 4.1. Since all of the parameters are now unknown, a joint prior for $\alpha$, $\mu_c$ and $\theta$ must be specified. The form for the prior given here is

$$p(\alpha,\mu,\theta)=p(\alpha)p(\mu)p(\theta)\propto p(\alpha)\mu^{-(k_1+2)}e^{-k_2/\theta}\theta^{-(K_1+2)}e^{-K_2/\theta},$$

where $p(\alpha)$ is any pdf for $\alpha$. This form assumes all the parameters are independent. The non-normalized joint posterior, making sure terms involving $\alpha$ are included, is proportional to

$$\frac{p(\alpha)\alpha^{N\alpha}\exp\{-(K_2/\theta)+\alpha(mg_x+ng_y)-[k_2+\alpha m\bar{x}+(\alpha n\bar{y}/\theta)]/\mu\}}{\Gamma(\alpha)^N\theta^{n\alpha+K_1+2}\mu^{N\alpha+k_1+2}},$$

where $N=n+m$ and $g_x$ and $g_y$ are the arithmetic means of the natural logarithms of the respective data sets.

The marginal posterior pdf for $\theta$ is the density of interest in determining the seeding effect. However, to obtain this marginal it is necessary to use numerical integration. This can be done directly on the above posterior or use can be made of the following marginal pdf's to determine the normalization constant and the required marginal for $\theta$:

$$\left.\begin{aligned} p(\alpha,\theta|\mathbf{x},\mathbf{y}) &\propto \frac{p(\alpha)\alpha^{N\alpha}\Gamma(N\alpha+k_1+1)\exp[\alpha(mg_x+ng_y)]}{\Gamma(\alpha)^N\theta^{n\alpha+K_1+2}[k_2+\alpha m\bar{x}+(\alpha n\bar{y}/\theta)]^{N\alpha+k_1+1}} \\ p(\alpha|\mathbf{x},\mathbf{y}) &\propto \frac{p(\alpha)\alpha^{N\alpha}\Gamma(n\alpha+K_1+1)\Gamma(m\alpha+k_1-K_1)\exp[\alpha(mg_x+ng_y)]}{\Gamma(\alpha)^N(k_2+\alpha m\bar{x})^{m\alpha+k_1-K_1}(\alpha n\bar{y})^{n\alpha+K_1+1}} \end{aligned}\right\}.$$

Assuming the numerical problems can be overcome, the remainder of the analysis is the same as in Section 3a. The procedures are demonstrated in Olsen *et al.* (1975) for the case when the joint prior is $p(\alpha,\theta,\mu) \propto (1/\theta)(1/\mu)$ for $\theta$, $\mu>0$ and $a\leqslant\alpha\leqslant b$. (Note this is obtained in the present setting by letting $k_1=K_1=-1$ and $k_2=K_2=0$.)

### b. Classical approaches

Two different hypothesis testing procedures have been proposed under the present assumptions. Schickedanz and Krause (1970) have derived a likelihood ratio test for determining if there is a difference in the scale parameters of two gamma distributions. This test is equivalent to testing if the seeding effect $\theta$ is equal to 1, i.e., no effect. The test is based on asymptotic theory for "large" sample sizes.

Another test has been given by Neyman and Scott (1967), called an optimal $C(\alpha)$ test, that also uses the same assumptions. Again the use of asymptotic theory is present in the development.

## 6. Summary

The preceding sections have presented both Bayesian and classical analysis procedures using the same assumptions concerning the form of the rainfall distributions, the use of a multiplicative effect, and selected cases of known parameters. The parallel development of the two methodologies was done to emphasize that inference procedures for both can be found and that the difficulties in applying either are comparable. No attempt was made to choose one methodology over the other.

For both of the methodologies the validity of the two basic underlying assumptions must be questioned when they are applied to a specific experimental analysis. In the case of many weather modification experiments, their validity will not be firmly established but neither will they be contradicted. If the analyses presented here are to be used, then a further study of the effect of particular types of departures from the assumptions should be made.

In the case of the Bayesian analyses presented, the analyses did not include a discussion of the problem of selecting an "appropriate" prior distribution. The priors used were selected for mathematical convenience and for their inclusion of a range of prior information. The selection of a prior to be used in a specific experimental analysis should be based on information obtained independent of the data. The selection can pose serious difficulties and there is no reason why two different analysts cannot use different priors for their individual analyses. Certainly, a justification for the selection should be made.

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