Methods of Computing the Power Spectrum for Equally Spaced Time Series of Finite Length

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ABSTRACT

Two conventional methods of computing the power spectrum, via the autocovariance function or via the fast Fourier transform (referred to as the lagged product method and the FFT method respectively for simplicity), have been examined analytically and numerically for equally spaced time series of finite length. It is found that the two methods are equivalent to each other, and that the only difference between them lies in regard to the spectral window. Spectral windows for the FFT method are superior to those for the lagged product method in that they do not show any negative values and that their influence is band-limited in frequency domain. There is little difference in spectral estimates between the two methods. In many cases the FFT method is economical in computation time, but for the case of large data points and small maximum lag the lagged product method is the more economical. It is proved that in the strict sense the power spectrum for higher frequencies than the Nyquist frequency is not folded linearly over lower frequencies both in the FFT method and the lagged product method. Finally it is discussed whether or not the use of original data repeatedly is consistent with the analysis of random phenomena.

1. Introduction

Spectral analysis of random phenomena has been an important tool in the fields of oceanography, meteorology, and many other branches of science and engineering. The practical method of measuring the power spectrum through the autocovariance function (hereinafter referred to as the lagged product method for simplicity), established theoretically by Blackman and Tukey (1958), has been used for a long time. However the rediscovery of the fast Fourier transform (Cooley and Tukey, 1965) has introduced another method (hereinafter referred to as the FFT method for simplicity) in which squared magnitudes of Fourier coefficients for original time series[1] are averaged with a weighting function to yield smoothed spectral estimates. Numerical comparison of spectral estimates between the two methods was made by Edge and Liu (1970), Taira (1971) and others. From theoretical points of view, however, it is important to investigate mathematically the essential or practical difference between the two methods.

In the practical situation, available records of natural phenomena are always limited in length. Moreover, in many cases, they are spaced at equal time intervals for digital computation. Blackman and Tukey (1958) discussed first the ideal case of a *continuous record* of infinite length in terms of Fourier integrals, and then proceeded to the case of an *equally-spaced record* of finite length by using Fourier series. Although they pointed out the limitation due to equally-spaced autocovariance function of finite length, they seem to have been confused between an equally spaced autocovariance function of finite length *obtained from a time series of infinite length* and that *obtained from an equally spaced time series of finite length*. For the former autocovariance function which may really include aperiodic and infinite information, their theoretical investigation is quite reasonable. However, for the latter autocovariance function which is the practical one, some of their theoretical conclusions cannot be justified.

In the following investigation, we start by expressing an original time series of finite length in terms of a finite Fourier series, because the use of Fourier integrals is not adequate for describing equally spaced time series of finite length. The interrelation of the FFT method to the lagged product method will be clarified, the limitation of the conventional analysis through the lagged product method (or the FFT method) will be discussed, and finally it will be concluded that the FFT method is superior to the lagged product method in the analysis of equally spaced record of finite length.

[1] The reader should note that in this paper the expression "time series" represents the original time sequence alone, and not the autocovariance function.

2. Spectral analysis by the FFT method

In Fig. 1 a continuous or equally spaced wind wave record of finite length is shown schematically. The continuous time series of finite length, $\eta(t)$, can be ex-

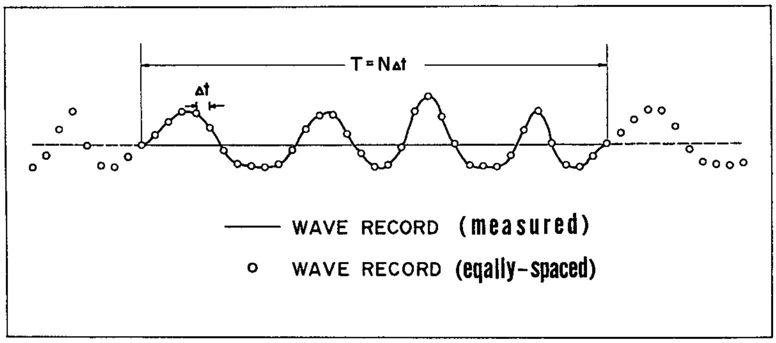


Fig. 1. Schematic representation of wind wave record. Solid line denotes continuous record of finite length, open circles equally-spaced record of finite length. $N\Delta t$, N and Δt are the total length of the record, the number of data points and sampling interval, respectively.

pressed by using an infinite Fourier series of the form

$$\eta(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos\frac{2\pi}{N}kt+b_k\sin\frac{2\pi}{N}kt\right),\quad 0\leqslant t\leqslant N,\quad (1)$$

where a_k, b_k are the Fourier coefficients, Δt (dimensional) and N (nondimensional) are arbitrary numbers within the constraint $T=N\Delta t$ (T, data interval), and real time is given by $t\Delta t$ (t; nondimensional variable number). The equally-spaced time series of finite length, $\eta'(t)$, can be expressed in terms of a finite Fourier series as

$$\eta'(t)=\frac{A_0}{2}+\sum_{k=1}^{N/2-1}\left(A_k\cos\frac{2\pi}{N}kt+B_k\sin\frac{2\pi}{N}kt\right)+\frac{A_{N/2}}{2}\cos\pi t,\quad t=0,1,\ldots,N-1,\quad (2)$$

where N is the number of data points and $\Delta t=T/N$ the sampling interval. Since real time is given again by $t\Delta t$, the subscript k corresponds to frequency such that $f=k/N\Delta t$. Fourier coefficients A_k, B_k for the time series $\eta'(t)$ are obtained by the fast Fourier transform provided, for example, $N=2^n$ (n, arbitrary integer).

In order to see the relation between (a_k,b_k) and (A_k,B_k), arbitrary numbers N and Δt in (1) are assumed to be identical with those in (2). By taking the Fourier transform, A_k and B_k are given by

$$A_k=\frac{2}{N}\sum_{t=0}^{N-1}\eta'(t)\cos\frac{2\pi}{N}kt,\quad k=0,1,\ldots,\frac{N}{2},\quad (3)$$

$$B_k=\frac{2}{N}\sum_{t=0}^{N-1}\eta'(t)\sin\frac{2\pi}{N}kt,\quad k=1,2,\ldots,\frac{N}{2}-1.\quad (4)$$

Since $\eta'(t)=\eta(t)$ for $t=0,1,\ldots,N-1$, $\eta'(t)$ in (3) (4) can be replaced by $\eta(t)$ in Eq. (1). Then

$$A_k=a_k+(a_{N-k}+a_{N+k}+a_{2N-k}+a_{2N+k}+\cdots)=a_k+\sum_{j=1}^{\infty}(a_{jN-k}+a_{jN+k}),\quad (5)$$

$$B_k=b_k+(-b_{N-k}+b_{N+k}-b_{2N-k}+b_{2N+k}-\cdots)=b_k+\sum_{j=1}^{\infty}(-b_{jN-k}+b_{jN+k}).\quad (6)$$

These equations indicate that the Fourier components (a_k,b_k) cannot be distinguished from $(a_{jN\pm k},b_{jN\pm k})$. This fact is well known as an aliasing effect or folding effect.

Now, the raw spectrum (or periodgram) for the FFT method is defined by[2]

$$P_k=\tfrac{1}{2}(A_k^2+B_k^2)=\tfrac{1}{2}\{[a_k+\sum_{j=1}^{\infty}(a_{jN-k}+a_{jN+k})]^2+[b_k+\sum_{j=1}^{\infty}(-b_{jN-k}+b_{jN+k})]^2\}.\quad (7)$$

In order to obtain reliable spectral estimates, P_k's are averaged over neighboring frequencies with a weighting function $h(k,k')$:

$$\bar{P}_k=\sum_{k'}h(k,k')P_{k'},\quad (8)$$

where $h(k,k')$ is a filter function of arbitrary shape and $\Sigma h(k,k')=1$. Typical examples of filter functions are shown in Fig. 2. Later in Section 3 these filter functions prove to be the same kind of spectral window which is well known for the lagged product method.

It is important to note that in (7) spectral estimates for higher frequency components than the Nyquist frequency are folded nonlinearly over lower frequency components: coupling terms between two different frequency components such as $a_{iN+k}a_{jN+k}$ $(i\neq j)$ do exist. These coupling terms are expected to decrease through the averaging process [Eq. (8)] because, while a_{iN+k}^2 and a_{jN+k}^2 are absolutely positive or zero, $a_{iN+k}a_{jN+k}$ may be positive or negative over a neighboring fre-

[2] Instead of the power spectrum or spectral density defined by $\Phi(k)=\frac{1}{2}(A_k^2+B_k^2)/\Delta f$ [cm² s], where Δf denotes the elementary frequency bandwidth, we use spectral estimates P_k [cm²] in order to relate them to the autocovariance function later in Section 3. However, all results of spectral analysis in this paper will be presented in the form of spectral density.

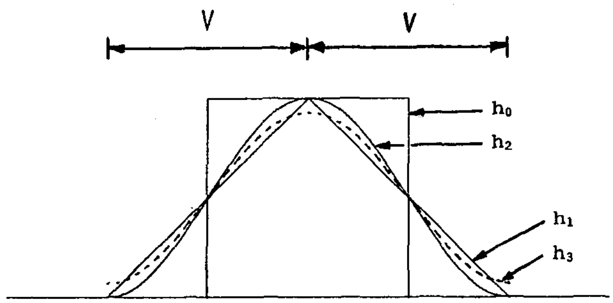


FIG. 2. Examples of filter functions for the FFT method: $h_0(k)=1/v$ $(|k|\leqslant/2)$, 0 (otherwise); $h_1(k)=[1-|k|/v]/v$ $(|k|\leqslant v)$, 0 (otherwise); $h_2(k)=[0.50+0.50\cos(\pi k/v)]/v$ $(|k|\leqslant v)$, 0 (otherwise); $h_3(k)=[0.54+0.46\cos(\pi k/v)]/v$ $(|k|\leqslant v)$, 0 (otherwise). Note that k's are integers and v is the nondimensional halfwidth of filter function for h_1, h_2 and h_3.

quency range. Nevertheless, in general, they do not vanish completely.

To evaluate the order of magnitude of the linear folding spectrum $\frac{1}{2}\sum_j(a^2_{jN\pm k}+b^2_{jN\pm k})$ and the nonlinear folding spectrum

$$\tfrac{1}{2}\{[a_k+\textstyle\sum_j(a_{jN-k}+a_{jN+k})]^2 + [b_k+\textstyle\sum_j(-b_{jN-k}+b_{jN+k})]^2\},$$

spectral analysis of laboratory wind wave data has been made by the FFT method smoothing with a triangular filter (half-width, $8/N\Delta t$; frequency bandwidth, $\Delta f=16/N\Delta t$). The results are shown in Fig. 3. First, spectral density ϕ_0 has been calculated from an equally-spaced wave record $\eta_0(t)$ [$N=2048$, $\Delta t=32/2048$, $f_N=\frac{1}{2}\Delta t=32$ Hz] and is assumed to have no spectral components for frequencies higher than Nyquist frequency. Then the spectral density ϕ_{nl} has been calculated from the same wave record[3] thinned out leaving one in two [$N=1024$, $\Delta t=64/2048$, $f_N=16$ Hz]. This is subjected to the nonlinear folding effect. The spectral density ϕ_l subject to the linear folding effect has been calculated as

$$\phi_l(k)=\phi_0(k)+\phi_0(N-k).$$

The ratio of $(\phi_l-\phi_{nl})$ to ϕ_l, that is, the ratio of the nonlinear folding spectrum to the linear folding spectrum proves to be roughly 10% on the average and 24% at most.

3. Spectral analysis by the lagged product method

If we let $X(t)$ be a continuous time series of infinite length, then the corresponding autocovariance function $R(\tau)$ is defined by

$$R(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}X(t)X(t+\tau)dt,\quad -\infty<\tau<\infty,\quad (9)$$

[3] There are two such time series available: $\eta_1(t)=\eta_0(0)$, $\eta_0(2)$, $\ldots$, $\eta_0(2046)$ and $\eta_2(t)=\eta_0(1)$, $\eta_0(3)$, $\ldots$, $\eta_0(2047)$. On denoting the spectral densities obtained from $\eta_1(t)$ and $\eta_2(t)$ by ϕ_{nl1} and ϕ_{nl2}, respectively, $\bar{\phi}_{nl}=(\phi_{nl1}+\phi_{nl2})/2$ proves to equal ϕ_l. In the practical situation, however, either $\eta_1(t)$ or $\eta_2(t)$ can be available to us, because we cannot have perfect information on the original time series. The spectral density ϕ_{nl} shown in Fig. 3 corresponds to time series $\eta_1(t)$.

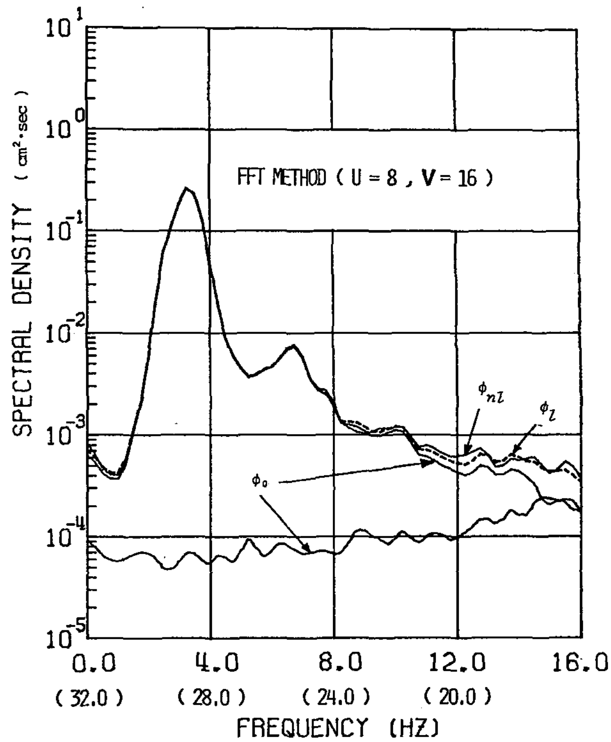


FIG. 3. Comparison of the linear folding effect to the nonlinear folding effect. Spectral density ϕ_0 is the reference spectrum calculated from an equally spaced wind wave record ($N=2048$, $\Delta t=32/2048$). Spectral densities ϕ_l and ϕ_{nl} (for explanation see the text) are subjected to linear and nonlinear folding effects, respectively.

where t and τ are dimensional numbers. The power spectrum $P(f)$ is related to the autocovariance function by the Fourier transform, i.e.,

$$P(f)=\int_{-\infty}^{\infty}R(\tau)e^{-2\pi if\tau}d\tau.\quad (10)$$

The above equation, referred to hereafter as the Wiener-Khintchine relation, gives the theoretical basis for the evaluation of the power spectrum through the autocovariance function. Therefore the most reasonable definition of the autocovariance function, the substitute for (9), is required so as to satisfy the above relation for equally spaced time series of finite length.

Blackman and Tukey (1958) defined such an autocovariance function as

$$R(\tau)=\frac{1}{N-\tau}\sum_{t=0}^{N-\tau-1}\eta(t)\eta(t+\tau),\quad \tau=0,1,\ldots,\quad (11)$$

while Akaike (1964) gave the definition[4]

$$R(\tau)=\frac{1}{N}\sum_{t=0}^{N-\tau-1}\eta(t)\eta(t+\tau),\quad \tau=0,1,\ldots.\quad (12)$$

[4] Akaike (1964) considered a time series $\eta''(t)$ [$\eta''(t)=\eta(t)$ for $t=0,\ldots,N-1$ and $\eta''(t)=0$ for otherwise] and obtained (12). However, this is unreasonable because the autocovariance function corresponding to $\eta''(t)$ should be equal absolutely to zero since $\overline{\eta''(t)}\equiv 0$ (the bar represents a time average).

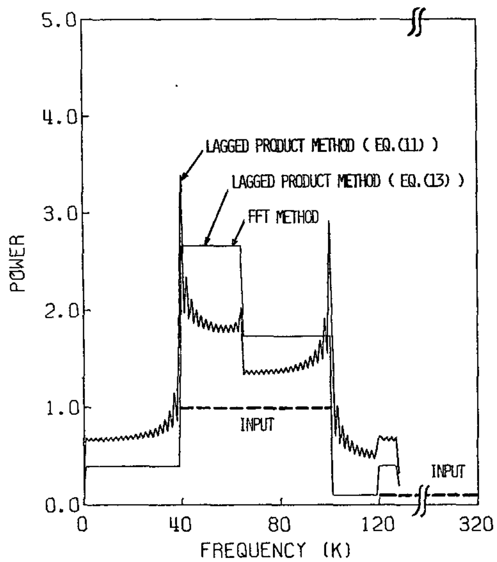


FIG. 4. Raw spectral estimates of artificial time series [Eq. (17)] by the FFT method and the lagged product method.

We now propose a third definition for the autocovariance function:

$$R(\tau)=\frac{1}{N}\sum_{t=0}^{N-1}\eta(t)\eta(t+\tau),\quad \tau=0,1,\ldots, \tag{13}$$

where time series is extended repeatedly such as $\eta(N+t)=\eta(t)$ for $t=1, 2, \ldots$.

The autocovariance function defined by (11) appears to be the most *natural* one, because it does not need any artificial modification to the original time series. As will be discussed in the following section, however, it does not satisfy the Wiener-Khintchine relation: the Fourier transform of $R(\tau)$ [Eq. (11)] does not give the true power spectrum appropriate to the original time series. The autocovariance function defined by (12) is somewhat biased by the artificial addition of zeros assuming $\eta(N+t)=0$ for $t=0, 1, \ldots$, and again does not satisfy (10). Only the autocovariance function defined by (13) proves to satisfy the Wiener-Khintchine relation. Since the relation is the theoretical basis for estimating the power spectrum through the autocovariance function, the author insists that (13) gives the most *reasonable* autocovariance function for determining the power spectrum. It will be shown in Section 5 that the cyclic extension of original data by (13) is not in contradiction to the concept of "random phenomena," *as well as* the other definitions.

a. Case of maximum lag number M_0 $(=N/2)$

First, we consider a special case of the maximum lag number M_0 $(=N/2)$, the reference maximum lag number[5] (N, number of data points). In this case, the elementary frequency bandwidth of spectral estimates is given by $\Delta f=1/N\Delta t$ for both the FFT method and the lagged product method, and then direct comparison of the two methods can be made. If we let $\eta(t)$, $t=0, 1, \ldots, N-1$, be an equally spaced record of wind waves, then $\eta(t)$ can be expressed in terms of a finite Fourier series as

$$\eta(t)=\sum_{n=1}^{N/2}\left(A_n\cos\frac{2\pi}{N}nt+B_n\sin\frac{2\pi}{N}nt\right),\qquad t=0,1,\ldots,N-1, \tag{14}$$

where $B_0=B_{N/2}=0$, and A_0 is assumed to be zero for simplicity. The lagged products are given by

$$\begin{aligned}\eta(t)\eta(t+\tau) &=\sum_{n=1}^{N/2}\sum_{m=1}^{N/2}\frac{A_nA_m}{2}\left\{\cos\frac{2\pi}{N}[(n+m)t+m\tau]+\cos\frac{2\pi}{N}[(n-m)t-m\tau]\right\}\\ &+\sum\sum\frac{-B_nB_m}{2}\left\{\cos\frac{2\pi}{N}[(n+m)t+m\tau]-\cos\frac{2\pi}{N}[(n-m)t-m\tau]\right\}\\ &+\sum\sum\frac{A_nB_m}{2}\left\{\sin\frac{2\pi}{N}[(n+m)t+m\tau]-\sin\frac{2\pi}{N}[(n-m)t-m\tau]\right\}\\ &+\sum\sum\frac{B_nA_m}{2}\left\{\sin\frac{2\pi}{N}[(n+m)t+m\tau]+\sin\frac{2\pi}{N}[(n-m)t+m\tau]\right\}.\end{aligned} \tag{15}$$

On substituting (15) into (11) or (12), lagged product sums of each term of (15) do not reduce to zero in general, and coupling terms between two different frequency components such as A_nB_m $(n\neq m)$ do remain. On substituting (15) into (13), on the other hand, every lagged product sum reduces to zero for $n\neq m$ and the sums of the third and forth terms of (15) in braces vanish for $n=m$. Thus (13) becomes

$$R_0(\tau)=\sum_{n=1}^{N/2}\tfrac{1}{2}(A_n^2+B_n^2)\cos\frac{\pi}{M_0}n\tau, \tag{16}$$

[5] Since the total number of degrees of freedom for time series $\eta(t)$ $(t=0, 1, \ldots, N-1)$ is given by N, the number of spectrum lines (each of which has two degrees of freedom at least) should be $N/2$ at most. In other words, M_0 is the largest maximum lag number. Although autocovariances can be calculated formally up to maximum lag $(N-1)$ by (11) or (12), these autocovariances are inconsistent with the Wiener-Khintchine relation.

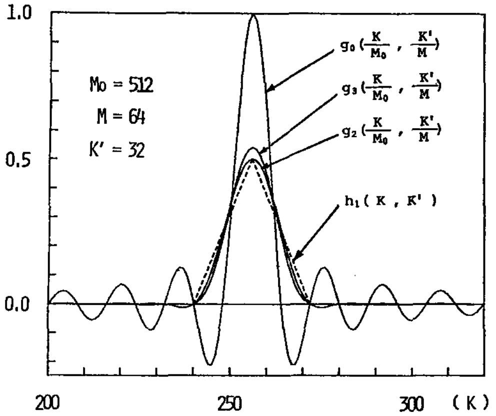


FIG. 5. Spectral windows g_0, g_2, g_3 ($M_0=512$, $M=64$, $k'=32$) for the lagged product method and h_1 ($v=16$) for the FFT method.

where τ is the nondimensional lag time ($\tau=0, 1, \ldots$). By taking the Fourier cosine transform of $R_0(\tau)$, one can obtain the same spectral estimates as the raw spectrum for the FFT method.

In order to see the differences in spectral estimates among these three definitions for autocovariance function, an artificial time series expressed by

$$\eta(t)=\sum_{k=40}^{100}\sqrt{2}\cos\frac{2\pi}{N}kt+\sum_{k=120}^{320}\sqrt{0.2}\cos\frac{2\pi}{N}kt,$$
$$t=0, 1, \ldots, 255, \quad (17)$$

has been analyzed. One should notice that the time series includes higher frequency components than the nondimensional Nyquist frequency ($=128$). The results shown in Fig. 4 indicate that spectral estimates through Eq. (13) are exactly the same estimates as those by the FFT method [Eq. (7)], and that spectral estimates through Eq. (11) are complicated and fluctuating with frequency, which may be attributed to the coupling of the two different frequency components. Spectral estimates through Eq. (12), not presented here, also fluctuate and differ little from those through (11).

Like the case of the FFT method, spectral estimates for the case of maximum lag M_0 should be smoothed with a spectral window (filter function) to obtain reliable estimates. Generally those complicated estimates becomes smooth through the averaging process described below.

b. Case of arbitrary maximum lag number M

Here we are concerned with the case of an arbitrary maximum lag number M ($M\leqslant M_0$). The discussion which follows is valid equally for the three kinds of autocovariance function. Since spectral estimates are obtained by the Fourier transform of autocovariance function, conversely autocovariances can be expressed in terms of spectral estimates. We let P_k ($k=0, 1, \ldots, M_0$) and $P'_{k'}$ ($k'=0, 1, \ldots, M$) be spectral estimates for the cases of maximum lag M_0 and M respectively; then autocovariance functions for the cases of maximum lag M_0 and M are, respectively,

$$R_0(\tau)=\sum_{k=0}^{M_0}P_k\cos\frac{\pi}{M_0}k\tau, \quad \tau=0, 1, \ldots, M_0 \quad (18)$$

$$R(\tau)=\sum_{k'=0}^{M}P'_{k'}\cos\frac{\pi}{M}k'\tau, \quad \tau=0, 1, \ldots, M. \quad (19)$$

It should be noted that spectral estimates P_k and $P'_{k'}$ correspond to *periodic* autocovariance function $R_0(\tau)$ and $R(\tau)$, respectively, as it is. Spectral estimates for the case of maximum lag M are given by

$$P'_{k'}=\frac{1}{M}\sum_{\tau=-M}^{M-1}R(\tau)\cos\frac{\pi}{M}k'\tau, \quad k'=1, 2, \ldots, M-1, \quad (20)$$

where $R(-\tau)$ is assumed to be equal to $R(\tau)$. P'_0 and P'_M can also be obtained by multiplying (20) by a factor of $\frac{1}{2}$ for $k'=0$ and M. Since $R_0(\tau)=R(\tau)$ for $\tau=-M, \ldots, 0, \ldots, M-1$, we can use (18) as a substitute for $R(\tau)$ in (20); then

$$P'_{k'}=\frac{1}{M}\sum_{\tau=-M}^{M-1}\sum_{k=0}^{M_0}P_k\cos\frac{\pi}{M_0}k\tau\cos\frac{\pi}{M}k'\tau$$
$$=\sum_{k=0}^{M_0}P_k g_0\left(\frac{k}{M_0}, \frac{k'}{M}\right), \quad (21)$$

where

$$g_0\left(\frac{k}{M_0}, \frac{k'}{M}\right)=\frac{1}{2M}\left[\sin\left(\frac{k}{M_0}+\frac{k'}{M}\right)\pi M\cot\frac{1}{2}\left(\frac{k}{M_0}+\frac{k'}{M}\right)\pi\right.$$
$$\left.+\sin\left(\frac{k}{M_0}-\frac{k'}{M}\right)\pi M\cot\frac{1}{2}\left(\frac{k}{M_0}-\frac{k'}{M}\right)\pi\right]. \quad (22)$$

Ordinarily, spectral estimates $P'_{k'}$ are smoothed in the frequency domain with Hanning's weights (0.25, 0.50, 0.25), for example. Then (21) becomes

$$\tilde{P}_{k'}=\sum_{k=0}^{M_0}P_k g_2\left(\frac{k}{M_0}, \frac{k'}{M}\right), \quad (23)$$

where

$$g_2\left(\frac{k}{M_0}, \frac{k'}{M}\right)=0.25g_0\left(\frac{k}{M_0}, \frac{k'-1}{M}\right)+0.50g_0\left(\frac{k}{M_0}, \frac{k'}{M}\right)$$
$$+0.25g_0\left(\frac{k}{M_0}, \frac{k'+1}{M}\right). \quad (24)$$

On replacing the coefficients (0.25, 0.50, 0.25) in Eq. (24) by Hanning's weights (0.23, 0.54, 0.23), we have a similar function, g_3. Here the weighting functions g_0, g_2, g_3 are equivalent to the spectral windows Q_0, Q_2, Q_3, respectively, described by Blackman and Tukey (1958). Eq. (23) means that smoothed spectral estimates for the case of maximum lag M are identical with running averages of raw spectral estimates for the case of maximum lag M_0. These equations are important in that they imply the interrelation between the FFT method and the lagged product method.

The same mathematical treatment as above can be easily applied to the cross-spectrum estimation. Again the same relation as (23) can be found between cross spectra for the cases of maximum lag M_0 and M, though the weighting function g_0 is replaced by a slightly different function.

Spectral windows g_0, g_2, g_3 for the case of $M_0=512$ and $M=64$ are shown in Fig. 5 together with the corresponding triangular filter h_1 used in the FFT method. Note that all these windows are defined for integers k. Spectral windows g_2 and g_3 are well-behaved compared to g_0. However, they show negative values, though not significant, and their influences are not band-limited perfectly in the frequency domain. These cause occasionally negative spectral estimates. In these respects, the triangular filter h_1 and other filters h_0, h_2, h_3 for the FFT method, shown in Fig. 2, are superior to the spectral windows g_0, g_2, g_3.

In Fig. 6 the spectral window g_2 for the case of $M_0=512$ and various maximum lag M is shown. It can be seen that the frequency range of the main lobe widens with decreasing M. This corresponds to the fact that reliable spectral estimates with high degrees of freedom can be obtained in the case of small maximum lag number. As an example, spectral densities estimated from laboratory wind wave records ($N=1024$) using Hanning's weights are shown in Fig. 7 for the case $M=64$ and $M=128$. One can see that spectral density for the case $M=64$ (dashed line) shows much greater smoothing over frequency compared to that for the case $M=128$ (solid line); the smaller the maximum lag, the smoother the resulting spectral density.

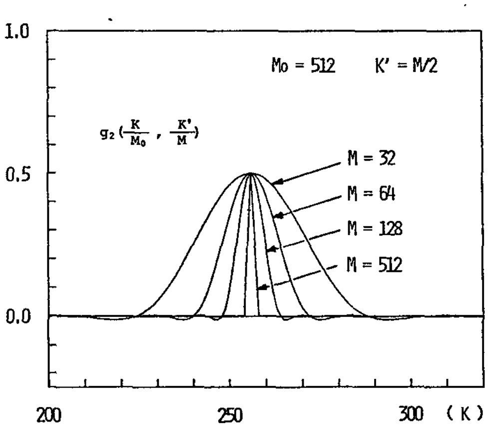


Fig. 6. Spectral window g_2 ($M_0=512$) for variable M.

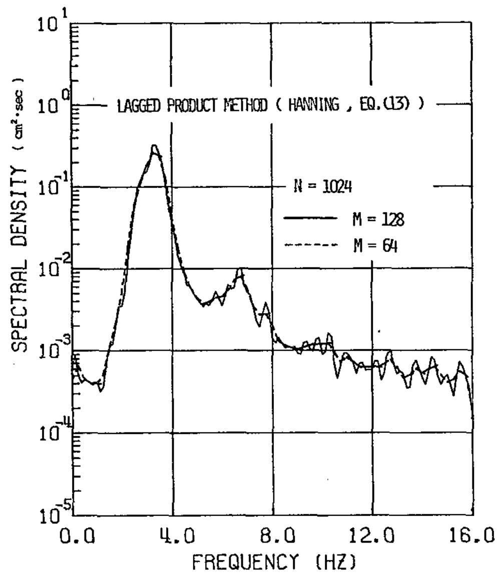
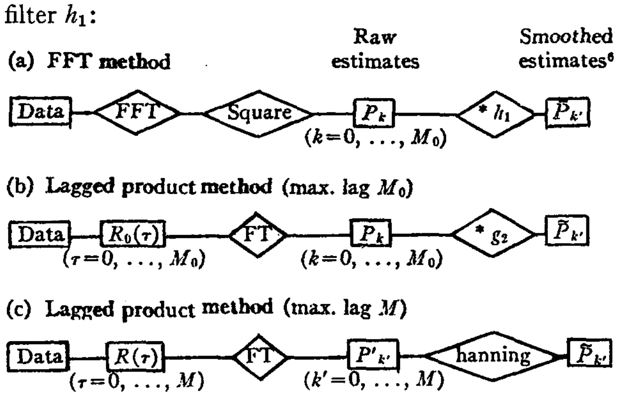


Fig. 7. Spectral densities of a wind wave record ($N=1024$) by the lagged product method [Hanning, Eq. (13)] for the case $M=128$ (solid line) and $M=64$ (dashed line).

4. Comparison of the FFT method to the lagged product method

The theoretical investigation described in preceding sections can be summarized as below for the case of Hanning's weights, spectral window g_2 and triangular filter h_1:



Here the asterisk denotes digital convolution, FT the Fourier transform and FFT the fast Fourier transform. These flow charts indicate that the smoothed estimates by methods (b) and (c) are identical to each other, and that the raw estimates by the FFT method are equal to those by method (b). Therefore, it can be concluded that spectral estimates by the lagged product method are identical with running averages of the raw estimates for the FFT method, and that the only difference between the two methods is that of the spectral window. As it turns out, filter functions h_i for the FFT method are the same as spectral windows g_i for the lagged product method.

[6] Smoothed estimates should be devided by Δf (frequency bandwidth) in order to give spectral density.

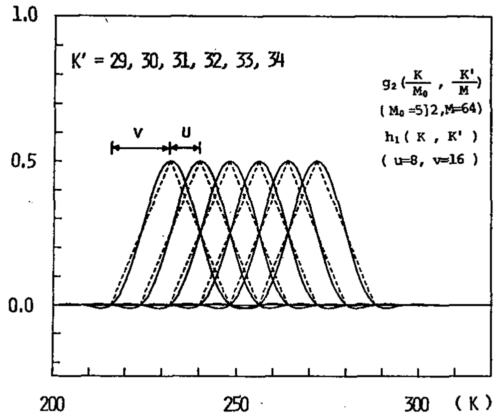


FIG. 8. Comparison of spectral window between the FFT method and the lagged product method. As an example spectral windows g_2 ($M_0=512$, $M=64$) and h_1 ($u=8$, $v=16$) are presented.

It is most important to note that spectral windows g_i ($i=0$, 2, 3) are digital filters defined at finite points $k=0, 1, \ldots, M_0$, and $k'=0, 1, \ldots, M$, unlike spectral windows Q_i ($i=0$, 2, 3) defined for infinite and continuous numbers. This is due to the fact that, since the original time series is limited in length, there is a reference maximum lag number. This is why we are able to use a non-negative and band-limited filter function for the FFT method.

We now proceed to compare the two methods in more detail for the case $N=2^n$ and $M=2^m$ (n, m; arbitrary integers). For the lagged product method, the elementary frequency bandwidth of spectral estimates and the halfwidth of the spectral window are given by $\Delta f=\frac{1}{2}M\Delta t$ and $2\Delta f$, respectively, whereas arbitrary integral multiples of $1/N\Delta t$ are available for the FFT method. In order to make an exact comparison between the two methods, the nondimensional elementary frequency bandwidth (normalized elementary frequency bandwidth by $1/N\Delta t$) and the nondimensional halfwidth of the spectral window (normalized halfwidth of spectral window by $1/N\Delta t$), hereinafter denoted by u and v, respectively, are determined as $u=N/2M$ and $v=2u$ for the FFT method.

In Fig. 8 the spectral window g_2 ($M_0=512$, $M=64$, $k'=29\sim33$, $k=200\sim320$) is shown with solid lines and the triangular filter function h_1 ($u=8$, $v=16$) with dashed lines. The figure symbolizes the difference between the two methods. A comparison of computed spectral estimates has been made by analyzing a laboratory wind wave record ($N=1024$) by the FFT method ($u=8$, $v=16$) and the lagged product method ($M=64$) through Eqs. (11)–(13). The results are presented in Table 1 for frequencies near the dominant frequency for which the spectral density is maximum. Good coincidence of spectral estimates between the FFT method and the lagged product method can be seen over the entire frequency range. This may be due to the fact that the spectral window h_1 for the FFT method resembles g_2 for the lagged product method. Closer coincidence can be expected by the spectral window h_2 for the FFT method. The coincidence of spectral estimates among the lagged product methods through Eqs. (11)–(13) can be explained by the fact that, for small lag number relative to data length, significant difference in autocovariances is not seen among these three definitions. In Fig. 9 the spectral densities by the FFT method and lagged product method through Eq. (11) are shown by the dashed and solid lines, respectively.

From an economic point of view, it may be useful to compare the computation time required for obtaining spectral density from original time series for the two methods. A number of experiments have been made by using time series of various numbers of N and M (and u, v) on the FACOM 230–60 of the Computer Center of Kyushu University. In order to make a reasonable or significant comparison, parametric conditions are retained such that $N=2^n$, $M=2^m$ and $v=2u=N/M$. The results, presented in Table 2, should allow for an inaccuracy of ±20 ms. By the lagged product method computation time increases considerably with increasing

TABLE 1. Comparison of spectral densities of a wind wave record ($N=1024$) using the FFT method ($u=8$, $v=16$) and the lagged product method ($M=64$) through Eqs. (11)–(13). Values in (cm^2 s).

Frequency (×0.25 Hz)	FFT method	Lagged product method		
		Eq. (11)	Eq. (12)	Eq. (13)
1	6.6214E-04	5.7410E-04	6.5490E-04	6.6545E-04
2	4.6206E-04	3.7708E-04	4.4950E-04	4.7682E-04
3	4.1610E-04	2.9330E-04	4.0365E-04	4.0399E-04
4	4.0664E-04	2.5063E-04	4.0427E-04	3.7975E-04
5	5.7682E-04	3.4328E-04	5.1242E-04	5.0392E-04
6	1.1700E-03	9.2105E-04	1.1244E-03	1.1206E-03
7	2.2077E-03	1.8527E-03	2.1512E-03	2.1376E-03
8	5.9634E-03	4.5528E-03	4.9705E-03	5.0205E-03
9	2.1746E-02	1.8213E-02	1.8922E-02	1.8879E-02
10	6.1208E-02	5.9674E-02	6.1246E-02	5.9971E-02
11	1.1224E-01	1.1566E-01	1.1458E-01	1.1557E-01
12	1.8557E-01	1.8744E-01	1.8232E-01	1.8664E-01
13	2.6202E-01	2.6576E-01	2.6453E-01	2.6309E-01
14	2.3605E-01	2.4385E-01	2.4562E-01	2.4144E-01
15	1.2794E-01	1.2832E-01	1.2787E-01	1.2830E-01
16	5.0432E-02	4.5492E-02	4.5579E-02	4.6379E-02
17	1.9574E-02	1.7388E-02	1.7924E-02	1.8008E-02
18	9.2553E-03	8.7125E-03	8.8454E-03	9.1022E-03
19	6.1181E-03	5.7076E-03	5.9338E-03	5.9617E-03
20	4.5504E-03	4.3314E-03	4.4788E-03	4.5122E-03
21	3.7868E-03	3.4388E-03	3.7462E-03	3.5883E-03
22	3.9810E-03	3.5771E-03	3.9610E-03	3.6905E-03
23	4.3831E-03	4.2280E-03	4.3804E-03	4.3091E-03
24	4.6481E-03	4.6908E-03	4.5587E-03	4.7605E-03
25	5.5490E-03	5.5115E-03	5.4317E-03	5.5708E-03
26	7.1178E-03	7.3585E-03	7.3325E-03	7.3683E-03
27	7.7573E-03	8.0520E-03	8.0025E-03	8.0285E-03
28	5.9471E-03	5.8338E-03	5.9081E-03	5.8536E-03
29	3.6041E-03	3.4007E-03	3.5137E-03	3.4571E-03
30	2.8513E-03	2.6818E-03	2.7484E-03	2.7281E-03

maximum lag number. Although the FFT method is more economical in many cases, the lagged product method needs less computation time in the case of smaller M (or larger v). The critical number of maximum lag that makes the two methods comparable in computation time is represented roughly by $M=64$ for $N=1024$, 2048, 4096 and $M=32$ for $N=256$, 512.

5. Discussion

In Section 3 the author has stressed that, since Wiener-Khintchine relation is the theoretical basis for power spectrum estimation by the lagged product method, the autocovariance function should be defined by (13) where the time series is extended repeatedly. In that case the lagged product method can be compared exactly with the FFT method: the two methods differ from each other only in the spectral window. As will be discussed below, the use of original data repeatedly by (13) is not inconsistent with the analysis of random phenomena as well as the use of (11) or (12).

Essentially aperiodic phenomena can be analyzed only by using time series of infinite length. If, by any chance, we could obtain an autocovariance function estimated from a time series of infinite length, we could then make an analysis of aperiodic phenomena, even if the autocovariances are equally spaced and of finite length. Blackman and Tukey (1958) discussed the measurement of power spectra based on this ideal autocovariance function. Spectral windows Q_i correspond to the ideal case. However, this is not the practical case. Infinite information (or aperiodic phenomenon) cannot be drawn from finite information,

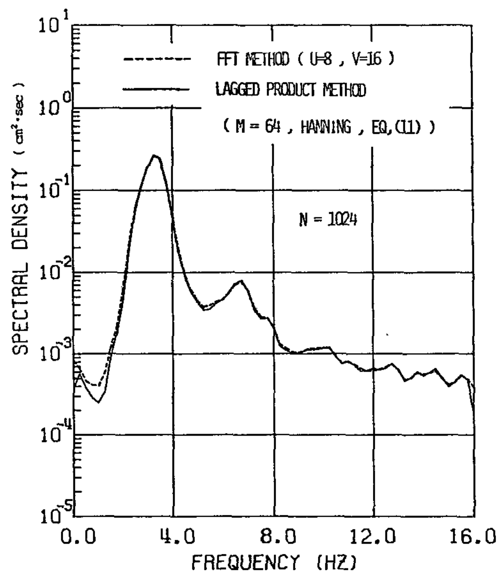


FIG. 9. Comparison of spectral densities of wind wave record ($N=1024$) between the FFT and lagged product methods. Solid line represents spectral density by the lagged product method [$M=64$, Hanning, Eq. (11)] and broken line by the FFT method ($u=8$, $v=16$).

TABLE 2. Comparison of computation time between the FFT and the lagged product method through Eq. (13). Numerical experiments were carried out on the FACOM 230-60 of the Computer Center of Kyushu University. The nondimensional frequency bandwidth for the FFT method is determined so that $u=v/2=N/2M$.

	Lagged product method [Eq. (13)] (Hanning's weights)		FFT method (triangular filter)	
Number of data points	Maximum lag	CPU time (ms)	Halfwidth of filter	CPU time (ms)
N=256	M= 16	162	v=16	422
	32	368	8	423
	64	1009	4	423
N=512	M= 32	600	v=16	935
	64	1514	8	907
	128	4092	4	916
N=1024	M= 32	1071	v=32	1944
	64	2370	16	1945
	128	6014	8	1944
N=2048	M= 64	4185	v=32	4020
	128	9416	16	4013
	256	23 292	8	3885
N=4096	M= 64	8753	v=64	8874
	128	18 419	32	8765
	256	41 266	16	8785

however elaborate the mathematical treatment may be. The addition of zeros at the both ends of autocovariance function (or time series) is essentially meaningless (Rikiishi and Mitsuyasu, 1973), since there is no significant increase in information. Thus, in a strict sense, the analysis of aperiodic phenomena is impossible practically.

In order to analyze random phenomena which are essentially aperiodic, it is necessary to assume that true autocovariance function for the phenomena can be given by the ensemble average of $\{R(\tau)\}$, where $R(\tau)$ is an autocovariance function estimated from a time series of finite length. Ordinarily we make use of only one time series of finite length,[7] and then we have to assume (or believe) additionally that the ensemble average of $\{R(\tau)\}$ *can be well represented* by an autocovariance function calculated from only one time series of finite length. In other words, conventional methods for the analysis of random phenomena are based on the *assumption* (or *belief*) that the true autocovariance function (weighted by lag window) for random phenomena can be represented by an autocovariance function (weighted by lag window) for one sample time series of finite length. Then how can we determine the most representative?

[7] Welch (1967) and Edge and Liu (1970) obtained spectrum estimates by sectioning a record and averaging periodgrams of sections. The total of the segment records still remain to be a time series of finite length.

Blackman and Tukey (1958) and Akaike (1964) proposed the autocovariance functions (11) and (12), respectively, as the best ones, while the author proposes (13). From a statistical point of view, the autocovariance functions (11)–(13) may be on an equal level with each other; we have no grounds for believing that one of them is more reasonable statistically than others. Indeed, significant differences are not seen between these three autocovariance functions, obtained from wind wave records, for small lag number τ in the case of a large number of data points N. From the viewpoint of relating the autocovariance function to the power spectrum, however, only (13) satisfies the Wiener-Khintchine relation as has been discussed. Accordingly (13) can be regarded as the most reasonable definition of the autocovariance function from the point of estimating the power spectrum.

It does not follow from the repeated use of the original record by (13) that phenomena continue to repeat, but only that the true autocovariance function for random phenomena *can be represented* by an autocovariance function obtained from a sample time series extended repeatedly. The application of (13) to meteorological or oceanographical data that can be best described stochastically may seem unreasonable for the reason that the repetition of an assumed time series is inconsistent with the conception of random phenomena. But this reason may be beside the point. What is most important in spectrum estimation is knowing how to determine the most reasonable autocovariance function for the stochastic process. If the total length of measured data is long enough, mean lagged products for small lags (which determine the gross features of the power spectrum) may not be affected so much by the cyclic use of measured data, because the end effect due to the repetition does not contribute so much to lagged products for small lags. It is likely that the autocovariance function for a stochastic process is well represented by that obtained from a time series of finite length extended repeatedly.

From the above discussion, the analysis of random phenomena by the lagged product method seems to be characterized only by the use of a lag window in comparison with the analysis of periodic phenomena. Since the use of a lag window in the lagged product method is equivalent to the procedure of smoothing periodgrams with a filter function in the FFT method, spectral analysis by the FFT method can be used for the analysis of random phenomena as well as the lagged product method.

Since the two methods are equivalent theoretically to each other, the discussion in Section 2 on the folding effect for the FFT method is also true for the lagged product method: the power spectrum for higher frequencies than the Nyquist frequency is folded nonlinearly over lower frequencies. The theoretical disagreement with the discussion made by Blackman and Tukey (1958) results from the fact that they erred in believing that an equally spaced autocovariance function was obtained from a time series of infinite length.

All that the author has discussed above is, in conclusion, that the FFT method is equivalent theoretically to the lagged product method, that filter functions for the FFT method are superior to spectral windows for the lagged product method, and that both methods are based on the analysis of periodic phenomena. The important point is that the results of spectrum estimation should be understood in terms of probabilities.

Finally we should refer briefly to the effect of the data window. In the FFT method use of both the data and spectral windows has been recommended (Tukey, 1967; Welch, 1967). Then, since the lagged product method is equivalent to the FFT method, it follows that data windows should be used in the lagged product method too. In the lagged product method, on the other hand, the data window has not been used. This implies that the data window is not necessary in the FFT method either. These statements contradict each other. The author is of the opinion that the theoretical basis for the use of the data window is not authentic, and that the data window makes an undesirable, though not significant, modification of the original record in estimating the *most reasonable* autocovariance function. The detailed discussion on this problem has been made elsewhere (Rikiishi and Mitsuyasu, 1973).

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