# Confidence Limits for Seeding Effect in Single-Area Weather Modification Experiments

EDWIN L. CROW

*National Center for Atmospheric Research,*[1] *Boulder, CO 80307*

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ABSTRACT

A recurring problem in single-area randomized seeding experiments has been the assessment of statistical significance over all experimental units, such as days, when some of the units receive no precipitation. The present paper solves the problem in two ways: 1) the likelihood ratio test of the complete model including both zero and positive observations and 2) an approximate confidence interval for the ratio of mean values over all seeded and non-seeded experimental units. The results are obtained for both log-normal and gamma distributions. They are illustrated by numerical examples from the 1972–74 National Hail Research Experiment randomized seeding experiment.

## 1. Introduction

A recurring problem in single-area randomized seeding experiments has been the assessment of statistical significance over all experimental units, such as declared hail days in the National Hail Research Experiment (NHRE), when some of the units receive zero precipitation. The present paper solves the problem in two ways: 1) the likelihood ratio test of the complete model including both zero and positive values of precipitation and 2) an approximate confidence interval for the ratio of mean values over all seeded and non-seeded experimental units, derived by linearizing the maximum likelihood estimator of the ratio. The results are obtained for both log-normal and gamma distributions.

In a single-area randomized seeding experiment an experimental area is instrumented for observation of some response variable(s), such as hail and/or rain mass. The variable is measured for each of a sample of experimental units, such as a day that is objectively determined (e.g., by radar) to have a high chance of producing precipitation. Seeding operations are carried out on a fraction of the days, the days being determined at random with prescribed probability (often 1/2). The other days form a non-seeded or control sample. Some days of both types may result in zero precipitation.

It is convenient to separate the data analysis into two parts: 1) a comparison of the proportions of seeded and control days with no precipitation, and 2) estimation of the ratio of mean values on seeded and control days with positive precipitation. However, a full assessment of the seeding effect must combine both parts. Neyman and Scott (1967) achieved this by linearly combining asymptotically optimal $C(\alpha)$ tests for the two parts.

In this paper the problem is solved by straightforwardly applying likelihood testing and estimation to the mathematical statistical model. The likelihood ratio test does not provide a convenient confidence interval, but an asymptotically optimal confidence interval for the ratio of mean values is obtained by linearizing the maximum likelihood estimator and deriving its asymptotic mean and variance.

The first part of the data analysis, i.e., that given by 1) in the preceding paragraph, is easily accomplished by a chi-squared test, or by Fisher's exact test, of a 2×2 contingency table. The second part of the analysis can be performed by well-known methods based on certain assumptions, the form of the distribution in particular. The results for log-normal and gamma distributions are stated in Section 2 in order to establish notation used in the combined analysis. The likelihood ratio test of the complete model including all experimental units is derived in Section 3. The approximate confidence interval for the ratio of mean values over all seeded and control experimental units is derived in Section 4. All of the methods are illustrated in Section 5 by numerical examples from the 1972–74 National Hail Research Experiment randomized seeding experiment (Crow *et al.*, 1976, 1978).

## 2. Confidence limits for ratio of means with positive precipitation

It is assumed that independent random samples from two populations with zero and positive values are observed, and it is desired to estimate the ratio of

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population mean values. In this section we restrict attention to the positive observations. In the specific weather modification application, a geographical experimental area is instrumented, by rain-gages, hailpads, hail/rain separators or radar, for example, and a sample of experimental units, such as individual storms or prospective hail days, is "declared" by the use of some predictors such as radar reflectivity. If the prediction process is good, almost all of the experimental units will produce some positive precipitation; if the prediction process is poor, many units will produce no precipitation. With specified probability (often 1/2) an experimental unit is subjected to a treatment (seeded), while the remaining units are left as controls. Thus, the sample sizes are random variables, but in this section the numbers of positive observations are considered fixed. All units are observed in the same way.

It is assumed that the positive values of precipitation mass (or any response variable in general) have a probability density function (pdf), designated $f(x)$ for the control units and $g(y)$ for the seeded units. For simplicity the corresponding random variables are designated $x$ and $y$ also, rather than the $X$ and $Y$ often used. It is assumed that the shape of the pdf is not altered by seeding, the only effect, if any, being a change in mean value (or scale) by a factor $\rho$. This means that

$$g(y)=\rho^{-1}f\left(\frac{y}{\rho}\right),\quad y>0,\ \rho>0. \tag{1}$$

If the mean values are designated by $E(x)$ and $E(y)$, it follows that

$$E(y)=\rho E(x). \tag{2}$$

The general approach of Sections 3 and 4 applies irrespective of the form of $f$ but the specific formulas there as well as here are restricted to the log-normal (LN) and gamma ($\Gamma$) distributions. The (two-parameter) log-normal pdf is given by

$$f_{\mathrm{LN}}(x)=(\sqrt{2\pi}\sigma x)^{-1}\exp[-(\ln x-\mu)/(2\sigma^2)],\quad x>0,\ \sigma>0, \tag{3}$$

where $\mu$ is the scale parameter and $\sigma$ the shape parameter. The gamma pdf is given by

$$f_{\Gamma}(x)=\frac{1}{\beta\Gamma(\gamma)}\left(\frac{x}{\beta}\right)^{\gamma-1}e^{-x/\beta},\quad x>0,\beta>0,\gamma>0, \tag{4}$$

where $\beta$ is the scale parameter and $\gamma$ the shape parameter.

All of these assumptions can be tested with the sample data by well-known methods, and this has been done for the National Hail Research Experiment (NHRE) 1972–74 data used as examples in Section 5; such testing, however, is not discussed in this paper.

It follows from (1) and (3) that

$$g_{\mathrm{LN}}(y)=(\sqrt{2\pi}\sigma x)^{-1}\exp[-(\ln y-\ln\rho-\mu)/(2\sigma^2)]. \tag{5}$$

The mean of $\ln y$ is thus $\mu+\ln\rho$. Since the mean of $x$ is

$$E(x)=\exp(\mu+\tfrac{1}{2}\sigma^2), \tag{6}$$

and $E(y)$ is similar, the validity of (2) may be confirmed.

Let the samples of control and seeded observations be $x_1,x_2,\ldots,x_m$ and $y_1,y_2,\ldots,y_n$, and their natural logarithms $u_i=\ln x_i$ and $v_j=\ln y_j$. Under the LN assumption it follows from normal theory that an optimal $1-2\alpha$ central confidence interval for $\rho$ is

$$\exp[\bar{v}-\bar{u}\pm t_{m+n-2,\alpha}s(m^{-1}+n^{-1})]^{\frac{1}{2}}, \tag{7}$$

where $\bar{u}$ and $\bar{v}$ are the sample means of $u$ and $v$, $s$ is the pooled estimate of the common standard deviation $\sigma$ of $u$ and $v$ with $m+n-2$ degrees of freedom (df), and $t_{m+n-2,\alpha}$ is the upper $\alpha$ probability point of the Student $t$ distribution with $m+n-2$ df. The null hypothesis that $\rho=1$ is rejected if the confidence interval does not include 1. The confidence interval has the advantage over the significance test format of giving *all* the values of $\rho$ with which the data are consistent.

For the gamma distribution it is further assumed for an exact result in this section (but not in Sections 3 and 4) that the shape parameter $\gamma$ is known. Eq. (2) applies again, and an optimal $1-2\alpha$ central confidence interval for $\rho$ is

$$\frac{\bar{y}}{\bar{x}}\frac{1}{F_{2n\gamma,2m\gamma,\alpha}}<\rho<\frac{\bar{y}}{\bar{x}}F_{2m\gamma,2n\gamma,\alpha}, \tag{8}$$

where $\bar{x}$ and $\bar{y}$ are the sample means of $x$ and $y$.

In practice $\gamma$ is unknown but can be estimated by maximum likelihood (ML) from the $x_i$ and $y_j$ (Greenwood and Durand, 1960; Mielke, 1976). By (1) the value of $\gamma$ is the same for both $x$ and $y$, so the ML estimate $\hat{\gamma}$ of Neyman and Scott [1967, Eq. (3.4)] from all of the data (allowing for the scale difference $\rho$) is appropriate. The confidence interval (8) is approximate if $\hat{\gamma}$ is substituted for $\gamma$. Although an exact confidence interval for $\rho$ is not available when $\gamma$ is unknown, an intuitive approximation is obtained by multiplying each of the limits in (8) by the ratio of the corresponding limit in (7) to the limit obtained by replacing $t_{m+n-2,\alpha}$ by $t_{\infty,\alpha}$, i.e., by the normal probability point.

The minimum variance unbiased estimate and a median-unbiased estimate of $\rho$ also can be calculated for each of the log-normal and gamma distributions (approximate for the gamma with unknown $\gamma$) (Crow, 1977). For the gamma distribution these estimates are factors (near 1) times $\bar{y}/\bar{x}$. Flueck, Holland, Mielke and Lee have considered properties of ratio estimates such as $\bar{y}/\bar{x}$, especially for bivariate gamma distributions of $x$ and $y$ [see Flueck and Holland (1976) and their references], whereas the present paper is limited to independent $x$ and $y$.

## 3. Likelihood ratio tests of means for all experimental units

In addition to the observations of positive precipitation analyzed in Section 2, we now take into account the numbers of experimental units experiencing no precipitation, denoted by $n'$ for the seeded units and $m'$ for the control units. The total sample size is fixed and denoted by $N$, so that $N=m+m'+n+n'$, but $m$, $m'$, $n$ and $n'$ are all random variables.

The probability that an experimental unit is seeded will be denoted by $p$. In the NHRE application, $p=1/2$. Let $\pi_c$ be the probability that an experimental unit designated as a control yields precipitation and $\pi_s$ the corresponding probability if the unit is seeded. Then $1-\pi_c$ and $1-\pi_s$ are the corresponding probabilities of no precipitation. The probability of the compound event that a unit is designated a control and that it yields precipitation is $(1-p)\pi_c$. Similar compound events have probabilities $(1-p)(1-\pi_c)$, $p\pi_s$ and $p(1-\pi_s)$.

With notation from Section 2, the probability that an experimental unit is designated a control, that there is precipitation, and that the precipitation mass is between $x$ and $x+dx$ is $(1-p)\pi_c f(x)dx$. The corresponding probability that a unit is seeded, that there is precipitation, and that its mass is between $y$ and $y+dy$ is $p\pi_s g(y)dy$. The experiment results in the multivariate random sample

$$(m,m',n,n';\,x_1,x_2,\ldots,x_m;\,y_1,y_2,\ldots,y_n),$$

the pdf or likelihood of which is consequently

$$\frac{N!}{m!m'!n!n'!}[(1-p)\pi_c]^m[(1-p)(1-\pi_c)]^{m'}(p\pi_s)^n \times[p(1-\pi_s)]^{n'}\prod_{i=1}^{m}f(x_i)\prod_{j=1}^{n}g(y_j). \quad (9)$$

### *a. The log-normal distribution*

In the case of LN distribution $f(x_i)$ and $g(y_j)$ are given by (3) and (5). The ML estimates of $\pi_c$, $\pi_s$, $\mu$, $\rho$ and $\sigma$ are the values that maximize (9) or, equivalently, its logarithm. Setting the partial derivatives equal to zero and solving yields the well-known ML estimates

$$\left.\begin{aligned} &\hat{\pi}_c=m/(m+m'),\quad \hat{\pi}_s=n/(n+n'),\\ &\hat{\mu}=\bar{u},\quad \hat{\rho}=e^{\bar{v}-\bar{u}},\\ &\hat{\sigma}^2=\frac{1}{m+n}\Big[\sum_i(u_i-\bar{u})^2+\sum_j(v_j-\bar{v})^2\Big]=\frac{m+n-2}{m+n}s^2 \end{aligned}\right\}. \quad (10)$$

It is desired to test the composite null hypothesis

$$H_0:\ \rho=1 \quad\text{and}\quad \pi_s=\pi_c \quad (11)$$

against the composite alternative hypothesis

$$H_1:\ \rho\neq 1 \quad\text{or}\quad \pi_s\neq\pi_c \text{ (or both).}$$

Thus, a test of $H_0$ will detect (with high probability given enough experimental units) any departure of the two means of positive precipitation from each other or of the two probabilities of precipitation from each other. If $H_0$ is true, $H_0$ will be rejected at a significance level that applies to the combined data, not merely the positive data alone or the zero data alone. The hypotheses can be expressed in terms of the ratio of mean precipitation mass per seeded experimental unit to the mean per control unit, which we denote by $\rho^*$. Since the expected number of control experimental units is $N_p$, and the expected total mass on control units is $N_p\pi_c E(x)$, the expected mass per expected control unit is $\pi_c E(x)$. Similarly, the expected mass per expected seeded unit is $\pi_s\rho E(x)$, so it follows that

$$\rho^*=\rho\frac{\pi_s}{\pi_c}. \quad (12)$$

Hence $H_0$ could equivalently be expressed as

$$H_0:\ \rho^*=1 \quad\text{and}\quad \pi_s=\pi_c \quad (13)$$

and similarly for $H_1$.

An asymptotically most powerful test of $H_0$ is provided by the likelihood ratio, the ratio of expression (9) maximized subject to $H_0$ to expression (9) maximized subject to $H_1$ (Wilks, 1962, pp. 402–422). In the case of the LN distribution the denominator of the ratio is thus Eq. (9) with the ML estimates (10) substituted. The estimates that maximize (9) with (11) substituted are easily found to be

$$\left.\begin{aligned} &\hat{\pi}_{c0}=(m+n)/N\\ &\hat{\mu}_0=(m+n)^{-1}(m\bar{u}+n\bar{v})\\ &\hat{\sigma}_0^2=(m+n)^{-1}\Big[\sum_i(u_i-\hat{\mu}_0)^2+\sum_j(v_j-\hat{\mu}_0)^2\Big] \end{aligned}\right\}. \quad (14)$$

Substituting (14) in (9) to get the numerator and (10) in (9) to get the denominator and simplifying yield the LN likelihood ratio

$$\lambda=\frac{(m+n)^{m+n}(m'+n')^{m'+n'}(m+m')^{m+m'}(n+n')^{n+n'}}{N^N m^m m'^{m'} n^n n'^{n'}} \times\left\{\frac{\sum(u_i-\bar{u})^2+\sum(v_j-\bar{v})^2}{\sum(u_i-\hat{\mu}_0)^2+\sum(v_j-\hat{\mu}_0)^2}\right\}^{(m+n)/2}. \quad (15)$$

This can be evaluated from the sample data. If $H_0$ is true, then $-2\ln\lambda$ is distributed asymptotically as chi-squared with df equal to the difference in number of independent parameters of the numerator and denominator, which is $5-3=2$ (Wilks, 1962, p. 419). If $H_0$ is not true, $-2\ln\lambda$ tends to be larger. Hence $H_0$ can be tested by comparing the sample value of $-2\ln\lambda$ with the upper 5% point (for example) of the chi-squared distribution with 2 df.

*b. The gamma distribution*

The likelihood ratio test under the assumption of gamma distributions can be similarly derived except that the estimates of $\gamma$ under $H_0$ and $H_1$, say $\hat{\gamma}_0$ and $\hat{\gamma}_1$, cannot be expressed explicitly. The unrestricted ML estimates (applicable to $H_1$) of $\hat{\pi}_c$ and $\hat{\pi}_s$ are the same as in (10). The unrestricted ML estimates of $\beta$, $\gamma$ and $\rho$ may be obtained from

$$\left.\begin{aligned} &\hat{\rho}=\bar{y}/\bar{x}, \quad \hat{\beta}_1=\bar{x}/\hat{\gamma}_1, \quad \hat{\gamma}_1=\eta_1\hat{\gamma}_1/\eta_1 \\ &\eta_1=(m+n)^{-1}[m(\ln\bar{x}-\bar{u})+n(\ln\bar{y}-\bar{v})] \end{aligned}\right\}. \qquad (16)$$

[Neyman and Scott, 1967, Eq. (3.4)], where $\eta_1\hat{\gamma}_1$ is found for given $\eta_1$ in Tables 1A or 1B of Greenwood and Durand (1960). The estimates maximizing (9) subject to (11) are $\hat{\pi}_{c0}$ as in (14) and

$$\left.\begin{aligned} &\hat{\beta}_0=\bar{z}/\hat{\gamma}_0, \quad \hat{\gamma}_0=\eta_0\hat{\gamma}_0/\eta_0 \\ &\bar{z}=(m+n)^{-1}(m\bar{x}+n\bar{y}), \quad \eta_0=\ln\bar{z}-\hat{\mu}_0 \end{aligned}\right\}, \qquad (17)$$

where $\eta_0\hat{\gamma}_0$ is found in the Greenwood and Durand (1960) tables. The likelihood ratio for gamma distributions is then

$$\lambda=\frac{(m+n)^{m+n}(m'+n')^{m'+n'}(m+m')^{m+m'}(n+n')^{n+n'}}{N^N m^m m'^{m'} n^n n'^{n'}} \times\frac{(\bar{x}^m\bar{y}^n)^{\hat{\gamma}_1}}{\bar{z}^{(m+n)\hat{\gamma}_0}}\left[\frac{\hat{\gamma}_0^{\hat{\gamma}_0}\Gamma(\hat{\gamma}_1)}{\hat{\gamma}_1^{\hat{\gamma}_1}\Gamma(\hat{\gamma}_0)}\right]^{m+n} \times\Big[e^{-(m+n)}\prod_i x_i\prod_j y_j\Big]^{\hat{\gamma}_0-\hat{\gamma}_1}. \qquad (18)$$

As for (15), $-2\ln\lambda$ is distributed asymptotically as chi-squared with 2 df if $H_0$ is true.

## 4. Confidence limits for ratio of means for all experimental units

It is of interest to see how a confidence interval for $\rho^*$, the ratio of means over all experimental units, can be derived from the likelihood ratio, although a more practical method will then be presented. Let the likelihood ratio (9) with lognormal distributions substituted be denoted by $L(\mu,\sigma,\rho,\pi_c,\pi_s)$. Then any hypothesis specifying $\rho$, $\pi_c$ and $\pi_s$ completely [rather than partly as in (11)] could be tested by the likelihood ratio

$$\lambda=\frac{L(\tilde{\mu},\tilde{\sigma},\rho,\pi_c,\pi_s)}{L(\hat{\mu},\hat{\sigma},\hat{\rho},\hat{\pi}_c,\hat{\pi}_s)}, \qquad (19)$$

where $\tilde{\mu}$ and $\tilde{\sigma}$ are the values of $\mu$ and $\sigma$ that maximize $L$ holding $\rho$, $\pi_c$ and $\pi_s$ fixed. This would result in a three-dimensional confidence region for $\rho$, $\pi_c$ and $\pi_s$. However, we are not interested in the absolute values of $\pi_c$ and $\pi_s$ so much as in their relative values, just as we are interested in the ratio $\rho$. Hence we let $\tau=\pi_s/\pi_c$

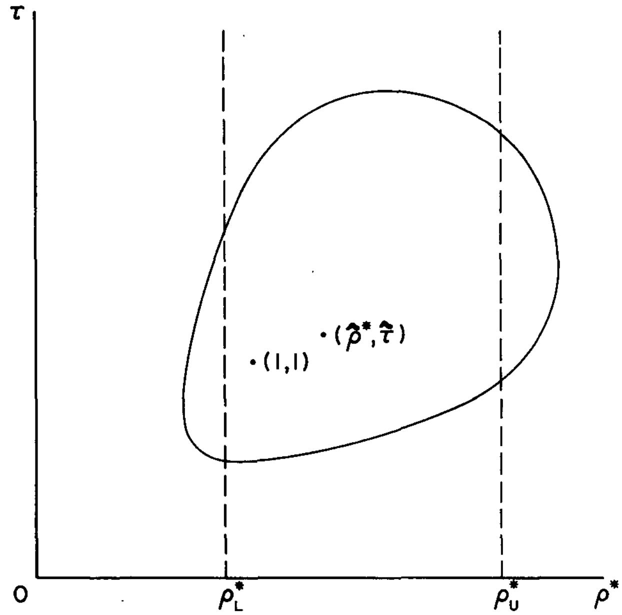


FIG. 1. Sketch of confidence region for $(\rho^*,\tau)$ with confidence level $1-2\alpha$ and of confidence interval $(\rho_L^*,\rho_U^*)$ for $\rho^*$ with confidence level $1-2\alpha$.

and $\rho^*=\rho\tau$, using (12), and consider the likelihood ratio

$$\lambda=\frac{L(\tilde{\mu},\tilde{\sigma},\rho^*,\tilde{\pi}_c,\tau)}{L(\hat{\mu},\hat{\sigma},\hat{\rho}^*,\hat{\pi}_c,\hat{\tau})}, \qquad (20)$$

where

$$\hat{\tau}=\hat{\pi}_s/\hat{\pi}_c, \quad \hat{\rho}^*=\hat{\rho}\hat{\tau}, \qquad (21)$$

the other unrestricted ML estimates are just as in (10), and the restricted ML estimates are easily found to satisfy the equations

$$\left.\begin{aligned} &\tilde{\mu}=(m+n)^{-1}\Big[\sum_i u_i+\sum_j v_j-n\ln(\rho^*/\tau)\Big] \\ &\tilde{\sigma}^2=(m+n)^{-1}\Big\{\sum_i (u_i-\tilde{\mu})^2 +\sum_j [v_j-\tilde{\mu}-\ln(\rho^*/\tau)]^2\Big\} \\ &N\tau\tilde{\pi}_c^2-[m+n+m'+\tau(m+n+n')]\tilde{\pi}_c +m+n=0 \end{aligned}\right\}. \qquad (22)$$

Thus, $\tilde{\mu}$, $\tilde{\sigma}$ and $\tilde{\pi}_c$ can be calculated for any specified values of $\rho^*$ and $\tau$, and (20) can be used to test the hypothesis that $\rho^*$ and $\tau$ have those values. In other words, (20) can be used to find *all* points $(\rho^*,\tau)$ that are consistent with the data at some significance level $2\alpha$, that is, the two-dimensional confidence region for $(\rho^*,\tau)$ with confidence level $1-2\alpha$ (Fig. 1). Any region containing this confidence region will be a confidence region for $(\rho^*,\tau)$ with level at least $1-2\alpha$. Hence, the tangents of this region perpendicular to the $\rho^*$ axis provide a conservative confidence *interval* for $\rho^*$ by itself, conservative in having confidence level at least $1-2\alpha$, the excess being unknown. If the point (1,1) lies in the 1–2$\alpha$ confidence region, then the null hypothesis that $(\rho^*,\tau)=(1,1)$ fails to be rejected.

Substitution of (10) and (22) in (20) yields, after simplification,

$$-2\ln\lambda(\rho^*,\tau)=(m+n)\ln[\tilde{\sigma}^2(\rho^*/\tau)/\hat{\sigma}^2] -2\ln\left\{\left[\frac{\tilde{\pi}_c(\tau)}{\hat{\pi}_c}\right]^m\left[\frac{1-\tilde{\pi}_c(\tau)}{1-\hat{\pi}_c}\right]^{m'} \times\left[\frac{\tau\tilde{\pi}_c(\tau)}{\hat{\pi}_s}\right]^n\left[\frac{1-\tau\tilde{\pi}_c(\tau)}{1-\hat{\pi}_s}\right]^{n'}\right\}. \quad (23)$$

The confidence region for $(\rho^*,\tau)$ then consists of all points satisfying

$$-2\ln\lambda(\rho^*,\tau)\leqslant\chi^2_{2,2\alpha}, \quad (24)$$

where $\chi^2_{2,2\alpha}$ is the upper $2\alpha$ probability point of the chi-squared distribution with 2 df.

Since (24) is tedious to evaluate and provides only a conservative confidence interval for $\rho^*$ by itself, it is worthwhile obtaining the approximate confidence interval available from the asymptotic normality and efficiency of $\ln\hat{\rho}^*$. The asymptotic normality and efficiency depends on conditions of differentiability, uniqueness and measurability (Wilks, 1962, pp. 380–381), which have been confirmed. The asymptotic expectation and variance of $\ln\hat{\rho}^*$ are needed and can be found by obtaining the asymptotic variance-covariance matrix of the maximum likelihood estimators. The details are given in the Appendix. The results for the LN distributions are

$$\left.\begin{aligned} &E(\ln\hat{\rho}^*)\sim\ln\rho^* \\ &\mathrm{Var}(\ln\hat{\rho}^*) \\ &\sim\frac{1}{N}\left[\left(\frac{1}{(1-p)\pi_c}+\frac{1}{p\pi_s}\right)(\sigma^2+1)-\frac{1}{p(1-p)}\right]\end{aligned}\right\}. \quad (25)$$

The parameter $p$ is chosen by the experimenter, and calculus confirms that $p=1/2$ is optimal [in that it minimizes $\mathrm{Var}(\ln\hat{\rho}^*)$] *when* $\pi_c=\pi_s$. The optimum value of $p$ is also very near 1/2 for $\pi_s$ substantially different from $\pi_c$ and the $\sigma^2$ experienced in NHRE.

The nuisance parameters $\pi_c$, $\pi_s$ and $\sigma$ can be approximated by their ML estimates $\hat{\pi}_c$, $\hat{\pi}_s$ and $\hat{\sigma}$. However it appears preferable to use $s$ from (10) instead of $\hat{\sigma}$ because the confidence interval (28) below will be seen to be exact when $m'=n'=0$ and the Student $t$ coefficient is used. Substituting in (25) from (10) and simplifying yields

$$s^2(\ln\hat{\rho}^*)=\frac{s^2}{N}\left[\frac{m+m'}{m(1-p)}+\frac{n+n'}{np}\right]+\frac{1}{N}\left[\frac{m'}{m(1-p)}+\frac{n'}{np}\right]$$
$$=\frac{2}{N}\left[2s^2+(s^2+1)\left(\frac{m'}{m}+\frac{n'}{n}\right)\right] \text{ when } p=\tfrac{1}{2}. \quad (26)$$

Since $(m+m')/N$ converges in probability to $1-p$ and $(n+n')/N$ to $p$, a simpler but possibly slightly less accurate approximation for $\mathrm{Var}(\ln\rho^*)$ for general $p$ is

$$s_1^2(\ln\hat{\rho}^*)=s^2\left(\frac{1}{m}+\frac{1}{n}\right)+\frac{1}{N}\left[\frac{m'}{m(1-p)}+\frac{n'}{np}\right]. \quad (27)$$

Consequently, at least for large $N$, a $1-2\alpha$ central confidence interval for $\rho^*$ is given by

$$\exp[\ln\hat{\rho}^*\pm t_\alpha s(\ln\hat{\rho}^*)], \quad (28)$$

where $t_\alpha$ is the upper $\alpha$ probability point of the standard normal distribution, or, intuitively more accurately, the upper $\alpha$ probability point of the Student $t$ distribution with $m+n-2$ df.

More generally, for large $N$, Eq. (28) applies to any distribution $f(x)$ of precipitation masses if the appropriate $s$ $(\ln\rho^*)$ is substituted. For the gamma distribution it is shown in the Appendix that $s^2(\ln\hat{\rho}^*)$ is the same as (26) with $s^2$ replaced by $1/\hat{\gamma}$.

A by-product of the derivation of the asymptotic confidence intervals for $\rho^*$ is an asymptotic confidence interval for $\tau$, i.e.,

$$\exp[\ln\hat{\tau}\pm t_\alpha s(\ln\hat{\tau})], \quad (29)$$

where

$$s^2(\ln\hat{\tau})=\frac{1}{N}\left[\frac{m'}{m(1-p)}+\frac{n'}{np}\right]. \quad (30)$$

The likelihood confidence interval for $\rho^*$ is asymptotically as short as possible, and the likelihood confidence region for $(\rho^*,\tau)$ is as small as possible (Wilks, 1962, pp. 375, 388). They provide asymptotically uniformly most powerful unbiased tests of the null hypotheses $\rho^*=1$ and $(\rho^*,\tau)=(1,1)$ respectively. They are not even asymptotically equivalent because one is one-dimensional and the other two-dimensional. However, as sketched in Fig. 1, the confidence interval for $\rho^*$ can be regarded as equivalent to a rectangular confidence region for $(\rho^*,\tau)$ with boundaries 0 and $\infty$ on $\tau$. To have the same confidence level as the nonrectangular likelihood ratio region for $(\rho^*,\tau)$ it necessarily extends no farther in the $\rho^*$ direction than the likelihood ratio region does.

## 5. Application to National Hail Research Experiment randomized seeding experiment

The confidence limits for ratios of means presented in Sections 2 and 4 have been used to analyze the data of the National Hail Research Experiment (NHRE) 1972–74 randomized seeding experiment (Crow *et al.*, 1976, 1978). Some of the results are presented in Tables 1 and 2 to illustrate the theory. A comparison of the numerical results of the likelihood theory of Sections 3 and 4 for $\rho^*$ is given later.

Table 1 refers only to the occurrence or nonoccurrence of rain or hail on each declared hail day as observed by a network of about 240 co-located hail/rain separators and hailpads distributed roughly uniformly over an experimental area of about 1600 km². The proportions of positive declared hail days are estimated as in (10), the seed/control ratio estimate $\hat{\tau}$ is calculated from (21), and the 90% confidence limits for $\tau$ are calculated from (29) and (30) with $p=1/2$. Since in each case the limits enclose the value $\tau=1$, the data are

TABLE 1. Frequencies of NHRE zero and positive declared hail days.*

| Response variable | Numbers of days c $m$ | $m'$ | s $n$ | $n'$ | Proportions of positive days $\hat{\pi}_c$ | $\hat{\pi}_s$ | Ratio $\tau=\hat{\pi}_s/\hat{\pi}_c$ | 90% limits on $\tau$ | |
|---|---|---|---|---|---|---|---|---|---|
| Separator rain | 27 | 3 | 24 | 3 | 0.90 | 0.89 | 0.99 | 0.85 | 1.16 |
| Separator hail | 16 | 14 | 17 | 10 | 0.53 | 0.63 | 1.18 | 0.80 | 1.73 |
| Hailpad | 14 | 16 | 14 | 13 | 0.47 | 0.52 | 1.11 | 0.70 | 1.76 |

* c = control day, s = seeded day.

consistent with the hypothesis of no seeding effect on the frequency of hail days or of rain days.

Table 2 refers to the amounts of positive response variables, the seed/control ratio $\rho$ of their mean values, and the seed/control ratio $\rho^*=\rho\tau$ of the mean values including zero amounts. Only four of the nine response variables analyzed by NHRE are presented here for illustration. The total amounts of rain or hail over the entire experimental area (the $x_i$ and $y_j$ of Section 2) were estimated in two ways: (i) simply by adding the amounts recorded by all instruments and multiplying by the ratio of experimental area to the sum of the instrument catchment areas and (ii) by a weighted interpolation to a rectangular grid of points and summing the interpolates, in order to account for the irregular spacing of the instruments. Only the "integrated" values (ii) are considered here. From the sample mean values and shape parameter estimates given in Table 2 the interested reader can confirm the maximum likelihood estimates $\hat{\rho}$ and $\hat{\rho}^*$, the 90% confidence limits for $\rho$ from (7) and (8) [as modified in the paragraph following (8)], and the 90% confidence limits for $\rho^*$ from (27)–(28) with $p=1/2$.

Since in each case the limits in Table 2 enclose the value 1, the data are consistent with the hypothesis of no seeding effect, whether considered over just positive hail or rain days or over all declared hail days. The widely separated confidence limits show that the data are also consistent with many possible seeding effects, both positive and negative. The estimates and limits for $\rho^*$ are little different from those for $\rho$ in the case of separator rain mass because there are so few declared hail days without some rain. For the hail response variables the point estimates $\hat{\rho}^*$ are not much different from the corresponding estimates $\hat{\rho}$ because the proportions $\hat{\pi}_c$ and $\hat{\pi}_s$ are not much different from each other. It is true that the nominal augmentation of hail by seeding shown by $\hat{\rho}$ for all three variables under the log-normal assumption is accentuated in the form of $\hat{\rho}^*$, but also the confidence intervals are in most cases lengthened by the added randomness in $\hat{\tau}$.

Since the confidence region (24) for $(\rho^*,\tau)$ [or an equivalent one for $(\rho,\tau)$] is tedious to calculate, the likelihood ratio test (Section 3) of the hypothesis $(\rho^*,\tau)=(1,1)$ will be compared with the likelihood estimation test (Section 4) of the hypothesis $\rho^*=1$ by comparing sample significance levels for the four response variables of Table 2. These levels are given in Table 3, along with the corresponding significance levels for the test of $\tau=1$ available by setting (29) equal to 1 and solving for $2\alpha$. (Two-tailed tests are appropriate because seeding effects in either direction are of interest

TABLE 2. NHRE seeding effect ratios for positive and all declared hail days.†

| Response variable | Distribution | Shape parameter | Means†† c | s | $\hat{\rho}$ | 90% limits for $\rho$ | $\hat{\rho}^*=\hat{\rho}\hat{\tau}$ | 90% limits for $\rho^*$ |
|---|---|---|---|---|---|---|---|---|
| Separator | LN | 2.320 | −0.3659 | 0.5456 | 2.49 | 7.41 | 2.46 | 7.40 |
| | | | | | | 0.84 | | 0.82 |
| rain mass | Γ | 0.4664 | 0.3478 | 0.5246 | 1.51 | 3.09 | 1.49 | 3.02 |
| | | | | | | 0.75 | | 0.74 |
| Separator | LN | 2.096 | 0.8597 | 1.2068 | 1.41 | 4.88 | 1.67 | 6.09 |
| | | | | | | 0.41 | | 0.46 |
| hail mass | Γ | 0.4211 | 1.3368 | 1.3249 | 0.99 | 2.54 | 1.17 | 3.14 |
| | | | | | | 0.38 | | 0.44 |
| Hailpad | LN | 1.888 | 0.1674 | 0.9718 | 2.24 | 7.55 | 2.48 | 9.13 |
| | | | | | | 0.66 | | 0.68 |
| mass | Γ | 0.4680 | 0.6712 | 0.7227 | 1.08 | 2.88 | 1.20 | 3.41 |
| | | | | | | 0.40 | | 0.42 |
| Hailpad | LN | 2.085 | 0.0684 | 0.9651 | 2.45 | 9.41 | 2.72 | 11.28 |
| | | | | | | 0.64 | | 0.66 |
| kinetic energy | Γ | 0.4051 | 0.8609 | 0.8356 | 0.97 | 2.82 | 1.08 | 3.28 |
| | | | | | | 0.33 | | 0.35 |

† c = control day, s = seeded day, LN = log-normal, Γ = gamma distribution.
†† The means for LN are $\bar{u}$ and $\bar{v}$ [Eq. (7)]; for Γ, $\bar{x}$ and $\bar{y}$ [Eq. (8)]. $\hat{\rho}=e^{\bar{v}-\bar{u}}$ for LN; $\hat{\rho}=\bar{y}/\bar{x}$ for Γ. The units of the means are omitted, being immaterial in the ratios. The shape parameter estimate for LN is $s$ [Eq. (10)]; for Γ, $\hat{\gamma}_1$ [Eq. (16)].

TABLE 3. Significance levels of NHRE sample seeding effect ratios

| Response variable | Distribution | Hypothesis tested $\tau=1$ | $\rho^*=1$ | $(\rho^*,\tau)=(1,1)$ |
|---|---|---|---|---|
| Separator | LN | 0.89 | 0.21 | 0.36 |
| rain mass | Γ | 0.89 | 0.34 | 0.32 |
| Separator | LN | 0.46 | 0.50 | 0.68 |
| hail mass | Γ | 0.46 | 0.79 | 0.76 |
| Hailpad mass | LN | 0.70 | 0.23 | 0.48 |
| | Γ | 0.70 | 0.77 | 0.91 |
| Hailpad | LN | 0.70 | 0.26 | 0.47 |
| kinetic energy | Γ | 0.70 | 0.91 | 0.92 |

and because the confidence region corresponds to a two-tailed test, as suggested in Fig. 1.) The levels for $\tau=1$ and $\rho^*=1$ were based on the normal distribution rather than the Student $t$ distribution because the normal approximation is comparable with the chi-squared approximation for the test of $(\rho^*,\tau)=(1,1)$. However, the effect on the level is only 0.01 or less.

It is seen that the significance levels in Table 3 for the test of $(\rho^*,\tau)=(1,1)$ [or $(\rho,\tau)=(1,1)$] are different from the corresponding levels for the test of $\rho^*=1$. They should be different because the hypotheses are different. They do arise from different asymptotic approximations, but they would not in general be the same even for very large samples.

It is of interest to compare the significance levels of Table 3 with those given by the asymptotically optimal $C(\alpha)$ tests of Neyman and Scott (1967). They give explicit formulas for the case of $x$ and $y$ gamma distributed. The $C(\alpha)$ criterion for the hypothesis $\tau=1$ is given by their Eq. (2.1), that for $\rho=1$ by (3.2), and that for $\rho^*=1$ by (4.5). Numerical comparisons are presented in Table 4. Evidently the two types of tests give closely similar results in these examples. The maximum likelihood and $C(\alpha)$ tests are both asymptotically unbiased and asymptotically optimal (and hence equal in power) but would be expected to differ in power for small sample sizes. Their bias and power for finite sample sizes can be investigated by simulation studies. Marcella Wells (private communication) did this some years ago for the $C(\alpha)$ test of $\rho^*=1$ and found substantial agreement with the asymptotic power.

TABLE 4. Comparison of maximum likelihood test significance levels of NHRE sample seeding effect ratios (from Table 3) in the gamma distribution case with those given by asymptotically optimal $C(\alpha)$ tests.

| Response Variable | Type of Test | Hypothesis tested $\tau=1$ | $\rho=1$ | $\rho^*=1$ |
|---|---|---|---|---|
| Separator | ML | 0.89 | 0.35 | 0.34 |
| rain mass | $C(\alpha)$ | 0.91 | 0.35 | 0.37 |
| Separator | ML | 0.46 | 0.98 | 0.79 |
| hail mass | $C(\alpha)$ | 0.46 | 0.99 | 0.82 |
| Hailpad | ML | 0.70 | 0.92 | 0.77 |
| hail mass | $C(\alpha)$ | 0.70 | 0.93 | 0.88 |

## APPENDIX

### Derivation of Asymptotic Variance of $\ln \hat{\rho}^*$

The asymptotic variance of $\ln\hat{\rho}^*$ follows from the asymptotic variance-covariance matrix of the maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ of the parameter vector $\boldsymbol{\theta}$ [$=(\pi_c,\pi_s,\mu,\sigma^2,\rho)$ in the lognormal case]. We use $\ln\hat{\rho}^*$ rather than $\hat{\rho}^*$ since

$$\ln\hat{\rho}^*=\ln\hat{\rho}+\ln\hat{\pi}_s-\ln\hat{\pi}_c, \tag{A1}$$

where $\ln\hat{\rho}$ is exactly normally distributed in the lognormal case and probably more nearly so than $\hat{\rho}$ in the gamma case, and $\ln\hat{\rho}$ is expected to influence the distribution of $\ln\hat{\rho}^*$ more than $\ln\hat{\pi}_s-\ln\hat{\pi}_c$ does. From (A1) we can obtain the asymptotic variance of $\ln\hat{\rho}^*$ immediately by the standard rule, without approximation, from the asymptotic variances and covariances of $\ln\hat{\rho}$, $\ln\hat{\pi}_c$ and $\ln\hat{\pi}_s$. Hence we alter our parameter vector slightly to

$$\begin{aligned}\boldsymbol{\theta}&=(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)\\&=(\ln\pi_c,\ln\pi_s,\mu,\sigma^2,\ln\rho)\end{aligned} \tag{A2}$$

in the lognormal case and seek the asymptotic variance-covariance matrix, say $\mathbf{V}$, of $\hat{\boldsymbol{\theta}}$ using Theorem 12.7.3 of Wilks (1962, p. 380). It is convenient to use the logarithm, say $L$, of the likelihood of the sample, given by (9). Then

$$\mathbf{V}=(B^{ij}/|\mathbf{B}|), \tag{A3}$$

the inverse of the symmetric matrix $(\mathbf{B}_{ij})$, where

$$B_{ij}=E\left(\frac{\partial L}{\partial\theta_i}\frac{\partial L}{\partial\theta_j}\right),\quad i,\ j=1,\ 2,\ \ldots,\ 5, \tag{A4}$$

and $B^{ij}$ is the cofactor of $B_{ij}$.

We proceed to outline the derivation for the LN case. From (9), (3) and (5),

$$\left.\begin{aligned}\frac{\partial L}{\partial\ln\pi_c}&=m-\frac{m'\pi_c}{1-\pi_c}\\ \frac{\partial L}{\partial\ln\pi_s}&=n-\frac{n'\pi_s}{1-\pi_s}\\ \frac{\partial L}{\partial\mu}&=(1/\sigma^2)[m(\bar{u}-\mu)+n(\bar{v}-\mu-\ln\rho)]\\ \frac{\partial L}{\partial\sigma^2}&=-[(m+n)/2\sigma^2]+(1/2\sigma^4)\\&\quad\times[\sum_i(u_i-\mu)^2+\sum_j(v_j-\mu-\ln\rho)^2]\\ \frac{\partial L}{\partial\ln\rho}&=(n/\sigma^2)(\bar{v}-\mu-\ln\rho)\end{aligned}\right\}. \tag{A5}$$

It follows from Section 3 that $m$, $m'$, $n$ and $n'$ are multinomially distributed with fixed total sample size $N$ and means $N(1-p)\pi_c$, $N(1-p)(1-\pi_c)$, $Np\pi_s$ and $N_p(1-\pi_s)$, respectively. Hence (Wilks, 1962, p. 139)

$$\left.\begin{aligned} E(m^2) &= N(1-p)\pi_c[1-(1-p)\pi_c] \\ &\qquad +[N(1-p)\pi_c]^2 \\ E(mm') &= N(N-1)(1-p)^2\pi_c(1-\pi_c) \end{aligned}\right\}, \quad \text{(A6)}$$

and similarly for expectations of other squares and products. Thus, after some algebra

$$B_{11}=\frac{N(1-p)\pi_c}{1-\pi_c}. \quad \text{(A7)}$$

Similarly

$$B_{22}=\frac{Np\pi_s}{1-\pi_s}, \quad B_{12}=0. \quad \text{(A8)}$$

In deriving the remaining matrix elements we take account of the fact that the $u_i$ and $v_j$ depend on $m$ and $n$ and first take the conditional expectation given $m$ and $n$, denoted by $E_{m,n}(\ )$. Thus

$$\begin{aligned} B_{33} &= (1/\sigma^4)E\{m^2E_{m,n}[(\bar{u}-\mu)^2] \\ &\quad +2mnE_{m,n}[(\bar{u}-\mu)(\bar{v}-\mu-\ln\rho)] \\ &\quad +n^2E_{m,n}[(\bar{v}-\mu-\ln\rho)^2]\} \\ &= (1/\sigma^4)E\{m^2\cdot\sigma^2/m+2mn\cdot 0+n^2\cdot\sigma^2/n\} \\ &= (1/\sigma^2)E(m+n)=(N/\sigma^2)[(1-p)\pi_c+p\pi_s]. \end{aligned} \quad \text{(A9)}$$

The last factor occurs often enough to warrant abbreviation. Let

$$A=(1-p)\pi_c+p\pi_s. \quad \text{(A10)}$$

The other $B_{ij}$ follow similarly. The matrix $(B_{ij})$ is

$$N\left[\begin{array}{cc|ccc} \dfrac{(1-p)\pi_c}{1-\pi_c} & 0 & & & \\ 0 & \dfrac{p\pi_s}{1-\pi_s} & & \mathbf{O} & \\ \hline & & \dfrac{A}{\sigma^2} & 0 & \dfrac{p\pi_s}{\sigma^2} \\ & \mathbf{O} & 0 & \dfrac{A}{2\sigma^4} & 0 \\ & & \dfrac{p\pi_s}{\sigma^2} & 0 & \dfrac{p\pi_s}{\sigma^2} \end{array}\right]. \quad \text{(A11)}$$

It follows from (A3), (A4) and (A11) that the asymptotic variance-covariance matrix $\mathbf{V}$ of the vector maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ of the $\boldsymbol{\theta}$ in (A2) (the lognormal case) is

$$N^{-1}\left[\begin{array}{cc|ccc} \dfrac{1-\pi_c}{(1-p)\pi_c} & 0 & & & \\ 0 & \dfrac{1-\pi_s}{p\pi_s} & & \mathbf{O} & \\ \hline & & \dfrac{\sigma^2}{(1-p)\pi_c} & 0 & -\dfrac{\sigma^2}{(1-p)\pi_c} \\ & \mathbf{O} & 0 & \dfrac{2\sigma^4}{A} & 0 \\ & & -\dfrac{\sigma^2}{(1-p)\pi_c} & 0 & \sigma^2\left[\dfrac{1}{(1-p)\pi_c}+\dfrac{1}{p\pi_s}\right] \end{array}\right]. \quad \text{(A12)}$$

Hence in (A1) the three terms are asymptotically uncorrelated, and the asymptotic variance of $\ln\hat{\rho}^*$ is found immediately to be that given in (25).

The asymptotic variance-covariance matrix of the vector maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ when $x$ and $y$ are gamma-distributed as in (4) and (1) and

$$\boldsymbol{\theta}=(\ln\pi_c,\ln\pi_s,\beta,\gamma,\ln\rho) \quad \text{(A13)}$$

follows similarly, with somewhat more algebra and a little calculus. Use is made of the digamma function

$$\psi(\gamma)=\frac{d\ln\Gamma(\gamma)}{d\gamma}, \tag{A14}$$

its derivative $\psi'(\gamma)$, and the easily derived results

$$\left.\begin{aligned} E[\ln x]&=\psi(\gamma)+\ln\beta \\ E[(\ln x)^2]&=\psi'(\gamma)+[\psi(\gamma)+\ln\beta]^2 \\ E[x\ln x]&=\beta\gamma[\psi(\gamma+1)+\ln\beta] \\ \psi(\gamma+1)&=\psi(\gamma)+1/\gamma \end{aligned}\right\}. \tag{A15}$$

The resulting asymptotic variance-covariance matrix $\mathbf{V}$ of $\hat{\boldsymbol{\theta}}$ is

$$N^{-1}\left[\begin{array}{cc|ccc} \dfrac{1-\pi_c}{(1-p)\pi_c} & 0 & & & \\ 0 & \dfrac{1-\pi_s}{p\pi_s} & & \mathbf{0} & \\ \hline & & \dfrac{\beta^2[A\gamma\psi'(\gamma)-p\pi_s]}{A\gamma\pi_c)1-p)[\gamma\psi'(\gamma)-1]} & -\dfrac{\beta}{A[\gamma\psi'(\gamma)-1]} & -\dfrac{\beta}{\gamma(1-p)\pi_c} \\ & \mathbf{0} & -\dfrac{\beta}{A[\gamma\psi'(\gamma)-1]} & \dfrac{\gamma}{A[\gamma\psi'(\gamma)-1]} & 0 \\ & & -\dfrac{\beta}{\gamma(1-p)\pi_c} & 0 & \dfrac{A}{\gamma p(1-p)\pi_c\pi_s} \end{array}\right]. \tag{A16}$$

Here the variance of $\ln\hat{p}$ is the same as in (A12) except that $1/\gamma$ replaces $\sigma^2$. Hence the variance of $\ln\hat{p}^*$ is the same as in (25) with that replacement.

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