Generalization of K Theory for Turbulent Diffusion. Part II: Spectral Diffusivity Model for Plume Dispersion

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ABSTRACT

Further development of the spectral turbulent diffusivity concept is presented with the aim of obtaining an Eulerian dispersion model applicable for multiple interacting sources. The theory is applied for studies of plume dispersion in a field of a homogeneous and stationary turbulence. A continuous plume is considered as consisting of an infinite number of expanding puffs. The puffs' center of mass fluctuates following the long-wave range of the turbulent velocity fluctuation spectrum. The center-of-mass fluctuations are assigned to phases of the Fourier coefficients of the concentration distribution. The standard deviation of the velocity of the phase fluctuations is dependent on the wave vector of the Fourier coefficient. Time-averaging results in a spectral phase diffusivity coefficient.

It is shown that the rate of growth and the center-line concentration obtained by the spectral diffusivity model are in agreement with results predicted by the Lagrangian statistical theory. For a narrow plume, it is shown that the plume width is proportional to the time of travel, while for a narrow puff, the $\frac{3}{2}$-power dependence is found. For a narrow distribution, the concentration shape deviates, however, from a Gaussian shape, in contradiction to results of the statistical theory.

It is shown that only two external parameters are required in the spectral turbulent diffusivity model. These are the long-wave range diffusivity coefficient K_0 and the wave vector k_m of the most energetic turbulent eddies. An Eulerian integro-differential transport equation is the final result of the model. This equation can also be used for dispersion in case of space- and time-dependent parameters. We suggest a procedure for a direct experimental test of the spectral turbulent diffusivity concept.

1. Introduction

In Part I (Berkowicz and Prahm, 1979), the spectral turbulent diffusivity theory was formulated and the dispersion of a single puff was treated. The present study (Part II) extends this formulation to the treatment of plume dispersion.

Dispersion of pollutants in the atmosphere is usually treated in terms of either gradient transfer theory or statistical theory. The latter has succeeded in explaining most features of dispersion in homogeneous stationary turbulent flows. But because of its Lagrangian character, the statistical theory is not suitable for studies of multiple interacting sources and time-dependent problems. In such cases, the diffusivity equation approach, based on the gradient transfer theory, is preferable. The gradient transfer theory, however, is known not to be in agreement with important physical features of turbulent dispersion. Therefore, a need arises for an improved Eulerian dispersion theory.

A continuous plume may be considered to be made up of an infinite number of puffs released sequentially with a vanishingly small time interval between puffs. Initially, each puff moves with the wind direction at the moment of release. As the puff moves, it will expand about its center owing to the action of turbulent fluctuations. The mean speed and direction of the puff can be expected to change from the original values during its travel, as the wind pattern in which it is embedded changes with time. The result is a progressive broadening of the cross-wind front over which material is spread at a given distance downwind of the source. Thus, the time-average concentration diminishes with distance from the source not only due to the diffusion of the virtual puffs but also because of the random fluctuations of the center of mass of the puffs.

The growth of a puff is governed by turbulent eddies of a scale smaller than the size of the puff, while the random fluctuations of the center of mass are essentially a consequence of the existence of dispersive motions on a scale larger than the actual plume cross section itself. Both processes depend on the size of the dispersed material but in a quite different way.

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The main idea of the spectral turbulent diffusivity concept for dispersion of a plume is to relate the quantities describing the cross-wind fluctuations of the center of mass to Fourier components of the cross-wind distribution of the plume. Expressions yielding the time-averaged downwind concentrations of a plume in a homogeneous and stationary turbulent flow are given. The dependence of the time-averaged concentrations on the averaging time is discussed.

2. Center of mass fluctuations in a turbulent flow

The cross-wind displacement of a cloud of material is a consequence of the action of eddies larger than the size of the cloud. The smaller eddies are essentially responsible for diffusion of the cloud but are not able to move the cloud as a whole. The ability of the turbulent eddies to displace the center of mass of a cloud thus decreases with increasing size of the cloud. If we regard a plume as consisting of a series of puffs, we can say that the large-scale turbulent fluctuations decrease as the width of the plume increases. A narrow plume fluctuates more than a broad plume does. The cross-wind fluctuations of a plume will depend on the velocity of the center of mass of the transverse cross section of the plume We consider the action of a one-dimensional turbulent velocity field $v(y,t)$ on a one-dimensional cloud of material homogeneously distributed in a box of a size l. The equation of motion is

$$\frac{\partial c}{\partial t}=-\frac{\partial}{\partial y}[v(y,t)c], \tag{1}$$

where c is the concentration inside the box and $c=Q/l$. In order to determine the velocity of the center of mass, we multiply both sides of (1) by y and integrate with respect to y. Integration by parts gives

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} ycdy=\frac{Q}{l}\int_{y_0-l/2}^{y_0+l/2} v(y,t)dy, \tag{2}$$

where y_0 is the position of the center of mass at time t. The velocity of the center of mass at time t can now be expressed as

$$v_0(t)=\frac{\partial y_0}{\partial t}=\frac{1}{l}\int_{y_0-l/2}^{y_0+l/2} v(y,t)dy, \tag{3}$$

where v_0 fluctuates in time and we assume that the time-mean value is zero. The quantity we are interested in is $\overline{v_0^2}$, the mean square deviation of v_0. From (3) it follows that

$$\sigma_{v_0}^2=\overline{v_0^2}=\frac{1}{l^2}\int_{y_0-l/2}^{y_0+l/2}\int_{y_0-l/2}^{y_0+l/2}\overline{[v(y_1)v(y_2)]}dy_1dy_2. \tag{4}$$

The overbar denotes a time average. The expression in the square brackets on the right-hand side of (4) is the familiar covariance which on division by the standard deviation coefficient σ_v^2 gives the Eulerian (fixed time) correlation coefficient $R(y_1,y_2)$. In the case of a homogeneous turbulence, $R(y_1,y_2)$ depends only on y_1-y_2 but not on the position in the space as discussed, e.g., by Pasquill (1974). Introducing a new variable $\xi=y_2-y_1$, Eq. (4) can be written in the form

$$\sigma_{v_0}^2=\sigma_v^2\int_{-l/2}^{l/2}\int_{-l/2-y_1}^{l/2-y_1} R(\xi)d\xi dy_1. \tag{5}$$

Regarding the overall variation as composed of a spectrum of fluctuations, the correlation coefficient $R(\xi)$ can be expressed as

$$R(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k_t)\exp(ik_t\xi)dk_t, \tag{6}$$

where $F(k_t)$ is the energy spectrum of the turbulent fluctuations (Pasquill, 1974) and represents the contribution of eddies of a wavelength $\lambda_t=2\pi/k_t$ to the turbulent energy. We use the suffix t to denote a wave vector of the turbulent eddies in order to avoid confusion with a wave vector of a concentration distribution which in the following is denoted just by k.

$F(k_t)$ represents the energy spectrum of fluctuations measured at a fixed time but with a lag in the space in direction perpendicular to the mean flow direction. Substituting (6) for $R(\xi)$ in (5) and integrating with respect to ξ and y_1 yields

$$\sigma_{v_0}^2=\sigma_v^2\int_{-\infty}^{\infty}(2\pi)^{-1}F(k_t)\{[\sin(k_tl/2)]/(k_tl/2)\}^2dk_t. \tag{7}$$

It follows from (7) that for $l\to 0$, $\sigma_{v_0}^2\to\sigma_v^2$, i.e., fluctuations of the center of the mass of a narrow cloud follow wind velocity fluctuations. On the other hand, when $l\to\infty$, we see that $\sigma_{v_0}^2\to 0$. It means that a broad cloud cannot, as a whole, follow the wind velocity fluctuations. The factor $[\sin(k_tl/2)/(k_tl/2)]^2$ acts as a low-pass filter on the turbulent energy spectrum. Only eddies represented by a wavelength $\lambda_t=2\pi/k_t$ larger than l can effectively contribute to the displacement of the center of mass of a cloud of a size l. The influence of the center of mass fluctuations on plume dispersion is discussed in the following sections.

3. Phase fluctuations in a spectral plume model

Diffusion of an instantaneous plume can be treated in the same way as a one-dimensional cloud or puff. According to the spectral turbulent diffusivity theory, developed in Part I, the concentration distribution in a one-dimensional puff can be written as

$$c(y,t)=\frac{Q_0}{\pi}\int_0^{\infty}\cos ky\exp[-k^2K_d(k)t]dk, \tag{8}$$

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Here t is the time of travel and the puff is assumed to move along the axis $y=0$; Q_0 is the strength of an instantaneous point source; and $K_d(k)$ is the spectral turbulent diffusivity coefficient introduced in Part I. We use here a subscript d in order to distinguish the spectral diffusivity coefficent for a puff from an analog quantity introduced in this paper for a plume.

The concentration in an idealized two-dimensional linear stationary plume consisting of all the emitted puffs is obtained by the conventional procedure where t and Q_0 are replaced in Eq. (8) by x/u and Q/u, respectively (see, e.g., Berlyand, 1975). But a real plume meanders. The concentration distribution of such a plume can be obtained by shifting the instantaneous position of the central axis from the mean position to the actual instantaneous position. According to this procedure, the downwind concentration distribution at a time t can be written as

$$c(x,y,t)=\frac{Q}{\pi u}\int_0^\infty \cos k(y-y_0) \times\exp[-k^2K_d(k)(x/u)]dk, \quad (9)$$

where u is the mean wind velocity, directed along the x axis, and $y_0=y_0(x,t)$ is the displacement of the center of mass of the plume at distance x from the source at a time t. Thus $y_0(x,t)$ fluctuates both in time and in space. This results in the broadening of a time-averaged plume cross section. As one can see from (9), the expression under the integral can be regarded as Fourier components of the concentration field. The amplitude is given by the exponential factor. According to the results of Part I, diffusion of an instantaneous plume can be related to attenuation of the amplitudes of Fourier components of the concentration distribution. The attenuation rate is determined by the spectral diffusivity coefficients $K_d(k)$, being a function of the wave vector k of the concentration distribution. On the other hand, as seen from (9), the fluctuation of a plume is related to the fluctuation of the phase of the Fourier components.

It was shown in Section 2 that the fluctuation of the center of mass depends on the width of the distribution. The shape of the distribution is uniquely defined by its Fourier coefficients. Encouraged by these relationships, we put forward the hypothesis that y_0 is a function of k, the wave vector of the concentration distribution. This is in analogy to the spectral diffusivity concept of Part I. The consequences of this hypothesis are the main subject of the present study. One should note that $y_0(x,t,k)$ now has to be interpreted not as the center of mass position, but as a phase factor as defined by (9).

4. A spectral fluctuating plume model

Eq. (9) expresses the possible realization of the concentration field in a continuous plume depending on the actual values of $y_0(x,t,k)$. Because of the turbulent character of the wind field, a plume must be described in terms of some statistical properties. In air pollution problems, time-averaged properties are usually considered. The question we are going to discuss now is what time-averaged concentrations we can expect in a continuous plume, taking into account both the small-scale plume expansion and the large-scale fluctuations of the plume as a whole.

In order to determine the time-averaged concentrations, we have to find the average of Eq. (9) with respect to all possible realizations of y_0. It has to be done independently for each Fourier component, as y_0 is a function of k. It is quite obvious to expect that y_0 has a Gaussian probability distribution. The weighting factor in this case is

$$\exp[-y_0^2/2\sigma_{y_0}^2],$$

where $\sigma_{y_0}^2$ is the mean-square deviation of y_0 and is a function of both x and the wave vector k. Later, we shall show that $\sigma_{y_0}^2$ also depends on the averaging time T. We now have to evaluate the expression

$$\int_{-y_{0m}}^{y_{0m}} \exp[-y_0^2/2\sigma_{y_0}^2]\cos ky_0 dy_0 \Big/ \int_{-y_{0m}}^{y_{0m}} \exp[-y_0^2/2\sigma_{y_0}^2]dy_0, \quad (10)$$

where y_{0m} is the maximum value of y_0 and we assume that the mean value of y_0 is zero. The term containing $\sin ky_0$ in expansion of Eq. (9) is not included in (10) because it disappears when integration with respect to y_0 is performed.

For $y_{0m}\gg\sigma_{y_0}^2$, the limits of the integrals can be replaced by infinity and (lo) can easily be evaluated. Finally, we obtain

$$\overline{c(x,y)}=\frac{Q}{\pi u}\int_0^\infty \cos ky \exp\{-k^2[K_d(k)(x/u) +\tfrac{1}{2}\sigma_{y_0}^2(x,k)]\}dk. \quad (11)$$

Comparing (11) with (8), we can see that, in a time-averaged plume, the amplitude of the Fourier components depends both on the small-scale diffusion, as in a puff, and on the large-scale diffusion of the center of mass. Introducing a quantity $\sigma_d(x,k)$ defined as

$$\sigma_d^2(x,k)=2K_d(k)(x/u), \quad (12)$$

we can write (11) in a more compact form

$$\overline{c(x,y)}=\frac{Q}{\pi u}\int_0^\infty \cos ky \exp[-k^2\tfrac{1}{2}(\sigma_d^2+\sigma_{y_0}^2)]dk. \quad (13)$$

In the case where both σ_d and σ_{y_0} are independent of the wave vector k, Eq. (13) reduces to the well-known Gaussian form with a standard deviation $\sigma^2=\sigma_d^2+\sigma_{y_0}^2$, equivalent to the Gaussian fluctuating plume model (Gifford, 1959).

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5. Spectral phase diffusivity coefficient

We introduce a spectral phase diffusivity coefficient $K_p(k)$ defined as

$$\sigma_{y_0}^2(x,k) = 2K_p(k)(x/u). \tag{14}$$

Expression (14) states that the mean-square standard deviation of the phase fluctuations of the Fourier components of the concentration distribution in a plume is proportional to the travel time $t=x/u$. This assumption needs some further justification.

Dispersion of a continuous plume in a field of a homogeneous turbulence was extensively treated in terms of the statistical theory (Taylor, 1921; Batchelor, 1964). The familiar result of the statistical theory is that the mean-square standard deviation of the cross-wind distance of particles emitted from a continuous source is given by

$$\sigma_y^2(t) = 2\sigma_v^2 \int_0^t \int_0^{t_1} R_L(\xi) d\xi dt_1, \tag{15}$$

where $t=x/u$ is the travel time and $R_L(\xi)$ is the Lagrangian correlation coefficient. σ_v^2 is the mean square deviation of the cross-wind fluctuations. The correlation coefficient decreases with increasing time lag ξ. The scale of correlation is given by the Lagrangian time scale t_L. After a travel time equal to t_L, the transverse velocity of particles is assumed to be completely uncorrelated with the initial transverse velocity. The correlation of the phase fluctuation velocity of the concentration's Fourier components depends on the correlation of the turbulent transverse velocities in both x and y directions. Therefore, we believe that the correlation of the velocity of phase fluctuations is characterized by a very short time scale. This means that by expressing $\sigma_{y_0}(x,k)$ in a way similar to (15), we can replace the correlation coefficient by a δ-like function. In this case, the mean-square standard deviation coefficient becomes proportional to the travel time $t=x/u$. This confirms the relation (14). However, if the hypothesis of the weak correlation of the velocities of phase fluctuation is not valid, the linear dependence of $\sigma_{y_0}^2$ on x is not justified. In spite of this, the spectral formulation given by (13) is still valid. An alternative form of (14) would result in a non-Eulerian description. In the following, we are going to use the linear dependence of $\sigma_{y_0}^2$ as given by (14).

Up to this point, no assumption has been made about the explicit dependence of $K_p(k)$ on the wave vector k. The considerations presented here yield a useful estimation of the form of the advective spectral diffusivity coefficient; however, one should notice that the model is based on some essential approximations and simplifications which are necessary in order to make the further progress of the theory possible. Final justification of the assumptions should be tested by direct experimental studies.

Batchelor (1949) has shown that if an effective diffusivity coefficient $\tilde{K}$ is defined as

$$\tilde{K} = \frac{1}{2}\frac{d\sigma_y^2(t)}{dt}, \tag{16}$$

the gradient transfer theory predicts the same plume dispersion results as the statistical theory. From (15) and (16), it follows that

$$\tilde{K} = \sigma_v^2 \int_0^t R_L(\xi) d\xi. \tag{17}$$

The diffusivity coefficient defined by (17) is, in general, a function of the travel time and cannot be related to an Eulerian coordinate system. Relation (17) leads to the idea that the spectral diffusivity coefficient of phase fluctuations should also be expressed as a product of the mean-square standard deviation of the speed of the phase fluctuations and a time factor depending also on the wave vector of the Fourier component. Thus,

$$K_p(k) = \sigma_{vph}^2(k)\tau(k). \tag{18}$$

Relation (18) is thus based on the assumption of similarity between diffusion of particles and diffusion of phases of Fourier components of the concentration distribution. However, the short correlation range of the velocity of phase fluctuations results in a constant time factor in (18), while in (17) it is a function of the travel time.

It seems now to be reasonable to relate $\sigma_{vph}^2(k)$ to the standard deviation coefficient of the velocity fluctuations of a center of mass of a cloud of a size corresponding to the wave vector k. It is easy to show that Fourier components of a homogeneous concentration distribution in a box of a size l are governed by a factor

$$A(k) \sim \sin(kl/2)/(kl/2). \tag{19}$$

Because (19) has a local extremum at $k=3\pi/l$, we conclude that a Fourier component with a wave vector

$$k = 3\pi/l \tag{20}$$

is the most representative for a box of a size l. This leads to the relation

$$\sigma_{vph}^2(k) = \sigma_{v_0}^2(3\pi/k). \tag{21}$$

A detailed knowledge of the turbulent energy spectrum $F(k_t)$ is necessary in order to compute $\sigma_{vph}^2(k)$ exactly according to (7) and (21). However, an approximate evaluation of $\sigma_{vph}^2(k)$ can be made on the basis of the general behavior of the energy spectrum function $F(k_t)$. We know that the turbulent energy has maximum at the long-wave range of the spectrum. Let us say that the most energetic eddies correspond to a wave vector k_m. If only those eddies would contribute to the turbulent fluctuations, $\sigma_{v_0}^2$ could be expressed as

$$\sigma_{v_0}^2 = \sigma_v^2[\sin(k_m l/2)/(k_m l/2)]^2, \tag{22}$$

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where $\sigma_{v_0}^2$ decreases as l increases, but for $(k_m l/2) > \pi$, the function starts to oscillate. The amplitude of the oscillations decreases as $(k_m l/2)^{-2}$. Because not only a single turbulent wave, but a quite broad spectrum, contributes to σ_{v_0}, we can expect, in fact, that σ_{v_0} should not oscillate, but for large l should decrease as $(k_m l/2)^{-2}$. On the other hand, for $l \to 0$, $\sigma_{v_0}^2 \to \sigma_v^2$. The following expression obeys these requirements:

$$\sigma_{v_0}^2 = \sigma_v^2 / [1 + (k_m l/2)^2]. \tag{23}$$

Taking (21) into account, we obtain

$$\sigma_{vph}^2(k) = \sigma_v^2 (\tfrac{2}{3}\pi k/k_m)^2 / [1 + (\tfrac{2}{3}\pi k/k_m)^2]. \tag{24}$$

We are going to use (24) in the following computations, but one should consider this relation as just a working approximation which, however, converges to proper values for small and large k values.

In order to determine the advective spectral turbulent diffusivity coefficient $K_p(k)$, we must still evaluate the time factor $\tau(k)$. The fluctuating plume model yields time-averaged concentration values. The results must thus depend on the averaging time T. When the averaging time is small, we can expect that no significant deviations from values of concentrations in an instantaneous plume can be detected. Dispersion of an instantaneous plume can be described by diffusion of puffs. In this case, we can expect that $K_p(k) \to 0$. Since our model shows that the only dependence of $K_p(k)$ on the averaging time T is given by the time factor $\tau(k)$, a necessary requirement is that

$$\tau(k) \to 0 \quad \text{for} \quad T \to 0. \tag{25}$$

On the other hand, when the averaging time is longer than some time T_m, corresponding to the slowest fluctuations, the time-averaged concentrations cannot depend on T, but should be determined by T_m. Thus,

$$\tau(k) \to a(k) T_m \quad \text{for} \quad T \gg T_m, \tag{26}$$

where $a(k)$ is some dimensionless function of the wave vector k of the concentration distribution; $a(k)$ takes care of a different behavior of the Fourier components with respect to the averaging time. As discussed in Section 3, the dispersion of a fluctuating plume can be related to phase fluctuations of the Fourier components of the concentration distribution. Because of the periodicity of the Fourier components, the change of the phase by a factor of 2π cannot contribute to the change of the concentration distribution. If during the averaging period T, the phase of a Fourier component changes by $k\,\delta y_0$, we can expect that the fluctuation of this Fourier component contributes to the dispersion only as long as

$$k\delta y_0 < 2\pi. \tag{27}$$

Assuming that the change of y_0 is proportional to the averaging time T, we can write

$$\delta y_0 = w^* T, \tag{28}$$

where w^* is some characteristic velocity. From (27) and (28), it follows that the dependence of $\tau(k)$ on the averaging time should be negligible when

$$k w^* T/(2\pi) \gg 1. \tag{29}$$

The most obvious interpretation of the velocity w^* is that it is the velocity of the eddies responsible for the low-frequency fluctuations. As the period of these eddies is T_m and the wave vector is k_m, we can write

$$w^* = 2\pi/(k_m T_m). \tag{30}$$

The simplest form of $\tau(k)$, which converges to the proper asymptotes given by (25) and (26) together with (29) and (30), is

$$\tau(k) = T \Big/ 1 + \frac{T}{T_m}\frac{k}{k_m}. \tag{31}$$

Expression (31) cannot be considered as an exact derivation of $\tau(k)$, but only as a first approximation based on the discussion of the behavior of $\tau(k)$ for large and small k and T values.

The final expression for the spectral phase diffusivity coefficient can now be written as

$$K_p(k) = \sigma_v^2 \frac{(\tfrac{2}{3}\pi k/k_m)^2}{1 + (\tfrac{2}{3}\pi k/k_m)^2} T \Big/ 1 + \frac{T}{T_m}\frac{k}{k_m}. \tag{32}$$

Introducing a ratio r, defined as $r = T/T_m$, Eq. (32) can be written in the more convenient form

$$K_p(k) = \sigma_v^2 T_m \frac{(\tfrac{2}{3}\pi k/k_m)^2}{1 + (\tfrac{2}{3}\pi k/k_m)^2} \frac{r}{1 + rk/k_m}. \tag{33}$$

6. Discussion of the plume and puff dispersion parameters

The spectral model of plume dispersion in a stationary homogeneous turbulent flow is completely determined when the spectral diffusivity coefficients $K_d(k)$ and $K_p(k)$ are specified. The spectral diffusivity coefficient $K_d(k)$, introduced in Part I, is given by

$$K_d(k) = \frac{K_0}{1 + Bk^{\frac{4}{3}}}. \tag{34}$$

The total spectral diffusivity coefficient of a plume

$$K(k) = K_d(k) + K_p(k) \tag{35}$$

thus depends on the following parameters:

- K_0 diffusivity of the long-wave range of the concentration spectrum $[= K_d(0)]$.
- B parameter determining the width of the spectral diffusivity coefficient $K_d(k)$
- σ_v^2 mean square deviation of the v component of the turbulent wind field

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T_m period of the slowest wind fluctuations; corresponds to the most energetic eddies
k_m wave vector of the most energetic eddies.

The averaging time T is an external parameter.

The most flexible model corresponds to the case where all the parameters specified above are considered as independent parameters. One could, for example, determine the parameters by fitting the results predicted by the model to empirical results. We would like to show, however, that the number of independent parameters can, in fact, be reduced to only two. The relationships are based, however, on some more or less empirical relations and evaluations, and therefore should not be considered as exact relations.

It was pointed out in Part I that the parameter B must be proportional to $l_m^{4/3}$, where l_m is the length of the most energetic eddies. Since $l_m \sim l/k_m$, we may write

$$B \sim k_m^{-4/3}. \tag{36}$$

The proportionality factor is still unknown, but it is of minor importance for the dispersion of a plume and in the following we shall use simply $B = k_m^{-4/3}$. The concentration distribution in a broad plume or a broad cloud is expressed by Fourier components with a small k value. From (33) and (34), it follows that the spectral diffusivity coefficient K_p tends to zero, while K_d tends to a constant value K_0 when $k \to 0$. The total diffusivity coefficient of a broad plume can thus be replaced by a constant value k_0. From (17), it follows that

$$\tilde{K} \to \sigma_v^2 t_L' \quad \text{for} \quad t \to \infty, \tag{37}$$

where t_L is the Lagrangian time scale. The Lagrangian time scale is mainly determined by the low-frequency part of the turbulent energy spectrum. Thus it does not seem to be too hazardous to assume that $t_L = T_m$. Expecting that the spectral diffusivity theory and the statistical theory should yield the same results in the limit $t \to \infty$, we must have that $K_0 = \tilde{K}$, and thus

$$K_0 = \sigma_v^2 T_m. \tag{38}$$

The relationships (36) and (38) reduce the number of the independent parameters to only three, e.g., K_0, k_m and T_m.

According to Hanna (1968), the long-wave diffusivity coefficient K_0 can be written as

$$K_0 = A\sigma_v \lambda_m, \tag{39}$$

where λ_m is the wavelength at which the turbulent energy has a maximum. In fact, the relationship (39) was given for the vertical diffusivity coefficient, but there is no reason not to believe that the same relation is true for the horizontal component. On the basis of experimental observations the proportionality coefficient A was estimated to be $A = 0.154$ (Smith, 1977). Comparing (38) and (39) and substituting $\lambda_m = 2\pi/k_m$, we obtain

$$\sigma_v T_m = (0.154) 2\pi/k_m. \tag{40}$$

Because (2π) 0.154 is, in fact, very close to 1 and recalling the experimental uncertainty in determining A, we can write

$$\sigma_v = (T_m k_m)^{-1}. \tag{41}$$

By means of (41), the number of independent parameters is now reduced to two. Eq. (41) can also be deduced in another way. On the basis of experimental observations, Markee (1963) found that the mean deviation of the horizontal wind fluctuations is nearly equal to one-sixth of the maximum range of the observed velocity fluctuations. The maximum velocity should be related to the long-wave eddies and according to (30) we can write

$$\sigma_v = \frac{2\pi}{6}(T_m k_m)^{-1}. \tag{42}$$

Because the factor $2\pi/6$ is also quite close to unity, the relation (42) is in fact identical with (41). The coincidence of the relationships (41) and (42) supports the interpretation of the velocity w^*, defined by (30), as the maximum of the turbulent velocity fluctuations.

Choosing K_0 and k_m as the two internal independent parameters and r, the ratio between the averaging time T and T_m as the external parameter, the total spectral diffusivity coefficient $K(k)$ can be written

$$K(k) = K_0\left[\frac{1}{1+(k/k_m)^{4/3}}\right] + K_0\left[\frac{(\frac{2}{3}\pi k/k_m)^2}{1+(\frac{2}{3}\pi k/k_m)^2} \times \frac{r}{1+rk/k_m}\right]. \tag{43}$$

The first term on the right-hand side of (43) represents the contribution from eddies smaller than the actual plume size to the plume expansion, while the second term is due to the eddies larger than the plume size and is responsible for the time-averaged broadening of the plume's cross section.

The time-averaged concentration in a continuous plume emitted from a continuous point source, located at (0,0), in a field of a homogeneous and stationary turbulence, can now be expressed as

$$c(x,y) = \frac{Q}{\pi u}\int_0^\infty \cos ky \cdot \exp[-k^2 K(k)(x/u)]dk, \tag{44}$$

where $K(k)$ is given by (43).

7. Discussion of the concentration shape

The behavior of a plume at distances close to and far from the source, in general, is quite different. This is caused by the relatively different role the small and the large-scale eddies play in dispersing a narrow and a broad plume, respectively. The cross-sectional size of a plume must be related to the typical length scale of the turbulence. A length scale X_T can be so defined that at a

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distance x from the source smaller than X_T, the plume dispersion is mainly determined by the small-scale eddies, while at x larger than X_T, the large-scale eddies play the most important role. The integral in (44), for $y=0$, can be divided in two parts:

$$I_1=\int_0^{k_m} \exp[-k^2K(k)(x/u)]dk,$$

$$I_2=\int_{k_m}^{\infty} \exp[-k^2K(k)(x/u)]dk.$$

Here X_T is defined as the distance at which $I_1=I_2$, i.e.,

$$\int_0^{k_m} \exp[-k^2K(k)(X_T/u)]dk = \int_{k_m}^{\infty} \exp[-k^2K(k)(X_T/u)]dk.$$

For $x \ll X_T$, we have

$$I_2 \gg I_1 \qquad (45)$$

and the integral of (44) can be fairly well approximated by I_2 only. The plume is narrow in this case.

For $x \gg X_T$ we have

$$I_1 \gg I_2. \qquad (46)$$

The integral of (44) can be approximated by I_1 only and the plume is broad. It is known that both the gradient-transfer theory and the statistical theory yield the same result for plume dispersion at large distances from the source where the plume is broad. In this case, the shape of the plume is expressed by a Gaussian distribution with a standard deviation given by

$$\sigma_y^2=2K_0\frac{x}{u}. \qquad (47)$$

The spectral diffusivity model predicts the same result in this case. We find from (46) that for $x \gg X_T$, only terms with $k \ll k_m$ contribute considerably to the integral of (44). In this case, however, $K(k) \to K_0$ and the plume becomes Gaussian. A similar result was presented in Part I for the cloud dispersion.

The disagreement between the statistical and the gradient transfer theory is serious in predicting dispersion of a narrow plume. Following the results of the statistical theory, the standard deviation of a plume is proportional to x, while the gradient transfer theory predicts that in the case of constant diffusivity, the standard deviation is proportional to $x^{\frac{1}{2}}$. Considering the spectral diffusion model, Eq. (45) shows that the terms with $k>k_m$ dominate dispersion of a plume at short distances from the source, i.e., for $x \ll X_T$. For $k \gg k_m$, the spectral diffusivity coefficient given by (43) can be approximated by

$$K(k) \approx K_0\frac{k_m}{k}. \qquad (48)$$

Substituting (48) into (44) yields

$$c(x,y) \approx \frac{Q}{\pi u}\int_0^{\infty} \cos ky \exp\left[-k^2K_0\frac{k_m}{k}\frac{x}{u}\right]dk = \frac{Q}{\pi u}\frac{K_0k_m(x/u)}{y^2+[K_0k_m(x/u)]^2}. \qquad (49)$$

Notice that (49) is valid only for $x \ll X_T$.

It is seen from (49) that the spectral diffusivity theory predicts a non-Gaussian concentration distribution in a plume close to the source. It is easy to show that the standard deviation coefficient of a distribution like (49) becomes infinite. Thus the standard deviation coefficient does not seem to be a good measure of a plume width in a case of a non-Gaussian distribution. In this case, we propose to define a parameter $y_0(x)$, the distance from a plume centerline, at which the concentration is equal to a certain ratio α, of the concentration at the centerline, In the case of a concentration distribution given by (49), we obtain

$$y_\alpha(x)=K_0k_m\left[\frac{1}{\alpha}-1\right]^{\frac{1}{2}}\frac{x}{u} \qquad (50)$$

and especially for $\alpha=\frac{1}{2}$, we have

$$y_{\frac{1}{2}}(x)=K_0k_m\frac{x}{u}. \qquad (51)$$

It follows from (50) that the spectral diffusivity theory predicts a linear expansion of a plume at short distances from the source. This is in accordance with results of the statistical dispersion theory for a continuous plume (Taylor, 9121). Furthermore, the statistical theory predicts that at short distances, the standard deviation of a plume is

$$\sigma_y(x)=\sigma_v\frac{x}{u}. \qquad (52)$$

From (38) and (41), it follows that

$$K_0k_m=\sigma_v. \qquad (53)$$

Substituting (53) into (51), we can see that, in fact, $y_{\frac{1}{2}}(x)$ is equal to $\sigma_y(x)$ from the statistical theory. The concentration shape of a plume predicted by the statistical and the spectral theory is, however, quite different.

Discussion of the dependence of the parameter y_α on the distance from the source can be extended to a more general form of the spectral diffusivity coefficient. We now consider a spectral diffusivity coefficient given by the expression

$$K(k)=ak^{-\gamma}, \qquad (54)$$

where a and γ are constants.

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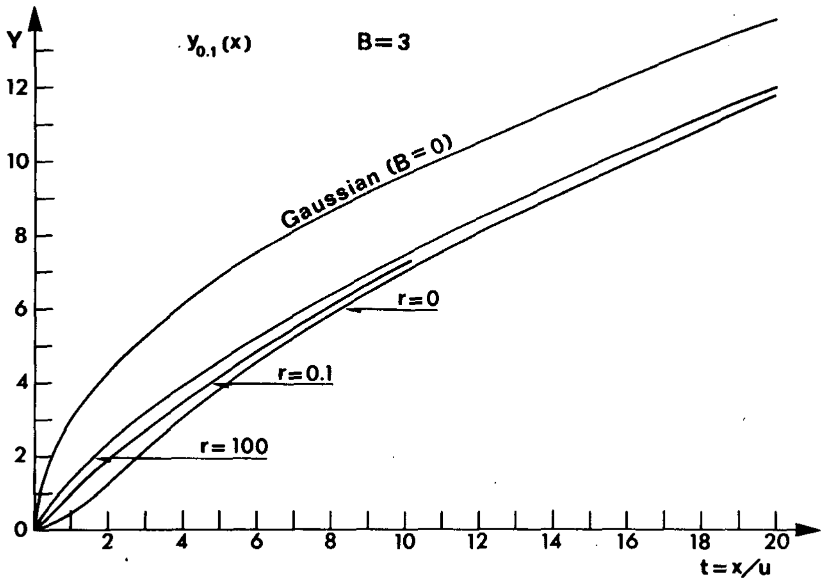

FIG. 1. Plume width given by $y_{0.1}(x)$ as a function of the time of travel $t=x/u\cdot B=k_m^{-\frac{4}{3}}$. The averaging time is given by $r=T/T_m$. The curve labeled $r=0$ is redrawn from Fig. 4 of Berkowicz and Prahm (1979). A Gaussian curve ($B=0$) is shown for comparison. Time is in units of $(\Delta y)^2/K_0$ and Δy is the length unit.

By definition, $y_\alpha(x)$ can be found from the relation

$$\alpha\int_0^\infty \exp[-ak^{2-\gamma}(x/u)]dk=\int_0^\infty \cos[ky_\alpha(x)]\times\exp[-ak^{2-\gamma}(x/u)]dk. \quad (55)$$

We make a substitution

$$ky_\alpha(x)=\theta.$$

Replacing in (55) the integration with respect to k by an integration with respect to θ, we obtain

$$\frac{\alpha}{y_\alpha(x)}\int_0^\infty \exp\left[-\frac{a\theta^{2-\gamma}}{u}\frac{x}{y_\alpha^{2-\gamma}(x)}\right]d\theta=\frac{1}{y_\alpha(x)}\int_0^\infty \cos\theta\times\exp\left[-\frac{a\theta^{2-\gamma}}{u}\frac{x}{y_\alpha^{2-\gamma}(x)}\right]d\theta. \quad (56)$$

In order to satisfy (56) for all x, we must have

$$y_\alpha(x)=(bx)^{1/(2-\gamma)}, \quad (57)$$

where b is some constant. In the case of a continuous plume $\gamma=1$, and we obtain again the relation (50) with

$$b=\left(\frac{1}{\alpha}-1\right)^{\frac{1}{2}}\frac{K_0k_m}{u}.$$

For a puff, the spectral diffusivity coefficient is given by (34), and for a small puff we can again consider the limit of $k\rightarrow\infty$. In this case, (34) can also be approximated by an expression like (54) with $\gamma=\frac{4}{3}$. From (57), it follows that the width of a puff, after a short time of travel, can be given by

$$y_\alpha(x)=(bx)^{\frac{3}{2}}. \quad (58)$$

A similar result for the expansion of a puff was found by Batchelor and Townsend (1956) on the basis of a dimensional analysis, and by Smith and Hay (1961) considering the relative velocities of particles in a puff. They found, however, that the $\frac{3}{2}$-power dependence of the puff width is valid only for a puff of a size smaller than the distance from a source. The $\frac{3}{2}$-power dependence was also predicted by Richardson (1926) on the basis of the relative diffusion model.

The most significant difference between results from the spectral diffusivity theory and the statistical theory is that the latter assumes a Gaussian shape of the concentration distribution while the spectral theory predicts, in general, a non-Gaussian distribution.

If the spectral diffusivity coefficient can be expressed by a relation like (54), it is easy to show that the centerline ($y=0$) concentration is given by

$$c(x,0)=\frac{Q}{\pi ua^{1/(2-\gamma)}}\frac{1}{2-\gamma}\Gamma\left(\frac{1}{2-\gamma}\right)\left(\frac{x}{u}\right)^{-1/(2-\gamma)}. \quad (59)$$

We can see again that at short distances the concentration of a continuous plume decreases as x^{-1}, while for a puff, the relation $x^{-\frac{3}{2}}$ is valid. The centerline concentration is thus proportional to $y_\alpha^{-1}(x)$. In a Gaussian plume, the centerline concentration is known to be proportional to σ_y^{-1}. In Fig. 1, the parameter $y_\alpha(x)$ for $\alpha=0.1$ is plotted as function of the time of travel $t=x/u$ for different averaging times. The curve corresponding to $r=0$ (instantaneous plume) is the same as presented in Part I for dispersion of a cloud. For computation of

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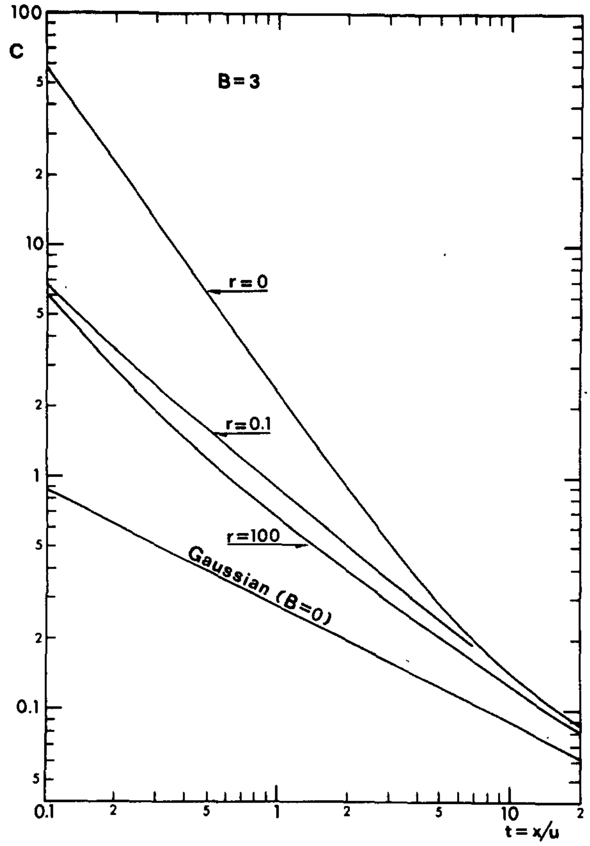

FIG. 2. The centerline concentration of a two-dimensional plume as a function of the time of travel $t=x/u$. Concentration is in units of $Q/(\Delta y \cdot u)$. See also explanation given in caption to Fig. 1. The curve labeled $r=0$ is redrawn from Fig. 3 of Berkowicz and Prahm (1978). A Gaussian curve ($B=0$) is shown for comparison.

$y_{0.1}$ Eq. (44) is used with the spectral diffusivity coefficient given by (43). The integral of (44) is replaced by a sum with a discrete sized k vector. The increment of k is $\pi/16$. For details, the reader is referred to Part I. In order to simplify the comparison with Part I, we use the parameter B instead of k_m [see relation Eq. (34)], and here $B=3\ (\Delta y)^{4/3}$ where Δy is the length unit. For comparison, the curve corresponding to $y_{0.1}$ from a Gaussian distribution, with $K=K_0(B=0)$, is also shown.

The centerline concentrations are shown in Fig. 2 for the same parameters as in Fig. 1. The Gaussian curve is shown for comparison.

8. Advection-diffusion equation in terms of the spectral turbulent diffusivity theory

It was shown in Part I that the introduction of the spectral diffusivity coefficient as a function of the k vector of the concentration distribution leads to a integro-differential diffusion equation. For a one-dimensional cloud, this equation is presented by Eq. (12) of Part I. A similar relation can also be given for dispersion of a plume including an advection term. The diffusivity transfer function is determined by Eq. (11) of Part I, but with spectral diffusivity coefficients now given by (43).

In general, the turbulent diffusivity transfer function is a second-order tensor and the integrals should be extended over the whole three-dimensional space. However, some preliminary considerations show that only non-localness in the direction parallel to the flux component is important so only one-dimensional integrals are necessary.

As discussed in Part I [Eq. (14)] the theory can be extended to treat dispersion in a nonhomogeneous turbulent flow. Also the variation of the dispersion parameters with time can easily be accounted for. Because no explicit dependence on the time of travel or the distance from a source is required, the theory results in an Eulerian model for plume dispersion. It is also easy to include source of sink terms describing, e.g., dry or wet deposition or chemical reactions. A similar extension is not possible in a statistical model. For a general form of diffusivity and wind fields the dispersion equation can be solved by numerical integration techniques. The most suitable method is the newly developed pseudo-spectral numerical models (Christensen and Prahm, 1976; Prahm and Christensen, 1977; Berkowicz and Prahm, 1978).

9. Suggestions for direct experimental test of the spectral turbulent diffusivity concept

The spectral turbulent diffusivity concept introduced in Part I can be directly tested by tracer studies. The main result of the theory is that in the case of diffusion in a field of homogeneous and stationary turbulence, the time-dependence of the Fourier coefficients of the concentration distribution is given by

$$A(k,t)=A(k,o)\exp[-k^2K(k)t]. \qquad (60)$$

If the experiment is designed in such a way that the time dependence of the Fourier coefficients is measured, the correctness of the relation (60) can be tested. The question which must be answered is: does $A(k,t)$ decrease exponentially with time? Furthermore, by studying the k-dependence of the proportionality factor, the spectral turbulent diffusivity coefficients can be measured. When puff and plume studies are made, the results should be compared with the relation presented in (34) or (33), respectively. As shown in this paper, two parameters are necessary in order to determine the spectral diffusivity coefficient $K(k)$—K_0 and k_m. The first corresponds to the "ordinary" eddy diffusivity and several methods can be used in order to determine this parameter (Pasquill, 1974). The wave vector k_m, corresponding to the most energetic turbulent eddies, can be obtained from spectra of turbulence (see, e.g., Busch and Panofsky, 1968).

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As the model presented in Part I and in the present study is designed for applied air pollution studies, the experiments should also correspond to conditions appearing in the real atmosphere. Preliminary studies of some Canadian field plume measurements reveal plume shapes confirming predictions from the present theory (Prahm and Berkowicz, 1979).

10. Conclusion

The scale-dependence of turbulent diffusion can be accounted for by introducing spectral diffusivity coefficients as functions of the k vector of the concentration distribution. This makes it possible to construct an Eulerian model which can easily be applied for both puff and plume dispersion in a turbulent field. In the case of a homogeneous turbulence, the results are consistent with the results of the Lagrangian statistical theory. The spectral turbulent diffusivity model is a significant improvement over the gradient transfer theory and is especially applicable in air pollution studies treating multiple interacting sources.

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