Approximation Formulas to Calculate Infrared Extinction
by an Aerosol Having a Junge Size Distribution

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ABSTRACT

The exact Mie curves for the extinction efficiency factor $Q_e$ as a function of the normalized size parameter $\rho = 2\alpha(n - 1)$ are approximated by eight continuous linear segments. These linear approximations for $Q_e$ are used to obtain an analytical approximation formula for the extinction coefficient $k_e$ in the infrared of an aerosol having a Junge size distribution, described by $dN/dr = A r^{-\nu}$. This formula expresses $k_e$ as an explicit function of $r$, propagation wavelength $\lambda$ and the real part $n$ and the imaginary part $\kappa$ of the complex index of refraction. For $\lambda$, $r$, $n$ and $\kappa$ in the ranges $1 \mu m \leq \lambda \leq 11 \mu m$, $3.6 \leq r \leq 4.4$, $1.2 \leq n \leq 1.65$ and $0 \leq \kappa \leq 0.15$, the values of $k_e$ calculated from the approximation formula agree to within 5% with those obtained by numerical integration using the exact Mie theory formulas for $Q_e$.

An analytical approximation formula for $k_e$ as a function of relative humidity is derived based on the assumption that the relationship between particle size $r$ in humid air and particle size $r_d$ in dry air can be described by $r/r_d = \Phi(S)$, where $S$ is the saturation ratio.

1. Introduction

It is quite well established (Junge, 1963) that the radius-number distribution of aerosol particles in continental air masses can often be adequately approximated by an equation of the form

$$n(r) = \frac{dN}{dr} = A r^{-\nu}. \quad (1)$$

Eq. (1) is the so-called Junge or inverse power-law size distribution. In Eq. (1), $n(r)$ is the number of particles per cubic centimeter per micron radius interval and $N$ is the total number density of particles with radius smaller than $r$. This equation is usually valid over the size range $0.07 \mu m \leq r \leq 10 \mu m$. Since Eq. (1) is an accepted model of the continental aerosol size distribution, it is frequently invoked in the calculation of the aerosol extinction coefficient $k_e$ for continental type air mass situations. An accurate analytical formula expressing $k_e$ as an explicit function of propagation wavelength, the real and imaginary parts of the complex index of refraction of the aerosol particles, and the parameters $A$ and $\nu$ of the Junge size distribution obviously would be very useful. Such an analytical expression would permit the calculation of the extinction coefficient on a hand calculator (thus avoiding the need to have access to a high speed computer to handle the complex Mie computations) and could result in substantial cost savings when, for example, repeated calculation of $k_e$ may be required in a subprogram of a larger program.

In this paper we shall derive approximate analytical formulas for the extinction coefficient at infrared wavelengths of an aerosol obeying Eq. (1). The formulas are valid for wavelengths between 1 and 11 $\mu m$, for values of $\nu$ between 3.6 and 4.4 (the value $\nu = 4.0$ is considered average for continental air), and for values of the real and imaginary parts of the refractive index in the range 1.2-1.65 and 0.0-0.15, respectively. For these ranges of values of the pertinent parameters, the value of $k_e$ calculated from the analytical formulas agrees to within 5% with that obtained by numerical integration of the general expression for $k_e$ using the exact Mie formulas for the extinction efficiency factor.

2. Equations for the aerosol extinction coefficient

The general expression for the aerosol extinction coefficient is

$$k_e = \int_0^\infty \pi r^2 Q_e(m, \alpha) n(r) dr. \quad (2)$$

where $Q_e(m, \alpha)$ is the extinction efficiency factor, $\alpha = 2\pi r/\lambda$ the particle size parameter and $m = n - i\kappa$ the complex index of refraction of the particles.

For $n(r)$ given by Eq. (1), the expression for $k_e$ (units: $cm^{-1}$) becomes

$$k_e = \pi A \times 10^{-8} \int_0^\infty r^{2-\nu} Q_e(m, \alpha) dr. \quad (3)$$

As pointed out by Fenn (1976), the limits of integration
in (3) are only approximately reached by the range of validity of the power law distribution. In terms of \( \alpha \), \( k_e \) may be expressed as

\[
k_e = \pi A \left( \frac{\lambda}{2\pi} \right)^{3-r} \times 10^{-8} \int_0^\infty \alpha^{2-n} Q_\alpha(\alpha, m) d\alpha.
\]

(4)

Mie theory gives \( Q_\alpha \) as an infinite series of mathematically complex terms; thus the integral in (4) cannot be evaluated analytically using the exact Mie formulas for \( Q_\alpha \). Only for a number of asymptotic cases has it been possible to obtain expressions for \( Q_\alpha \) in finite form. Asymptotic expressions for \( Q_\alpha \) have been obtained, for example, in the case of \( \alpha \ll 1 \) (van de Hulst, 1957) and for \( \alpha > 1 \) (Shifrin, 1968; van de Hulst, 1957). Because the functional dependence of \( Q_\alpha \) on \( \alpha \) and \( m \) is so complex, it is even difficult to obtain accurate approximation expressions for \( Q_\alpha \) over the range of \( \alpha \) and \( m \) of importance in extinction by atmospheric aerosols, which would permit analytical evaluation of the integral in (4).

Consider, however, the normalized size parameter

\[ \rho = 2\alpha (n-1) \]

which is the real part of \( 2\alpha (m-1) \). Physically, \( \rho \) is the actual phase shift experienced by the central ray through the particle. \( Q_\alpha \) is a less complicated function of \( \rho \) than of \( \alpha \) since the maxima and minima of \( Q_\alpha \) occur at approximately the same values of \( \rho \) irrespective of the index of refraction of the particle. In terms of \( \rho \), Eq. (4) may be written

\[
k_e = \pi A \left[ \frac{\lambda}{4\pi (n-1)} \right]^{3-r} \times 10^{-8} \int_0^\infty \rho^{2-n} Q_\rho(m, \rho) d\rho.
\]

(5)

Defining \( k = \lambda / 4\pi (n-1) \), we have

\[
k_e = \pi A k^{3-r} \times 10^{-8} \int_0^\infty \rho^{2-n} Q_\rho(m, \rho) d\rho.
\]

(6)

Deirmendjian (1960) has developed empirical approximation formulas for \( Q_\rho(m, \rho) \) which are accurate to within 4% for any \( \rho \) and for \( m \) satisfying the condition

\[
1 < n \leq 1.50,
\]

\[
0 < \kappa \leq 0.25.
\]

Unfortunately, the functional dependence of \( Q_\rho \) on \( \rho \) expressed by these formulas precludes analytical integration of Eq. (6).

We see from an examination of Eq. (6) that if the integral were replaced by the summation

\[
\sum_{n=0}^\infty \rho^{2-n} Q_\rho d\rho + \sum_{m=1}^\infty \rho^{2-n} Q_\rho d\rho + \ldots + \sum_{n=1}^\infty \rho^{2-n} Q_\rho d\rho
\]

and if \( Q_\rho \) were approximated by a linear function of \( \rho \) within each interval of \( \rho \), then the integration could be accomplished analytically and one could obtain an analytical expression for \( k_e \) as an explicit function of \( \lambda \), \( n \), \( \kappa \), \( \nu \) and \( A \). In the following section a linear approximation formula for \( Q_\rho(m, \rho) \) for each of eight specified ranges of \( \rho \) will be developed.

### 3. Approximation formulas for \( Q_\rho \)

Fig. 1 contains plots of the cumulative percent contribution to \( k_e \) as a function of \( \rho \) for several sets of values of the parameters \( \nu \), \( n \), \( \kappa \) and \( \lambda \). This figure
shows that at infrared wavelengths and for an aerosol obeying a Junge size distribution, a very high percentage of the total extinction coefficient is contributed by particles having $\rho \leq 7$. It is noted that even in the case of an aerosol distribution having the relatively flat slope of $\nu = 3.6$ and composed of non-absorbing particles, more than 70% of the extinction is due to particles for which $\rho \leq 7$. Therefore, in developing an accurate analytical expression for $k_e$, it is necessary for the approximation formulas for $Q_e$ to be highly accurate in the region $\rho \leq 7$. In the region $\rho > 7$ we can tolerate considerably less accuracy in $Q_e$ without greatly affecting the accuracy of the value of $k_e$.

The continuous curves in Fig. 2 show the exact variation of $Q_e$ over the region $\rho < 9$ for four different values of $m$. In the approximation scheme developed the exact curve of $Q_e$ as a function of $\rho$ is approximated by eight straight-line segments. The ranges of $\rho$ of the eight continuous linear segments are as follows: $1.0 < \rho < 0.75; 0.75 < \rho < 1.4; 1.4 < \rho < 2.4; 2.4 < \rho < 3.5; 3.5 < \rho < 4.5; 4.5 < \rho < 6.5; 6.5 < \rho < 7; \rho > 7$. We will now briefly outline the derivation of the approximation formulas for $Q_e$. If one closely examines the dependence of $Q_e(m, \rho)$ on $m$ for a given $n$ (see Figs. 2a and 2b), it is seen that the value of $Q_e$ in the region $2.3 < \rho < 2.45$ is quite insensitive to the imaginary part of the index of refraction. Therefore, the value of $Q_e$ at $\rho = 2.4$ was taken to be a function of $n$ only. This value is denoted by $C$. It is also seen from Fig. 2 that the first maximum of $Q_e$ occurs at approximately $\rho = 4$, irrespective of the index of refraction. The value of $Q_e$ in the range $3.5 < \rho < 4.5$ was therefore taken to be independent of $\rho$.

![Fig. 2. Comparison of exact (continuous curves) and approximation (dashed curves) values of $Q_e$ for four values of $m$.](image-url)
We denote the value of \( Q_e \) in this range by \( \Lambda(n, \kappa) \). Inspection of the exact curves of \( Q_e \) as a function of \( \rho \) further shows that to a suitable degree of accuracy the value of \( Q_e \) at \( \rho = 6.5 \) can be taken to be equal to the value of \( Q_e \) at \( \rho = 2.4 \). In the region \( 6.5 < \rho < 7 \) we approximate \( Q_e \) by a straight line having a slope equal to \( (C - \Lambda)/3 \), which is two-thirds of the value of the slope of the linear approximation to \( Q_e \) in the region \( 4.5 < \rho < 6.5 \). The value of \( Q_e \) at \( \rho = 7 \) is then equal to \( (7C - \Lambda)/6 \). In the region \( \rho > 7 \), \( Q_e \) is set equal to the value of \( Q_e \) at \( \rho = 7 \). The value of \( Q_e \) at \( \rho = 1.4 \) is denoted by \( C_1 \) and is a function of \( n \) and \( \kappa \). Between the limits of \( \rho = 0.75 \) and \( \rho = 1.4 \), \( Q_e \) is approximated as a straight line with a slope equal to \( C_1/(1.4 - x_1) \), where \( x_1 \) is a function of \( n \) and \( \kappa \). Finally, the value of \( Q_e \) is assumed to be zero when \( \rho \leq x_1 \). Equations describing the dependence of \( x_1, x_2, C, C_1 \) and \( \Lambda \) on \( n \) and \( \kappa \) were arrived at empirically.

Equations expressing \( Q_e \) as a linear function of \( \rho \) over each of the eight \( \rho \)-intervals were constructed from the values of \( Q_e \) at the end points of the intervals. The resulting linear approximation formulas for \( Q_e \) are as follows:

\[
\begin{align*}
Q_e &= \frac{C_1(0.75 - x_1)}{1.05 - 1.4x_1 - 0.75x_2 + x_1x_2}, \\
& \quad \text{if } x_1 \leq \rho < 0.75 \quad (7a) \\
Q_e &= \frac{-C_1x_2}{1.4 - x_2}, \\
& \quad \text{if } 0.75 \leq \rho < 1.4 \quad (7b) \\
Q_e &= \frac{C_1 + (C - C_1)(\rho - 1.4)}{10}, \\
& \quad \text{if } 1.4 \leq \rho < 2.4 \quad (7c) \\
Q_e &= \frac{C + (\Lambda - C)(\rho - 2.4)}{11}, \\
& \quad \text{if } 2.4 \leq \rho < 3.5 \quad (7d) \\
Q_e &= \frac{\Lambda}{1.4 - \rho}, \\
& \quad \text{if } \rho \geq 7 \quad (7e)
\end{align*}
\]

These formulas are valid for \( n \) and \( \kappa \) in the range

\[
1.2 \leq n \leq 1.65, \\
0 \leq \kappa \leq 0.15
\]

and are not meant to be applied outside these limits. These limits encompass most measured values of the real and imaginary index of refraction of average atmospheric particulate matter in the 1–11 \( \mu \)m wavelength region. However, there are a few notable exceptions. Water possesses values of \( n \) and \( \kappa \) outside these limits in the 2.7–3.1 \( \mu \)m wavelength band (Hale and Querry, 1973) and values of \( \kappa \) in excess of 0.15 have been reported for desert aerosol (Fischer, 1975) in the 9–11 \( \mu \)m band.

The dashed curves in Fig. 2 were computed from Eqs. (7a)–(7h) for the indicated values of \( m \). It is seen that the approximation formulas for \( Q_e \) possess a fairly high degree of accuracy for \( \rho \leq 7 \), which was our stated objective. The accuracy of these formulas in the region \( \rho > 7 \) is sufficient to yield an analytical formula for \( k_e \), which is accurate to within 5% since the contribution to \( k_e \) from particles with \( \rho > 7 \) can be expected to be less than 30% (and as little as 1%, depending on the value of \( m \)) and since the exact curve of \( Q_e \) oscillates around the constant approximation value of \( Q_e \) so that positive and negative errors in \( Q_e \) tend to cancel each other in the process of integrating.

4. **Analytical approximation formula for \( k_e \)**

The Junge size distribution is normally valid over a limited portion of the size range, say from \( r_{\text{min}} \) to \( r_{\text{max}} \). Typically, \( r_{\text{min}} \) lies in the range 0.07–0.10 \( \mu \)m, while the upper limit \( r_{\text{max}} \) is in the range 5–15 \( \mu \)m (Junge, 1963). Strictly speaking, therefore, the extinction coefficient for the Junge distribution should be expressed

\[
k_e = \pi A \int_{r_{\text{min}}}^{r_{\text{max}}} r^{2-n} Q_e(m, \alpha) \, dr,
\]

or, in terms of \( \rho \), as

\[
k_e = \pi A k^{\rho-\eta} \int_{\rho_{\text{min}}}^{\rho_{\text{max}}} \rho^{2-n} Q_e(m, \rho) \, d\rho,
\]

where \( \rho_{\text{min}} = 4\pi(n-1)r_{\text{min}}/\lambda \) and \( \rho_{\text{max}} = 4\pi(n-1)r_{\text{max}}/\lambda \). Since a continental aerosol size distribution usually peaks at a value of \( r \) just below \( r_{\text{min}} \), since the value of \( Q_e \) is quite small for particles smaller than \( r_{\text{min}} = 0.07 \) \( \mu \)m and since the number of giant particles with radius \( > r_{\text{max}} \) is so low, the total extinction coefficient of a continental aerosol is normally not much larger than that computed from Eq. (9) for the “straight-line” region alone.

At this point we express the integral in (9) as a sum of eight integrals, the limits of integration chosen to coincide with the end points of the eight intervals of
\[ k_e = \pi Ak^{3-\gamma} \times 10^{-8} \left[ \int_{\rho_{\text{min}}^*}^{0.75} \rho^{\mu - \gamma} Q_d \rho^\prime d\rho + \int_{0.75}^{1.4} \rho^{\mu - \gamma} Q_d \rho^\prime d\rho + \ldots + \int_{\rho_{\text{max}}^*}^{\rho_{\text{max}}} \rho^{\mu - \gamma} Q_d \rho^\prime d\rho \right], \]  
where \( \rho_{\text{min}}^* \) is defined as \( \rho_{\text{min}} + 16(0.16 - \kappa)^{2.15}(\lambda/11)^{0.3} \), and where
\[ \chi_1 \leq \rho_{\text{min}}^* \leq 0.75, \]
\[ \rho_{\text{max}} \geq 7. \]

If we now perform the integration in (10) using the newly derived approximation formulas for \( Q_d \) given by Eqs. (7a)–(7h) we obtain, after some rearrangement,
\[ k_e = \pi Ak^{3-\gamma} 10^{-8} \sum_{i=1}^{9} F_i, \]  
where
\[ f(\nu) = 12 - 7\nu + \nu^2 \]
and the values of \( F_i \) are as follows:
\[ F_1 = C_1(0.75 - X_2)B_1^{-1} [X_3 \rho_{\text{min}}^{\mu - 1} (4 - \nu) - (3 - \nu)]^{4 - \nu}_{\rho_{\text{min}}} \]
\[ = D_1 [X_3 \rho_{\text{min}}^{\mu - 1} (4 - \nu) - (3 - \nu)]^{4 - \nu}_{\rho_{\text{min}}} \]
\[ F_2 = C_2(X_2 - X_3)B_1^{-1} (0.75)^{4 - \nu} = D_2 (0.75)^{4 - \nu} \]
\[ F_3 = \left[ C - C \left(1 + \frac{1}{1.4 - X_2}\right) \right]^{1.4 - \nu} = D_3 (1.4)^{4 - \nu} \]
\[ F_4 = \left( C_1 + \frac{21}{11} - C \right) 2.4^{4 - \nu} = D_4 (2.4)^{4 - \nu} \]
\[ F_5 = \left( C_1 + \frac{21}{11} - C \right) 2.4^{4 - \nu} = D_5 (2.4)^{4 - \nu} \]
\[ F_6 = \frac{(C - \Lambda)3.5^{4 - \nu} - D_3 3.5^{4 - \nu}}{11} \]
\[ F_7 = \frac{(C - \Lambda)4.5^{4 - \nu} - D_4 4.5^{4 - \nu}}{11} \]
\[ F_8 = \frac{(C - \Lambda)6.5^{4 - \nu} - D_5 6.5^{4 - \nu}}{11} \]
\[ F_9 = \frac{(C - \Lambda)7.5^{4 - \nu} - D_6 7.5^{4 - \nu}}{11} \]
\[ F_0 = \frac{7C - \Lambda}{6} (4 - \nu)^{3 - \gamma} \rho_{\text{max}}^3 = D_9 (4 - \nu)^{3 - \gamma} \rho_{\text{max}}^3. \]

In these expressions \( B_1 = 1.05 - 1.4X_2 - 0.75X_3 + X_1X_2 \) and \( X_1, X_2, C_1, C_1, \) and \( \Lambda \) are as defined earlier.

When \( \nu = 4 \), \( \sum_{i=1}^{9} F_i \) and \( f(\nu) \) both assume a value of zero, so that \( k_e \) has the indefinite value of 0/0. We evaluate \( k_e \) in the limit \( \nu \to 4 \) by applying L'Hopital's law. Thus, denoting \( \sum_{i=1}^{9} F_i \) by \( G(\nu) \) we have for the value of \( k_e \) when \( \nu = 4 \)
\[ k_e = \pi Ak^{3-\gamma} 10^{-8} \lim_{\nu \to 4} \frac{G(\nu)}{f(\nu)}, \]
where the prime denotes differentiation with respect to \( \nu \). Differentiating the functions \( G(\nu) \) and \( f(\nu) \) and evaluating the resulting limit gives
\[ k_e = -\pi Ak^{3-\gamma} 10^{-8} \left[ D_1 (\ln \rho_{\text{min}}^* + \chi_3 \rho_{\text{min}}^{\mu - 1} - 1) - 0.288D_2 + 0.336D_3 + 0.875D_4 + 1.253D_5 + 1.504D_6 + 1.872D_7 - 1.946D_8 + D_{\rho_{\text{max}}}^{\prime \prime \prime \prime} \right], \]

where the \( D \)'s are defined by Eqs. (12a)–(12i). We note from Eqs. (12e)–(12h) that
\[ D_6 = 0.55D_6, \]
\[ D_7 = -0.183D_6, \]
\[ D_8 = -0.366D_6, \]
so that (14) can be simplified to
\[ k_e = -\pi Ak^{3-\gamma} 10^{-8} \left[ D_1 (\ln \rho_{\text{min}}^* + \chi_3 \rho_{\text{min}}^{\mu - 1} - 1) - 0.288D_2 + 0.336D_3 + 0.875D_4 + 1.253D_5 + 1.504D_6 + 1.872D_7 - 1.946D_8 + D_{\rho_{\text{max}}}^{\prime \prime \prime \prime} \right], \]

Eq. (11) applies when \( \rho_{\text{max}} \geq 7 \) and \( \chi_1 \leq \rho_{\text{min}} \leq 0.75 \). Depending on the index of refraction, the value chosen for \( r_{\text{max}} \) and the wavelength of interest, \( \rho_{\text{max}} \) may be \( < 7 \). If the value of \( \rho_{\text{max}} \) is between 6.5 and 7.0 (and \( \chi_1 \leq \rho_{\text{min}} \leq 0.75 \)), then \( k_e \) is given by
\[ k_e = \pi Ak^{3-\gamma} 10^{-8} \left[ \frac{1}{f(\nu)} \sum_{i=1}^{7} F_i + \left(\frac{7/6(A - C)}{3 - \nu} \right) \rho_{\text{max}}^{3 - \gamma} + \left(\frac{1/3(A - C)}{4 - \nu} \right) \rho_{\text{max}}^{4 - \nu} \right]. \]

If \( 4.5 \leq \rho_{\text{max}} \leq 6.5 \) (and \( \chi_1 \leq \rho_{\text{min}} \leq 0.75 \)), then \( k_e \) is given by
\[ k_e = \pi Ak^{3-\gamma} 10^{-8} \left[ \frac{1}{f(\nu)} \sum_{i=1}^{6} F_i + \left(\frac{3.25A - 2.25C}{3 - \nu} \right) \rho_{\text{max}}^{3 - \gamma} + \left(\frac{1/3(A - C)}{4 - \nu} \right) \rho_{\text{max}}^{4 - \nu} \right]. \]

If \( 3.5 < \rho_{\text{max}} \leq 4.5 \), then \( k_e \) is given by
\[ k_e = \pi Ak^{3-\gamma} 10^{-8} \left[ \frac{1}{f(\nu)} \sum_{i=1}^{5} F_i + \frac{A}{3 - \nu} \rho_{\text{max}}^{3 - \gamma} \right]. \]

Also, if \( 2.4 < \rho_{\text{max}} \leq 3.5 \), then \( k_e \) is given by
\[ k_e = \pi Ak^{3-\gamma} 10^{-8} \left[ \frac{1}{f(\nu)} \sum_{i=1}^{4} F_i + \left(\frac{3.18C - 2.18A}{3 - \nu} \right) \rho_{\text{max}}^{3 - \gamma} + \left(\frac{10/11}(A - C) \right) \rho_{\text{max}}^{4 - \nu} \right]. \]

Finally, if \( 1.4 < \rho_{\text{max}} \leq 2.4 \), then the approximation
It is possible that at times the range of validity of the Junde size distribution is such that the value of \( r_{\text{min}} \) results in a value of \( \rho_{\text{min}} > 0.75 \). If \( 0.75 < \rho_{\text{min}} \leq 1.4 \), then \( k_e \) is given by the appropriate one of Eqs. (11) or (16a)–(16e) [depending on the value of \( \rho_{\text{max}} \)] with \( F_1 = 0 \) and \( F_2 \) given by

\[
F_2 = \frac{\lambda_2 \rho_1 (4 - \nu) - C_1 (3 - \nu)}{(1.4 - \lambda_2) \rho_1} \left[ \begin{array}{c} \lambda_2 \rho_1 - C_0 A_\lambda \\ 1.4 - \lambda_2 \end{array} \right].
\]

Fig. 3 shows a comparison between values of the extinction coefficient calculated from the approximation formula [Eq. (11) in this case] and those calculated from Eq. (8) using the exact Mie values of \( Q_e \). The values of \( k_e \) are plotted as a function of the real part of the index of refraction for two sets of values of \( \kappa \), \( \nu \) and \( \lambda \). For the purpose of these calculations we used \( r_{\text{min}} = 0.07 \, \mu \text{m}, r_{\text{max}} = 15 \, \mu \text{m} \) and \( A = 1.087 \). In numerically evaluating the integral in (8) the region from \( r = 0.07 \) to \( r = 15 \, \mu \text{m} \) was divided into 68 discrete radius values. We see from Fig. 3 that the exact values of \( k_e \) and those calculated from the approximation formula are in excellent agreement.

Figs. 4 and 5 present the results of a detailed calculation of the relative error in the value of \( k_e \) computed from the approximation formula. Relative error is here defined as the difference between the approximation value of \( k_e \) and the exact value of \( k_e \) all divided by the exact value of \( k_e \). The relative error is plotted as a function of the slope of the Junde distribution for four different values of \( n \). Fig. 4 is for the case when

\[
k_e = \pi A \kappa_b \nu \sum_{i=1}^{3} \frac{F_i \left( \frac{2.4 C_1 - 1.4 C_0}{3 - \nu} \rho_{\text{max}}^2 - \rho_{\text{max}}^4 \left( \frac{C - C_0}{4 - \nu} \rho_{\text{max}}^2 \right) \right)}{f(\nu)}\times 10^{-3}.
\]

Fig. 4. Relative error in the values of \( k_e \) calculated from the approximation formulas as a function of \( \nu \) and \( n \) for \( \lambda = 1 \, \mu \text{m} \) and \( \kappa = 0 \).
5. Dependence for \( \kappa \) on relative humidity

We can extend the above results to yield an analytical formula for the extinction coefficient as a function of relative humidity.

Let the dry aerosol size distribution be given by

\[
\frac{dN(r_d)}{dr_d} = cr_d^{-\gamma},
\]

(18)

where \( r_d \) is the dry particle radius and \( N(r_d) \) the number of particles per cubic centimeter which have a radius \( \leq r_d \). We consider a chemically homogeneous aerosol. For such an aerosol, a one-to-one correspondence exists between the dry particle radius and the radius \( r \) of the particle in humid air.

The relationship between particle size in humid air and the particle size in dry air is usually expressed by an equation of the form

\[
\frac{r}{r_d} = \Phi(S),
\]

(19)

where \( S \) is the saturation ratio. This relationship implies that all particle sizes experience the same fractional change in radius for a given change in relative humidity. Using a combination of theoretical reasoning and an analysis of observational data on the mass of water uptake of a given mass of dry aerosol as a function of relative humidity, Winkler (1973) found that the growth of polluted aerosols of central Europe could be described using \( \Phi(S) \) given by

\[
\Phi(S) = \left[1 + \frac{0.084F(S)}{1 - S + \phi(S)}\right],
\]

(20)

where \( F(S) \) is the fraction of soluble material which is dissolved at humidity \( S \) and \( \phi(S) \) is a correction term which allows for deviations from the dilute solution approximation. When \( S \geq 0.8 \), \( F(S) = 1 \), \( \phi(S) = 0 \) and Eq. (20) reduces to

\[
\Phi(S) = \left[1 + \frac{0.084}{1 - S}\right].
\]

(21)

Hanel (1976) developed the following expression for \( \Phi(S) \) to describe the behavior of natural aerosol particles:

\[
\Phi(S) = \left[1 + \mu(S)\frac{\rho_d}{\rho_w} S\right]\]

(22)

where \( \mu(S) \) is called the mass increase coefficient and \( \rho_d \) and \( \rho_w \) are the density of the dry particle and of
water, respectively. Hanel gives \( \mu(S) \) as a function of \( S \) for both continental and maritime aerosols.

For the case of partly soluble particles in which the soluble material is pure ammonium sulfate, Fitzgerald (1975) derived the expression

\[
\Phi(S) = 1.2 \left[ 1 - k_1(1 - e) - k_2(1 - e^2) \right] \exp \left( \frac{0.066S}{1.058 - S} \right),
\]

(23)

where \( \epsilon \) is the mass fraction of soluble material in the particle and \( k_1 \) and \( k_2 \) are functions of \( S \). This equation is valid for \( 0.81 \leq S \leq 0.95 \).

From Eqs. (18) and (19) we have

\[
\frac{dN}{dr} = c\Phi^{-1}r^4.
\]

(24)

Combining Eqs. (9) and (24) we have for \( k_e \) (units: \( \text{cm}^{-1} \))

\[
k_e = ck^{-2}n\Phi^{-1} \times 10^{-8} \int_{\rho_{\text{min}}}^{\rho_{\text{max}}} \rho^2 \gamma Q_\rho(m, \rho) d\rho,
\]

(25)

where \( c \) is defined by Eq. (18) and

\[
\rho_{\text{min}} = 4\pi(n-1)\Phi(S)r_{\text{dmin}}/\lambda,
\]

\[
\rho_{\text{max}} = 4\pi(n-1)\Phi(S)r_{\text{dmax}}/\lambda,
\]

(26)

where \( r_{\text{dmin}} \) and \( r_{\text{dmax}} \) are the limits of dry particle radius over which Eq. (18) is valid.

Analytical approximation formulas for \( k_e \) as a function of \( S \) for an aerosol having a Junge size distribution are given by Eqs. (11) and (16a)–(16e) (the appropriate equation to use being determined by the values of \( \rho_{\text{min}} \) and \( \rho_{\text{max}} \)) with \( A \) replaced by \( c\Phi^{-1}r^4 \) and \( \rho_{\text{min}} \) and \( \rho_{\text{max}} \) given by Eq. (26). For a complete description of the dependence of \( k_e \) on relative humidity it is necessary to take into account the variation of the complex index of refraction with relative humidity. Hanel (1968) has proposed that the dependence of the complex index of refraction on relative humidity can be described by

\[
n(S) = n_w + (n_d - n_w) \left( \frac{r_d}{r} \right)^3,
\]

\[
k(S) = k_w + (k_d - k_w) \left( \frac{r_d}{r} \right)^2,
\]

(27)

where the subscripts \( d \) and \( w \) refer to the dry aerosol and water, respectively. With the relationship between particle size in dry and humid air being given by Eq. (19), Eq. (27) becomes

\[
n(S) = n_w + (n_d - n_w) \Phi^{-3},
\]

\[
k(S) = k_w + (k_d - k_w) \Phi^{-2},
\]

(28)

Fig. 6 shows the results of a calculation of \( k_e \) as a function of relative humidity. It was assumed that the growth of continental aerosol particles with relative humidity obeys Eq. (20). In this example we chose \( \nu = 4, \epsilon = 1, \lambda = 7 \mu\text{m}, n_d = 1.4, \epsilon_d = 0.06 \). These
values of $n_d$ and $\kappa_d$ represent an average of the refractive index values of water soluble material and fine dust reported by Volz (1972). The material analyzed by Volz was obtained from continental rainwater. For water at 7 $\mu$m wavelength, Hale and Querry (1973) give $n_w=1.317$ and $\kappa_w=0.0322$. If we compare the variation of $k_e$ with $S$ as shown in Fig. 6 with the dependence of particle size on $S$ as expressed by (19) and (20), then we find that $k_e$ is roughly proportional to $\Phi^{2.55}$.

6. Summary

The exact Mie curves for the extinction efficiency factor $Q_e$ as a function of the normalized size parameter $\rho=2\alpha(n-1)$ are approximated by eight continuous straight-line segments. The approximation formulas expressing $Q_e$ as a linear function of $\rho$ in each of the eight intervals of $\rho$ serve as the basis for the derivation of an analytical expression for the extinction coefficient $k_e$, at infrared wavelengths, of an aerosol having a Junge size distribution described by $dN/dr=Ar^{-\nu}$. This formula gives $k_e$ as an explicit function of $\nu$, wavelength $\lambda$, and the real part $n$ and the imaginary part $\kappa$ of the index of refraction. A formula for $k_e$ as a function of relative humidity is also derived based on the assumption that particle size $r$ in humid air is related to particle size $r_d$ in dry air by an expression of the form $r/r_d=\Phi(S)$, where $S$ is the saturation ratio.

The approximation formulas for $k_e$ are valid for $\lambda$, $\nu$, $n$ and $\kappa$ in the ranges $1 \mu$m $\leq \lambda \leq 11 \mu$m, $3.6 \leq \nu \leq 4.4$, $1.2 \leq n \leq 1.65$ and $0 \leq \kappa \leq 0.15$. For these ranges of values of the parameters, the values of $k_e$ calculated from the approximation formulas agree to within 5% with those obtained by means of numerical integration using the exact Mie theory formulas for $Q_e$. The use of the approximation formulas presented here can reduce the cost and labor of computing values of $k_e$.

REFERENCES


Hanel, G., 1968: The real part of the mean complex refractive index and the mean density of samples of atmospheric aerosol particles. Tellus, 20, 371–379.


