Lanczos Filtering in One and Two Dimensions

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ABSTRACT

A Fourier method of filtering digital data called Lanczos filtering is described. Its principal feature is the use of "sigma factors" which significantly reduce the amplitude of the Gibbs oscillation. A pair of graphs is developed that can be used to determine filter response quality given the number of weights and the value of the cutoff frequency, the only two inputs required by the method. Examples of response functions in one and two dimensions are given and comparisons are made with response functions from other filters. The simplicity of calculating the weights and the adequate response make Lanczos filtering an attractive filtering method.

1. Introduction

The general purpose of filtering time series is to predictably alter the Fourier amplitudes that describe the series. This is accomplished by modifying a given data sequence with a set of weights, called the filter weight function, to produce a new data sequence. The filter weight function is related to the variation with frequency of the ratio of the Fourier amplitude of the modified data sequence to that of the given data sequence. The latter function is called the filter response function.

In this paper the filter response function is expressed as an infinite Fourier series so that the weights become the Fourier coefficients. In practice, a finite or truncated Fourier series is used, with the result that if a response function with a step change in response were desired, the computed response function would exhibit an oscillation called the Gibbs phenomenon. The fewer the number of weights the larger the oscillation. Lanczos (1956, p. 219) showed that the error in a truncated Fourier series has the form of a "modulated carrier wave." The carrier frequency is equal to the frequency of the first term neglected and its amplitude contributes significantly to the amplitude of the Gibbs oscillation. Therefore, as Lanczos proposed, one should filter out the carrier frequency. This can be done conveniently by convolving a rectangular function, whose width is the period (in the frequency domain) of the carrier wave, with the desired response function. The Fourier coefficients for the smoothed response function are determined by multiplying the original weight function by a function that Lanczos called the "sigma factor." Because of the key role that Lanczos played in the development of this method of filtering it is called Lanczos filtering.

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The objectives of this paper are to demonstrate the simplicity of Lanczos filtering, to develop graphs that can be used to predict the main characteristics of the response function, to compare Lanczos response functions to those from other types of filters, and to extend the analysis to two dimensions.

2. Mathematical formulation

Digital filtering involves transforming an input data sequence x_t, where t is time, into an output data sequence y_t using the linear relationship

$$y_t = \sum_{k=-\infty}^{\infty} w_k x_{t-k}, \qquad (1)$$

in which the w_k are suitably chosen weights. The effect of filtering the data is best observed in the frequency domain. The relationship between the input and output Fourier amplitude density functions $X(f)$ and $Y(f)$, where f is frequency, is obtained by taking the Fourier transform of (1). The result is [see a text on linear systems analysis, e.g., Jenkins and Watts (1968)]

$$Y(f) = R(f) \cdot X(f), \qquad (2)$$

where $R(f)$ is the frequency response function. The weight function and response function comprise a Fourier series transform pair such that

$$R(f) = \sum_{k=-\infty}^{\infty} w_k \exp(i2\pi fk\Delta), \qquad (3)$$

$$w_k = \frac{1}{2f_N}\int_{-f_N}^{f_N} R(f)\exp(-i2\pi fk\Delta)df,$$

$$k = \ldots, -1, 0, 1, \ldots, \qquad (4)$$

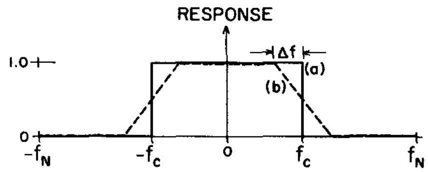

FIG. 1. Curve (a) is an ideal low-pass response function with cutoff frequency f_c. Curve (b) is the smoothed ideal response function given by Eq. (6) and whose transition band is $2\Delta f = 2f_N/n$.

where Δ is the sampling interval and f_N the Nyquist frequency with value $\frac{1}{2}$ cycle per data interval.

In practice, what is done is to first decide on the form of the response function, then determine the weight function but limit the number of weights in consideration of the length of the data sequence, and finally use (3) to calculate the actual response due to using a finite number of weights. Since an ideal filter is being considered, the first step is to decide the value of the cutoff frequency f_c, i.e., the frequency at which the response drops from one to zero as shown by curve (a) in Fig. 1. If there is a total of $2n-1$ weights in the weight function, then, following Lanczos' suggestion, in order to suppress the Gibbs oscillation the ideal response function is convolved with the rectangular function

$$h(f) = \begin{cases} n/2f_N, & |f| \leqslant f_N/n \\ 0, & |f| > f_N/n. \end{cases} \quad (5)$$

The convolution here is the same as averaging so that the smoothed version of $R(f)$ is

$$\bar{R}(f) = (n/2f_N) \int_{-f_N/n}^{f_N/n} R(f+v)dv, \quad (6)$$

as illustrated by curve (b) in Fig. 1.

If $R_n(f)$ represents the partial sum of the Fourier series obtained by replacing the infinite limits in (3) by the finite limits $-(n-1)$ and $(n-1)$, then the partial sum of the Fourier series of $\bar{R}(f)$ can be written

$$\bar{R}_n(f) = \sum_{k=-n}^{n} \bar{w}_k \exp(i2\pi fk\Delta)$$

$$= \bar{w}_0 + 2\sum_{k=1}^{n} \bar{w}_k \cos 2\pi fk\Delta, \quad (7)$$

where, by analogy with (4),

$$\bar{w}_k = \frac{1}{2f_N} \int_{-k_N}^{f_N} \bar{R}(f) \exp(-i2\pi fk\Delta)df. \quad (8)$$

Substituting (6) into (8) and recognizing that $R(f)$ is periodic results in

$$\bar{w}_k = w_k \frac{\sin 2\pi k f_N \Delta/n}{2\pi k f_N \Delta/n},$$

which for unit sampling interval becomes

$$\bar{w}_k = \frac{\sin 2\pi f_c k}{\pi k} \frac{\sin \pi k/n}{\pi k/n}, \quad k = -n, \ldots, 0, \ldots, n. \quad (9)$$

Thus, it can be seen that the truncated weight function for the smoothed response is the product of that for the ideal filter and a $\sin X/X$ term denoted by σ and called the "sigma factor" by Lanczos.

Now in (7) the logical limits of the summation would have been $-(n-1)$ and $(n-1)$ in parallel with those for $R_n(f)$. However, from (9) it can be seen that $\bar{w}_{\pm n} = 0$, so that for convenience $-n$ and n were used. Consequently, if from data length considerations the number of weights is constrained to $2p+1$, it will be advantageous to let $n = p+1$.

Before proceeding it is possible to anticipate a general feature of the Lanczos response function by recalling a property of the Gibbs oscillation. Even as the number of terms in the Fourier series representation of a step function becomes very large, its amplitude has a lower bound of 9% (Hsu, 1970). The Lanczos filter smooths the discontinuity (as in Fig. 1) with the consequence that the above property is again present except that the amplitude of the Gibbs oscillation is significantly reduced.

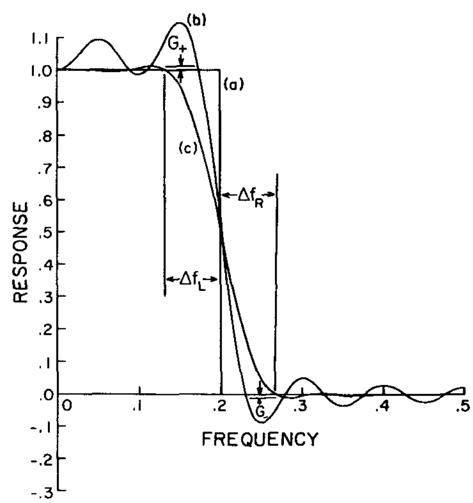

FIG. 2. Curve (a) is an ideal low-pass response function. Curve (b) is the observed response function using 21 weights. Curve (c) is the result of Lanczos filtering wherein the weight function is multiplied by a sigma factor. The other symbols are defined in the text.

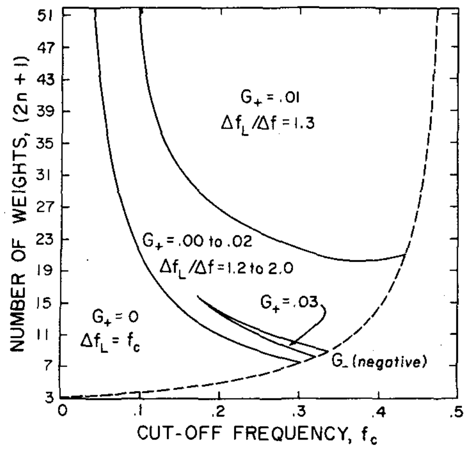

FIG. 3. The magnitudes of the maximum positive Gibbs oscillation (G_+) and left bandwidth ratio ($\Delta f_L/\Delta f$) as a function of the number of weights and cutoff frequency for Lanczos filtering. In the G_- (negative) region the response function never passes through zero.

3. Results

a. Low-pass filtering

Curve (a) in Fig. 2 is an ideal response function in which the cutoff frequency f_c is 0.2 cycles per data interval (from this point on all frequencies will have these same units so that only the numerical value will be given). Curve (b) is the response function computed from (3) for $2n-1=19$ weights whose values were computed from (4) with integral limits -0.2, 0.2. The Gibbs phenomenon is seen to be quite pronounced. Curve (c) is the response function computed from (7) in which the weights are obtained by multiplying the weight function above by the sigma factor. For curve (c) there are $2n+1=21$ weights but from Eq. (9) the two end weights are zero. The advantage of using the sigma factor is clearly evident in the reduced Gibbs phenomenon. At the same time the width of the transition band, i.e., the frequency interval between the nearest unit and zero responses about f_c, increases. The wider band in curve (c) corresponds to the transition band $2\Delta f$ in Fig. 1. One can anticipate that the use of a sigma factor to a power greater than 1 will result in further suppression of the Gibbs phenomenon coincident with an even wider transition band.

The properties of an observed Lanczos response function are completely determined by the ideal cutoff frequency f_c and the number of weights $2n+1$ and can be presented graphically. The properties of interest are as follows:

1) $G_+(G_-)$: the maximum value of the Gibbs oscillation for frequencies lower (higher) than f_c (see Fig. 2).

2) $\Delta f_L/\Delta f(\Delta f_R/\Delta f)$: the ratio of the bandwidth between f_c and the frequency of the nearest unit (zero) response of the observed response function to that of the smoothed response function (see Figs. 2 and 1).

Fig. 3 shows the response properties for frequencies $<f_c$. These results were obtained from computations of a dense matrix of cutoff frequencies and numbers of weights. There are four distinguishable regions:

1) Between the inner solid curve and the dashed curve to the right the bandwidth ratio is 1.3 and maximum Gibbs oscillation is 0.01 (a 1% error).

2) Between the inner and outer solid curves and the dashed curve to the right the bandwidth ratio varies from 1.2–2.0, the latter value occurring coincidentally with the outer solid curve. The maximum Gibbs oscillation varies from 0–0.02 except within the curved wedge-shaped region where it is 0.03. The zero value coincides with the outer solid curve.

3) Within the outer solid curve, the dashed line and the left coordinate axis the Gibbs oscillation is zero. The response is unity at the origin and decreases with increasing frequency.

4) Outside the dashed curve (in Fig. 4, also) the response function never passes through zero response. The use of a response function for a combination of number of weights and cutoff frequency that is in this region is not recommended.

Fig. 4 shows the response characteristics for frequencies higher than f_c. The regions of interest are as follows:

1) Within the inner curve the bandwidth ratio is 1.3 and the maximum Gibbs oscillation is 0.01 (a 1% error).

2) In the area bounded by the inner solid curve, the outer solid curve on the left and the dashed curve

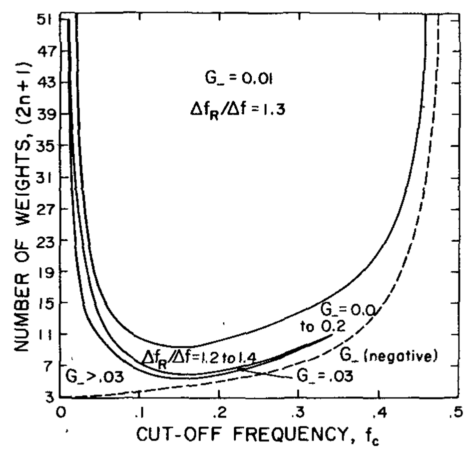

FIG. 4. The magnitudes of the maximum negative Gibbs oscillation (G_-) and right bandwidth ratio ($\Delta f_R/\Delta f$) as a function of the number of weights and cutoff frequency for Lanczos filtering. In the G_- (negative) region the response function never passes through zero.

on the right beginning at $f_c \approx 0.2$, the bandwidth ratio varies from 1.2–1.4 and the maximum Gibbs oscillation from 0–0.02, except in the crescent-shaped region where the latter quantity is 0.03. The zero value of the Gibbs oscillation coincides with the dashed curve.

3) In the remaining area above the dashed curve the maximum Gibbs oscillation is greater than 0.03 and the bandwidth ratio can be greater than 1.4 so that the use of a response function for a combination of number of weights and cutoff frequency that is in the region is not recommended.

As an illustration of the application of these graphs we consider the problem of assessing how the response function changes when the number of weights varies and $f_c = 0.15$. With 31 weights the maximum Gibbs oscillation will be 0.01 and the bandwidth ratio will be 1.3. Increasing the number of weights to 51 will not change appreciably either the magnitude of the Gibbs oscillation or the bandwidth ratios. On the other hand, the width of the transition band $\Delta f_L + \Delta f_R$ decreases since $\Delta f = 1/(2n)$.

If between 15 and 31 weights are chosen G_+ lies between 0 and 0.02 and $\Delta f_L/\Delta f$ between 1.2 and 2.0, the exact values determined by the actual number of weights. There is no change in G_- or $\Delta f_R/\Delta f$ until the number of weights is less than 11. From 5–15 weights $G_+ = 0$ and $\Delta f_L = 0.15$. From 5–9 weights G_- lies between zero and 0.03, and $\Delta f_R/\Delta f$ between 1.2 and 1.4.

In general, there is comparatively little change in the magnitude of either the Gibbs oscillations or the bandwidth ratios for values of $2n+1$ and f_c above the dashed lines. The most dramatic change is in the transition bandwidth $\Delta f_R + \Delta f_L$, which is approximately inversely proportional to n. Also, Figs. 3 and 4 can be used to reasonably infer the response properties when the total number of weights exceed 51.

b. *High-pass filtering*

The response function for a high-pass filter can be obtained from that for a low-pass filter [Eq. (7)] by subtracting the latter from 1 to get

$$\bar{R}'_n(f) = \bar{w}'_0 + 2\sum_{k=1}^{n} \bar{w}_k \cos 2\pi f k, \qquad (10)$$

where $\bar{w}'_0 = 1 - \bar{w}_0$ and $\bar{w}'_k = -\bar{w}_k$.

Recognizing that (i) $\Delta f_L(\Delta f_R)$ is now the bandwidth between f_c and the frequency of the nearest zero (unit) response, (ii) $G_+(G_-)$ is the maximum value of the Gibbs oscillation below zero (above unit) response, and (iii) if G_- is negative the observed response curve never passes through unit response, then Figs. 3 and 4 can be applied directly to high-pass filters.

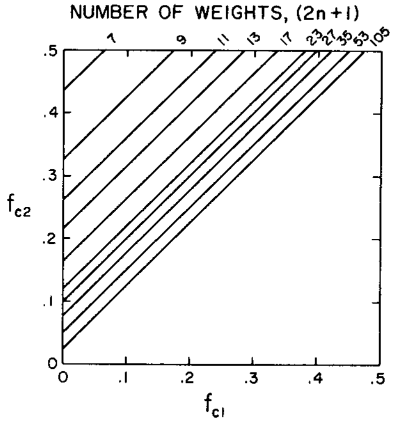

FIG. 5. The minimum number of weights [see Eq. (14)] required to achieve unit response for a Lanczos band-pass filter given the cutoff frequencies f_{c1} and f_{c2}.

c. *Band-pass filtering*

The ideal band-pass filter would show zero response from the frequency origin to the cut-in frequency f_{c1}, unit response from f_{c1} to the cut-out frequency f_{c2}, and zero response from f_{c2} to the Nyquist frequency. From Eq. (8) the smoothed weights would become

$$\bar{w}_k = \left(\frac{\sin 2\pi f_{c2} k}{\pi k} - \frac{\sin 2\pi f_{c1} k}{\pi k}\right)\sigma, \quad k = n, \ldots, 0, \ldots n. \qquad (11)$$

Eq. (11) represents the difference in weight functions for two low-pass filters, with cutoffs f_{c2} and f_{c1}. With this view in mind the shape of the response function can be anticipated by recalling the shapes of low-pass and high-pass filters. Also, if the criterion

$$f_{c2} - \Delta f_{L2} \geqslant f_{c1} + \Delta f_{R1}, \qquad (12)$$

where the Δ terms have meanings similar to those given earlier, is applied to the two low-pass filters then the response at the center of the pass-band will be very close to 1. If the criterion is not met, then the response at the center of the pass-band will be attenuated. Of course, it is desirable to satisfy this criterion. This can be done using the method described below, the result of which relates the number of weights common to both filters to the difference between the given cutoff frequencies, i.e., the ideal pass band. From Eq. (5) and the definitions given in Section 3a,

$$\Delta f_{L2} = K_2 \Delta f = K_2/2n,$$
$$\Delta f_{R1} = K_1 \Delta f = K_1/2n.$$

Their sum is

$$\Delta f_{L2} + \Delta f_{R1} = (K_1 + K_2)/2n.$$

Figs. 3 and 4 each show a large region in which $K_1 = K_2 = 1.3$. For this condition,

$$\Delta f_{L2} + \Delta f_{R1} = 1.3/n. \qquad (13)$$

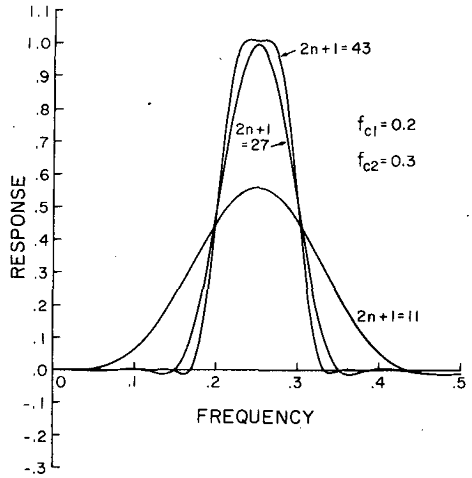

FIG. 6. Lanczos band-pass filters showing the effects on response of using less than and more than the number of weights given in Fig. 5.

Substituting (12) into (13) yields

$$n \geqslant 1.3/(f_{c2}-f_{c1}), \tag{14}$$

the equality portion of which is plotted in Fig. 5. For example, if $f_{c2}=0.3$, $f_{c1}=0.2$ then the minimum total number of weights required to achieve unit response at the band center is 27. The observed response function is shown in Fig. 6 along with those for $2n+1=11$ and 43. The response at the band center ($f=0.25$) for 27 weights is 0.995 and for 43 weights is 1.005. That the maximum amplitude in the pass-band is less than 1 when the number of weights is fewer than 27 is due to the fact that the two terms on the right side of (11) are individually normalized when computing the weight function (in order to insure zero response at the frequency origin).

d. Comparison with other filters

Of course, there are a plethora of digital filters that have been designed and used over the years—a number of them recently discussed by Hamming (1977). In order that a comparison between a Lanczos filter and any other filter be fair there need be only the same number of weights in each. Then the only reason the response functions can differ is the values of the weights.

A commonly used filter is the running mean. As this represents a filter with "poor" response, in contrast, the von Hann or Hanning filter is representative of a filter with "good" response. The weight and response functions for the von Hann filter are, respectively (Hamming, 1977)

$$w_k=\begin{cases}\dfrac{1+\cos\pi k/n}{2}, & |k|\leqslant n\\[2mm] 0, & |k|\geqslant n\end{cases}$$

$$R(f)=\frac{\sin 2\pi nf}{2\tan\pi f}\left[1-\left(\frac{\sin\pi f}{\sin\pi/2n}\right)^2\right]^{-1}, \tag{15}$$

and the response function for the running mean is (Burroughs, 1978)

$$R(f)=\sin(m\pi f)/m\sin\pi f, \tag{16}$$

where m is the total number (odd) of weights.

The comparison among the above response functions and the response function for Lanczos filtering [Eq. (7)] is shown in Fig. 7 where each filter has 11 nonzero weights (so $m=11$ and $n=6$) and the cutoff frequency for the Lanczos filter is 0.065. The figure shows that the running mean has the first zero crossing (or tangent) at $f=0.09$ but that it has much larger side lobes than the other two response functions. Clearly, one pays a heavy price in the frequency domain for its simplicity in the time domain. In contrast, there are only minor differences between the von Hann and Lanczos responses. The advantage of the Lanczos filter over the von Hann filter is that the cutoff frequency can be controlled independently of the number of weights (Figs. 3 and 4).

As a second example, we consider a comparison between a Craddock band-pass filter and a Lanczos band-pass filter. The development and application of the former filter is discussed by Craddock (1969, 1965). In brief, "elementary" filters whose response functions are determined exactly by the number of weights are combined to produce a desired response function. It is in the latter procedure that considerable effort can be expended. The values of the 21-weight Craddock filter chosen for comparison are given in a paper by Carrea (1978). The response function is given by the thin curve in Fig. 8 and has a peak value of 1.006 at a frequency of 0.07. The thick curve is the Lanczos response function and was obtained

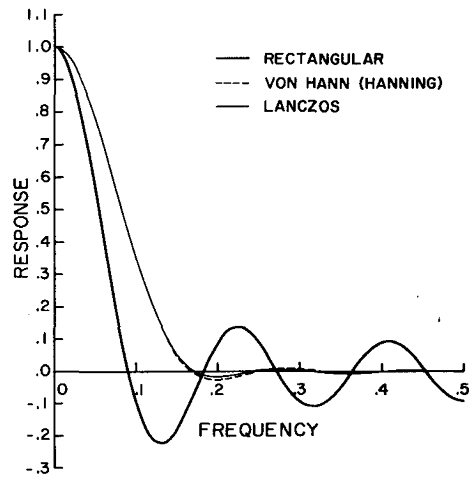

FIG. 7. Comparison among the rectangular (moving average), von Hann and Lanczos filtering in which each uses 11 nonzero weights.

as follows. The difference between the cutoff frequencies $f_{c1}-f_{c2}$ was computed from (14) with $n=11$ (21 nonzero weights). To get the cutoff frequencies f_{c1} and f_{c2} when the band-pass filter has unit response at a single frequency we let $(f_{c1}-f_{c2})/2=0.06=\Delta f_{L2}=\Delta f_{R1}$. Then, since both sides of (12) equal 0.07, the desired central frequency, $f_{c1}=0.01$ and $f_{c2}=0.13$. Using (11) and (7), the resulting Lanczos filter has a peak response of 0.998 at a frequency of 0.08. As seen in Fig. 8 the response functions are quite similar except that the Lanczos filter has a wider passband and smaller side lobes. This example demonstrates that the Lanczos approach to the design of a band-pass filter is quite easy and yields good response characteristics.

4. Two-dimensional filtering

a. Mathematical formulation

Based on the formulation for Lanczos smoothing derived in Section 2, Justice (1976) has developed a generalization to N dimensions. From this generalization or by analogy with (7), the response function for two dimensions becomes

$$\bar{R}_{n_x,n_y}(f_x,f_y)=\sum_{k_x=-n_x}^{n_x}\sum_{k_y=-n_y}^{n_y}\bar{w}_{k_x,k_y}\times\exp[i2\pi(f_xk_x+f_yk_y)], \quad (17)$$

where the smoothed weight function is, by analogy with (9),

$$\bar{w}_{k_x,k_y}=w_{k_x,k_y}\frac{\sin\pi k_x/n_x}{\pi k_x/n_x}\frac{\sin\pi k_y/n_y}{\pi k_y/n_y}=w_{k_x,k_y}\sigma_x\sigma_y. \quad (18)$$

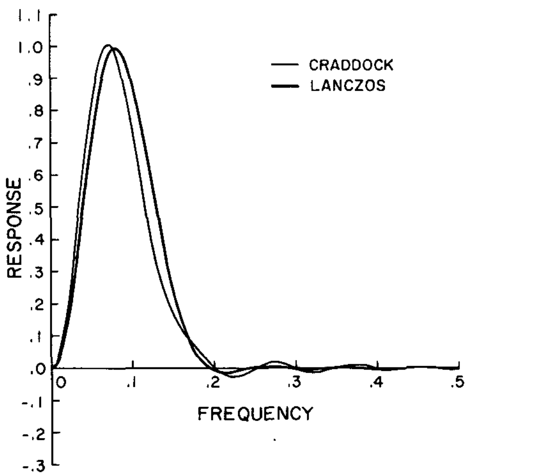

FIG. 8. Comparison between Craddock and Lanczos band-pass filtering in which each uses 21 nonzero weights.

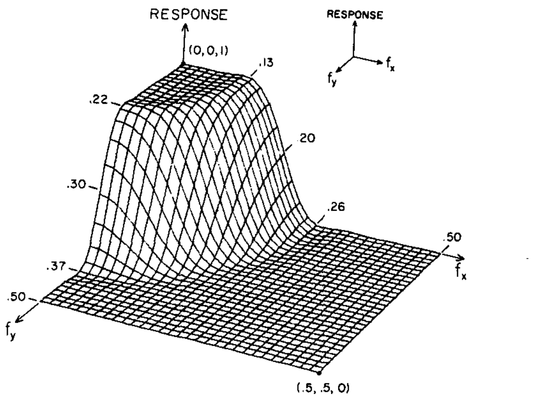

FIG. 9. Response function for a two-dimensional Lanczos filter in which the cutoff frequencies are 0.2 and 0.3 and each direction has 21 weights.

As before, frequency is in cycles per data interval and the data interval is unit length. In accord with previous notation k_x, n_x, f_x, σ_x and k_y, n_y, f_y, σ_y denote the weight number, number of weights, frequency and sigma factor in the orthogonal x and y directions, respectively.

The general relationship between the weight function and response function can be written

$$w(x,y)=\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}R(f_x,f_y)\times\exp[-i2\pi(f_xx+f_yy)]df_xdf_y. \quad (19)$$

This is analogous to (4) except that for convenience in what is to follow the weight numbers have been expressed as continuous variables.

As in Section 2, we let $R(f_x,f_y)$ represent the ideal response function which then will have unit value between the cutoff frequencies in the x and y directions, i.e., from $-f_{c_x}$ to f_{c_x} and from $-f_{c_y}$ to f_{c_y}, and has the shape of an elliptic cylinder.

In order to integrate (19) the following changes of variable are made:

$$f_x'=f_x/f_{c_x}, \qquad f_y'=f_y/f_{c_y},$$
$$x'=f_{c_x}\cdot x, \qquad y'=f_{c_y}\cdot y.$$

The former change of variable transforms the elliptic cylinder into a circular cylinder. As shown by Goodman (1968) the next step is to place the primed variables in cylindrical coordinates and, then, because of the circular symmetry the integration is straightforward. The result of the integration (see previous reference for details) is

$$w(x,y)=f_{c_x}f_{c_y}J_1(2\pi u)/u, \quad (20)$$

where $u=(f_{c_x}^2x^2+f_{c_y}^2y^2)^{\frac{1}{2}}$ and J_1 is a Bessel function of the first kind, order one. After reverting back to weight numbers where

$$x=k_x\Delta x,\quad k_x=0, 1, 2, \ldots, n_x,\quad \Delta x=1,$$
$$y=k_y\Delta y,\quad k_y=0, 1, 2, \ldots, n_y,\quad \Delta y=1,$$

the weight function becomes

$$w_{k_x,k_y}=f_{c_x}f_{c_y}J_1(2\pi z)/z, \tag{21}$$

where

$$z=(f_{c_x}^2k_x^2+f_{c_y}^2k_y^2)^{\frac{1}{2}}.$$

By specifying the two cutoff frequencies f_{c_x} and f_{c_y} and the number of weights n_x and n_y, the values of the weights and the frequency response function for two-dimensional Lanczos filtering can be determined.

b. Example

Fig. 9 shows the response function for a two-dimensional low-pass Lanczos filter in which $f_{c_x}=0.2$, $f_{c_y}=0.3$, $2n_x+1=21$, and $2n_y+1=21$. The magnitude of the Gibbs oscillations and the bandwidth ratios along the RESPONSE-f_x and RESPONSE-f_y planes can be determined directly from Figs. 3 and 4. The response function in the RESPONSE-f_x plane is the same as curve (c) in Fig. 2. The values 0.22 and 0.13, and 0.37 and 0.26 are the frequencies of unit response and zero response, respectively, nearest the cutoff frequencies.

5. Summary

A Fourier method of filtering digital data called Lanczos filtering is presented. The principal feature of the method is the use of "sigma factors" which significantly reduces the amplitude of the Gibbs oscillation. A pair of graphs has been designed that show the magnitude of the Gibbs oscillation and the width of the pass band after Lanczos filtering, given the number of filter weights and the cutoff frequency, the only two inputs required. For low-pass filtering the weight function is given by (9) and the response function by (7). The weight function is easily modified to get high-pass and band-pass filtering.

The methodology is extended to two dimensions and a computer program is available from the author to calculate the weight function (21) and response function (17).

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