The Use of the Weibull Three-Parameter Model for Estimating Mean Wind Power Densities

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ABSTRACT

The Weibull three-parameter model is discussed for estimation of mean wind power densities. This probability density function is a generalization of a number of more conventional density functions. Using wind speed observations, it is shown that this model generally gives a more reliable fit to the empirical wind speed frequency data than the density functions with one or two parameters. Wind power density estimations turn out to be strongly dependent on the hypothesized probability density function. The variation with height of the three parameters of the discussed model is investigated; no simple height dependence can be proposed.

1. Introduction

Wind power studies have appeared more frequently in the scientific literature in recent years, since there is an increased interest in alternative and less expensive energy resources. Previous wind power investigations made in Belgium relied mostly on mean wind speed values at various sites (Malet, 1978); wind power estimates should be based on elaborated wind speed statistics. Several probability density functions (pdf) such as the log-normal, gamma (Pearson-type III) and Rayleigh distributions can be tried *a priori* as possible useful fits to empirical wind speed frequency classes. Justus *et al.* (1976, 1978) advocated for the use of the two-parameter Weibull distribution in wind energy applications. Hennessey (1977) showed that it is a practical model for wind speed statistics and also that it leads to another Weibull two-parameter model for the distribution of the cube of the wind speed.

The Weibull three-parameter density function is a straightforward generalization of the former density curves; the purpose of this paper is to point out that it gives a better fit to empirical wind speed frequency data. A number of coherent and remarkable conclusions can be drawn from the possible values of the parameters of this pdf.

2. The Weibull three-parameter model

The instantaneous wind power contained in a flow of air through a unit surface perpendicular to the air stream is given by

$$P = \tfrac{1}{2}\rho v^3 \quad [\mathrm{W\ m^{-2}}], \tag{1}$$

where v is the wind speed (m s^{-1}) and ρ the density of the air. If the density is supposed to be independent of the wind speed cubed, the statistically expected value of the wind power density is then calculated from

$$\bar{P} = E\{P\} = \tfrac{1}{2}\bar{\rho}E\{v^3\}, \tag{2}$$

with $\bar{\rho}$ the mean air density (1.293 kg m^{-3} at sea level). In this way the mean power density $E\{P\}$ becomes proportional to the third noncentral moment of the wind speed, which is treated as a random variable.

The Weibull two-parameter family of curves is generally accepted as providing adequate probability density functions for the description of wind speed statistics. The reader is referred to the papers of Hennessey (1977) and Justus *et al.* (1976, 1978) for a fairly complete description and application of the Weibull two-parameter model.

The Weibull three-parameter (hypergamma or modified gamma) pdf is a more powerful family of probability density curves; it is mathematically expressed by the form

$$f(x) = \begin{cases} \dfrac{ab^{c/a}}{\Gamma(c/a)} x^{c-1} e^{-bx^a}, & a > 0,\ b > 0,\ c > 0,\ x < 0, \\ 0, & x \geqslant 0, \end{cases} \tag{3}$$

for any random variable X, where Γ is the gamma function, $b^{-1/a}$ is a scale factor which has the same units as the random variable, a and c are shape factors for the pdf. The corresponding cumulative distribution function is given by

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TABLE 1. Probability density functions derivable from Weibull three-parameter pdf.

pdf	Parameters		
Weibull-3	a	b	c
Weibull-2	c	b	c
Gamma	1	b	c
Chi	2	$cb^2/2$	c
Rayleigh	2	$b^2/2$	2
Exponential	1	b	1

$$F(x) = \frac{1}{\Gamma(c/a)} \int_0^{bx^a} t^{(c/a-1)} e^{-t} dt = P(c/a, bx^a), \quad (4)$$

where $P(\cdot,\cdot)$ is the incomplete gamma function (Abramowitz and Stegun, 1966, p. 260). It is easily verified that the expression (3) is identical to the Weibull two-parameter pdf when $a = c$.

Since the Weibull-three pdf is a generalization of more conventional pdf's, it is of interest to summarize in Table 1 the relation between the Weibull-three pdf and some of the probability density functions which have been used with more or less success to fit empirical wind speed distributions. Transformation of Eq. (3) using the expression in Table 1 leads to the standard form of the corresponding pdf.

Insofar as the Weibull-3 model adequately describes the wind speed distribution, the frequency distribution of the wind speed cubed can immediately be obtained. In general, if the random variable X follows a Weibull-3 pdf with parameters (a,b,c), the probability density function of the random variable $Y = X^r$ is another Weibull 3-parameter pdf with parameters $(a/r,b,c/r)$, respectively. The noncentral moment of order r of the Weibull-3 pdf is given by

$$E\{X^r\} = b^{-r/a}\Gamma[(c + r)/a]/\Gamma(c/a). \quad (5)$$

Therefore, the skewness, the kurtosis, the coefficient of variation and the proportionality factors $\gamma_r = E\{X^r\}/E\{X\}^r$ are all functions of the parameters a and c only. This is illustrated in Fig. 1 where the isolines of the factor γ_3, which is important in wind power studies, are shown in the (a,c) plane. When $\gamma_3 = 1$, we apparently can treat the problem of using annual wind speeds and cubing them to calculate the mean annual wind power. It is apparent that $\gamma_3 = 1$ only if the wind speed is constant for the entire year. This never happens, so γ_3 must be greater than unity. Hence, the correction factor needed to convert the cube of the mean annual wind speed to the annual mean of individual speeds cubed can be found in Fig. 1. The positions in the (a,c) plane of the probability density functions of Table 1, relative to the Weibull-3 pdf, are also pointed out. This representation is of great practical interest with regard to a fixed property for different distribution functions. For instance, without calculating the value γ_3 for the Rayleigh distribution, it is obvious that it is a constant which is equal to ~1.9.

Furthermore, for a certain calculated (a,c) combination of the hypothesized Weibull three-parameter pdf, the location of the representative point in the (a,c) plane illustrates the possible goodness of fit of the five other pdf's of Table 1.

Estimates of the Weibull parameters (a,b,c) can be obtained by the maximum likelihood technique. For n data $(x_1, \ldots, x_n)$ of the random variable X, the maximum of the function

$$\ln L = \sum_{i=1}^{n} \ln f(x_i) = n\left[\ln a + \frac{c}{a}\ln b - \ln\Gamma\left(\frac{c}{a}\right)\right] + (c-1)\sum_{i=1}^{n}\ln x_i - b\sum_{i=1}^{n} x_i^a \quad (6)$$

is localized by solving the system of equations

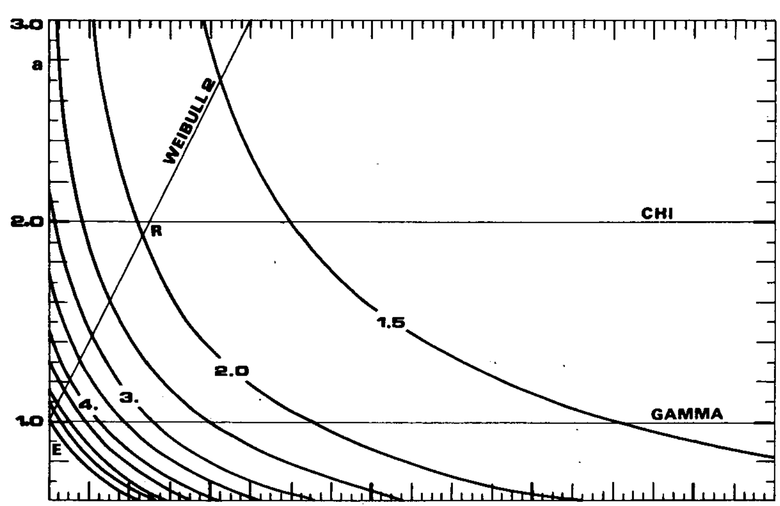

FIG. 1. Isopleths of the factor γ_3 on the (a,c) plane for the Weibull-3 pdf and locations of the derived pdf of Table 1. (R = Rayleigh, E = exponential).

$$\frac{\partial \ln L}{\partial a} = \frac{\partial \ln L}{\partial b} = \frac{\partial \ln L}{\partial c} = 0,$$

yielding the following set of maximum likelihood equations:

$$aT_3 + \ln b - \Psi(c/a) = 0, \quad (7)$$

$$abT_1 - c = 0, \quad (8)$$

$$abT_2 - cT_3 - 1 = 0, \quad (9)$$

with Ψ the digamma function (Abramowitz and Stegun, 1966, p. 258), i.e.,

$$\Psi(x) = \frac{d}{dx} \ln\Gamma(x) \quad (10)$$

and

$$T_1 = n^{-1} \sum_{i=1}^{n} x_i^a, \quad T_2 = n^{-1} \sum_{i=1}^{n} x_i^a \ln x_i,$$

$$T_3 = n^{-1} \sum_{i=1}^{n} \ln x_i. \quad (11)$$

The parameters b and c can be eliminated from (8) and (9) as functions of a

$$b = 1/[a(T_2 - T_1T_3)], \quad c = T_1/(T_2 - T_1T_3). \quad (12)$$

Substitution of (12) into (7) gives the following nonlinear equation for the parameter a:

$$aT_3 - \ln[a(T_2 - T_1T_3)] - \Psi\left[\frac{T_1}{a(T_2 - T_1T_3)}\right] = 0, \quad (13)$$

which can be solved by Mueller's iteration scheme of successive bisection and inverse parabolic interpolation (Kristiansen, 1963). The solution of (13) for the parameter a then gives the estimates of the parameters b and c in (12). It is also remarked that b is completely determined by a and c:

$$b = c/(aT_1). \quad (14)$$

The log-normal pdf with standard form

$$f(x) = \frac{1}{xb\sqrt{2\pi}} \exp[-(\ln x - a)^2/2b^2] \quad (15)$$

is not directly related to the family of pdf's in Table 1; this pdf is also used as a statistical model to estimate mean wind power densities.

3. Description of the data and statistical analysis

To perform the wind power statistical analysis we used the wind speed data from three meteorological towers: Mol, Tihange and Zwijndrecht, the characteristics of which are given in Table 2.

For Tihange the observations for 1971 were missing, as were the data for Zwijndrecht in 1970. The tower data were 10 min mean values given in meters per second, taken at the end of each hour. We also used 3 h wind speed data, measured in knots, at 21 synoptic stations in Belgium.

TABLE 2. Period and level heights used at each tower.

Station	Level (m)						Available period
Mol		24	49	78	114		1966–1976
Tihange	8	24	48	80	130	200	1967–1975
Zwijndrecht			51		154		1968–1975

The hybrid density function, suggested by Takle and Brown (1978),

$$f_H(x) = F_0\delta(x) + (1 - F_0)f(x), \quad (16)$$

where F_0 is the probability of observing zero wind speed, $\delta(x)$ the Dirac delta function and $f(x)$ one of the density functions mentioned in Table 1 or the log-normal pdf, is fitted to the empirical wind speed frequency distributions derived from the data. This means that in fact $f(x)$ is fitted to the empirical frequency classes of the nonzero wind speed data. A few selected results are tabulated in Table 3, showing the calculated parameters for each of the $f(x)$ in Eq. (16), together with two criterions for comparing the accuracy of fit of the empirical data and the corresponding estimated mean wind power density.

As could be expected, it was found that the Weibull 3-parameter pdf almost always gives the best fit to the empirical distributions, at least in the sense of the root-mean-square error $\bar{\epsilon}$. However, regarding the position of the (a,c) points of this pdf in Fig. 1, one of the mentioned pdf's of Table 1 may be slightly better as far as the χ^2 criterion is concerned.

It is evident from Table 3 that the exponential, log-normal and Rayleigh distributions can give completely misleading estimates of the mean power density $\bar{P}$. The optimal pdf's in these four cases are selected from the minimum values of the χ^2 and $\bar{\epsilon}$ criteria and are indicated by the asterisks. In all cases, the Weibull-3 model gives the best fit when compared to the six other pdf's; in particular it is obvious that the Weibull two-parameter distribution may give different estimates as compared to the Weibull-3 pdf, especially when the a and c parameters in the Weibull-3 model are definitely unequal.

These conclusions are confirmed by fitting the Weibull three-parameter pdf to the wind speed data taken from 21 synoptic stations distributed over Belgium. The stations are all equipped with anemometers placed at a height of 20 to 24 m, depending on the heights and the distances of the obstacles surrounding the stations in order to conform to the WMO regulations relative to surface wind observations. Each sample consisted of 25000 synoptic 3 h wind speed data, covering about 10 years of observations. In Fig. 2 one remarks the relatively large deviations of the (a,c) combinations from the

TABLE 3. Example of the fitting of several distribution functions to wind speed data. Asterisks indicate optimal pdf's.

Mol (1971: 78 m)	a	b	c	χ^2	$\bar{\epsilon}$	$\bar{P}$
Exponential	1	0.4480	1	26997	813.0	43.2
Log-normal	1.6181	0.4630		2325	256.9	217.8
Rayleigh	2	0.0280	2	1856	204.9	183.1
Chi(*)	2	0.0456	3.2537	384	109.4	167.0
Gamma	1	0.9932	5.5009	587	136.4	177.1
Weibull-2	2.6360	0.0081	2.6360	822	112.3	166.7
Weibull-3(*)	2.0090	0.0445	3.2433	386	109.3	166.9
Tihange (1975: 200 m)	a	b	c	χ^2	$\bar{\epsilon}$	$\bar{P}$
Exponential	1	0.2505	1	5309	318.7	246.1
Log-normal	1.6900	0.7590		893	88.1	1371.7
Rayleigh	2	0.0163	2	449	58.9	413.8
Chi	2	0.0130	1.6050	118	37.8	435.6
Gamma	1	0.3612	2.4367	161	55.3	508.8
Weibull-2(*)	1.7336	0.0300	1.7336	87	32.9	442.1
Weibull-3(*)	1.7582	0.0277	1.7178	89	33.3	441.5
Tihange (1968: 130 m)	a	b	c	χ^2	$\bar{\epsilon}$	$\bar{P}$
Exponential	1	0.1312	1	5510	388.4	106.4
Log-normal	1.4368	0.6522		500	85.9	325.6
Rayleigh	2	0.0285	2	677	113.3	178.1
Chi	2	0.0245	1.7185	424	145.3	184.4
Gamma(*)	1	0.5495	2.7615	263	75.8	200.5
Weibull-2	1.7821	0.0443	1.7821	350	83.6	186.9
Weibull-3(*)	0.9985	0.5446	2.7642	262	75.8	200.6
Tihange (1975: 130 m)	a	b	c	χ^2	$\bar{\epsilon}$	$\bar{P}$
Exponential	1	0.2796	1	3458	285.6	171.7
Log-normal	1.3290	1.0561		1694	175.6	5098.0
Rayleigh	2	0.0241	2	2941	131.6	222.3
Chi	2	0.0140	1.1628	157	58.1	255.2
Gamma	1	0.2944	1.5771	468	94.6	356.4
Weibull-2	1.4223	0.0811	1.4223	261	71.3	277.1
Weibull-3(*)	2.4211	0.0043	1.0874	167	56.1	248.3

Weibull-2 line $a = c$ for most stations, demonstrating the superior fit of the Weibull-3 model over the other pdf's of Table 1. The factor $\hat{\gamma}_3$ of the hybrid density function $f_H(x)$ is related to the γ_3 factor of $f(x)$ by the relationship (Takle and Brown, 1978)

$$\hat{\gamma}_3(f_H(x)) = \gamma_3(f(x))/(1 - F_0)^2. \qquad (17)$$

The arithmetic mean value of γ_3 for the 21 synoptic stations is

$$\gamma_3(a,c) = 2.05 \pm 0.23.$$

Since the real mean value $\hat{\gamma}_3(a,c,F_0)$ for each station must be calculated from Eq. (17), the mean of these 21 values is equal to

$$\hat{\gamma}_3(a,c,F_0) = 2.35 \pm 0.36.$$

4. Variability of the wind power density with respect to time

Whenever statistics are applied in estimating mean wind power densities from the available wind speed data, it is always implicitly assumed that the observations are stationary and that they are drawn from the same population, i.e., that the data do not show a different behavior in time. Before making conclusions about the available mean wind power density at certain locations, it is necessary to have an idea about the variability of the calculated results with time. The tower data were used to compute the mean power density at each level and for each available year, by using Eqs. (2) and (5) with $r = 3$. The results are summarized in Table 4; the Weibull 3-parameter pdf was systematically fitted to at least 6319 hourly data out of a maximum of 8760 observations for each year.

As can be seen in Table 4, the results show an appreciable scatter. Although the standard deviation at each level is only 10–13% of the overall yearly mean for the towers of Mol and Zwijndrecht, which dominate a flat landscape, it is much higher for Tihange, which lies in the river Meuse valley.

Differences of 40% may exist between the computed mean wind power density relative to one year and that relative to another year despite the fact that the corresponding annual mean wind speeds

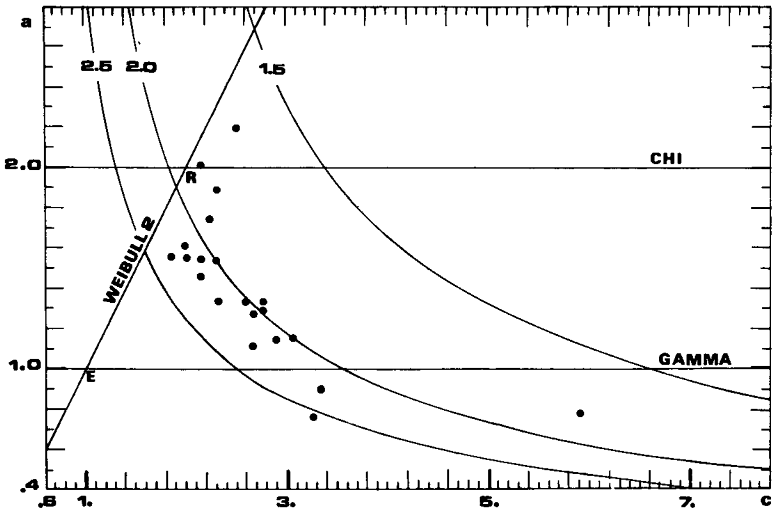

FIG. 2. (a,c) points of the Weibull three-parameter model fitted to the 3 h wind data at 21 synoptic stations and three-isopleths of $\gamma_3(a,c)$.

differ by <1 m s^{-1}. It is important to remark that statistical analysis of data from non-overlapping time intervals can lead to different estimates of the "mean" wind power density at a given location and height.

5. Wind power variation with respect to height

The variation with height of the characteristics of the probability density functions for wind speed values has been extensively studied. Justus and Mikhael (1976) propose two expressions for the shape and scale parameters of the Weibull two-parameter pdf as a function of height. Their forms have been verified as leading to the same conclusions as Doran and Verholek (1978). The data which were used here indicate that these formulae give acceptable though slightly underestimated wind power density extrapolations; however, their application to individual cases can lead to completely erroneous extrapolated Weibull-2 pdf's. Table 4 gives the variation of the mean power density with height for the three towers; the relationship seems to be almost linear.

TABLE 4. Variability of mean wind power densities with respect to height and time (W m^{-2}).

Level (m)	1966	1967	1968	1969	1970	1971	1972	1973	1974	1975	1976	Overall yearly mean
Mol												
24	43.7	52.7	49.6	47.5	46.2	35.2	44.4	41.2	49.3	41.9	42.7	44.9
49	103.6	130.9	119.8	118.8	126.7	90.8	99.9	109.3	125.8	94.2	105.7	111.4
78	191.8	238.6	218.3	221.0	220.5	166.9	194.9	188.6	234.4	181.8	202.5	205.4
114	284.2	374.1	333.8	334.7	348.6	257.6	300.1	304.9	353.6	261.7	311.6	315.0
Level (m)		1967	1968	1969	1970	1971	1972	1973	1974	1975		Mean
Tihange												
8		50.7	33.1	31.3	45.4	—	45.2	38.5	49.0	36.7		41.2
24		70.7	58.9	57.0	92.3	—	72.3	72.2	93.5	60.3		72.2
48		104.3	87.8	79.7	139.8	—	128.2	116.7	150.7	98.4		113.2
80		153.9	129.5	117.5	208.0	—	197.5	174.5	261.2	145.5		173.5
130		235.0	200.6	180.8	332.0	—	330.7	283.9	383.3	248.3		274.3
200		292.6	249.2	246.1	409.6	—	444.9	274.2	337.1	441.5		336.9
Level (m)			1968	1969	1970	1971	1972	1973	1974	1975		Mean
Zwijndrecht												
51			222.3	224.0	—	180.0	210.2	192.3	233.1	163.8		203.7
154			451.9	428.5	—	399.7	450.1	422.8	505.2	380.8		434.1

Fig. 3 shows the calculated (a,c)-combinations of the Weibull-3 pdf for different levels on the three towers and three isolines of γ_3, respectively, equal to 1.55, 1.75 and 2.55; each run consists of a statistical analysis using 25000 hourly data points. For Zwijndrecht and Tihange there are two and for Mol there are three non-overlapping periods available. From this figure several conclusions are obtained.

From the analysis of the Mol tower data it may be considered that the a and c parameters of the Weibull-3 pdf are not necessarily equal, as is imposed by the Weibull-two pdf, nor even close to each other. It is also clear that a given empirical wind frequency distribution may happen to be much better fitted by another pdf of Table 1 than by the Weibull-two model.

The points in the (a,c) plane, representing wind speed distribution parameters at the different heights, are situated in a region extending from the lower right corner (small a, large c) for the lower levels, to the upper left corner (large a, small c) for the highest level; in a more or less hyperbolic way.

It is only at the greatest height (114 m for the data of Mol) that the a and c values become more or less equal, which means that only the data at the highest level can be adequately fitted by a Weibull-2 pdf. This fact has an important consequence on the relationships proposed by Justus and Mikhael (1976), who have explicitly stated that their formulas are only valid up to 100 m.

As a matter of fact, it is easy to observe from Fig. 3 that the Weibull-2 pdf is generally not a good fit for the empirical wind distributions for heights below 100 m. This questions the adequacy of the vertical extrapolation scheme of Justus and Mikhael.

These conclusions are confirmed by the results of the meteorological tower of Zwijndrecht, although there are only two levels of observation. Regarding the Tihange data, there is a large randomness, of course, conditioned by the mean power density values, in the successive locations of the (a,c) points and it may be thought that this different behavior is due to some kind of valley effect. More tower data are needed in order to find a consistent height-dependence relation, if any, for the parameters of the Weibull-3 pdf.

It can also be seen from Fig. 3 that for a specific tower, all calculated (a,c) combinations appear to have approximately the same γ_3 values, at least for the data of Mol and Zwijndrecht; the γ_3 values for the Tihange meteorological tower, which is situated in a valley, show a decrease with height. The fact that this tower is located in more irregular surroundings also explains the larger magnitude of the values. It is therefore concluded that rougher terrain affects the wind to produce larger γ_3 magnitudes for areas with similar mean wind speeds. In this respect the numerical value of γ_3 could possibly be used in classifying terrain effects on wind energy.

6. Conclusions

The Weibull three-parameter density function is proposed as a straight-forward generalization of the Weibull two-parameter and other models for statistical wind power analysis. At the expense of estimating a third parameter by solving a nonlinear equa-

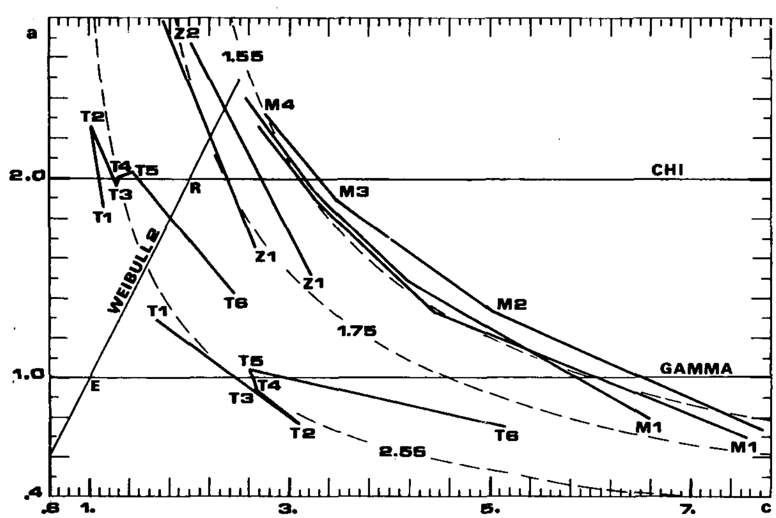

FIG. 3. (a,c) points of the Weibull three-parameter model at different heights and non-overlapping time intervals (ex. M3: Mol, 78 m level; Z: Zwijndrecht; T: Tihange: see Table 2) and three isopleths of the γ_3 factor equal to 1.55, 1.75, 2.55, respectively.

tion a more flexible model can be used. Analysis of tower data and synoptic observations show that the Weibull-three pdf gives a more reliable fit to empirical wind speed frequency data. The variability of the wind power with respect to time shows an appreciable scatter depending on the individual locations. This time variation can even be larger than the difference originating in the estimation of the wind power by different pdf's. No coherent vertical extrapolation scheme for the parameters of the Weibull three-parameter function can be proposed, but there seems to be no simple height dependence as far as indicated by our data.

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