A Simple Analytic Model for Transport by Fluctuating Winds

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ABSTRACT

The statistical properties of airborne tracer concentrations, and the time-lagged correlation between concentration and wind velocity, are calculated from a simple model for the fluctuations in wind velocity. The mean concentration distribution thus calculated is compared with that obtained using the usual diffusivity formulation for transport, and the limitations on the diffusivity approximation discussed for various sources and sinks. It is shown that the model provides a method for calculation of the concentration statistics in the presence of velocity-dependent sources.

1. Introduction

Material transport by fluctuating winds has been actively studied, both theoretically and observationally, since the first investigations of turbulent fluids.

In general, there are two sorts of approaches to this problem. The first is to assume that the wind velocity is deterministic, to parameterize the effects of fluctuations by a diffusivity and thence to derive concentrations downstream of a source via a semi-empirical "plume" or "box" model. The second method is to write the conservation equation for the transported substances as a function of a fluctuating velocity **u** and to evoke a full turbulence theory to compute downstream concentrations in terms of random variables which cannot, in general, be described analytically. Although the second is more promising, at present neither approach affords a great deal of insight into the relationships between the noise or fluctuations in **u**, and the statistical properties of the advected concentrations.

Here we investigate the implications of a simple model for the wind velocity, which is considered to be a stochastic variable, and calculate the statistical properties of transported concentrations in terms of the velocity distribution properties. The mean concentration thus calculated reduces to that which results from a diffusivity approximation in certain cases. The main purpose of this work is to compute those measurable parameters of the concentration distribution which cannot be derived from a diffusivity formulation.

In Section 2 we present simplified continuity equations and define the quantities to be calculated. A multivariate Gaussian model for the distribution of the fluctuating wind velocity is given in Section 3 and in Sections 4–6 this model is used to calculate the statistical properties of the transported quantity for several types of source and sink functions.

2. Statement of the problem

The conservation equation for a concentration $c(\mathbf{r},t)$ [kg m^{-3}] can be written

$$\frac{\partial c(\mathbf{r},t)}{\partial t} = -\mathbf{u}(\mathbf{r},t)\cdot\nabla c(\mathbf{r},t) + q(\mathbf{r},t) - s(\mathbf{r},t), \quad (1)$$

where q [kg m^{-3} s^{-1}] and s [kg m^{-3} s^{-1}] are source and sink functions for the concentration $c(\mathbf{r},t)$.[1] In previous work (Baker *et al.*, 1979) we have investigated the effects on $c(\mathbf{r},t)$ of limited classes of non-velocity-dependent fluctuations in q and s. For the present, we wish to concentrate on the effects of fluctuations in **u** [m s^{-1}]. We shall focus on the special case

$$\frac{\partial c(\mathbf{r},t)}{\partial t} = -\mathbf{u}(t)\cdot\nabla c(\mathbf{r},t) + g(\mathbf{r})q(t) - \frac{a(\mathbf{r})}{T(t)}c(\mathbf{r},t) \quad (2)$$

in which the source and sink terms are separable functions of **r** and t, and we assume the variability is only in the temporal and not the spatial coordinate. In exchange for the loss of generality induced by

[1] The symbols used are defined in the order of their appearance in a table at the end of this article.

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this assumption, we gain the insights inherent in analytic solution to the continuity equations.

For the moment we let $a(\mathbf{r}) = 1$, a constant. It will be useful to assume a Gaussian distribution for $g(\mathbf{r})$:

$$g(\mathbf{r}) = \frac{\exp(-r^2/2w^2)}{(2\pi)^{3/2}w^3}. \tag{2'}$$

For $w \to 0$ this yields an almost point source approximation and for finite w it yields an analytically tractable representation of a source of width w.

We define $\bar{f}(\mathbf{k},t)$, Fourier transform of a function $f(\mathbf{r},t)$ in the spatial domain, where $\mathbf{k}$ has components (k_x,k_y,k_z):

$$\bar{f}(\mathbf{k},t) \equiv \int_0^\infty \exp(i\mathbf{k}\cdot\mathbf{r})f(\mathbf{r},t)d^3r, \tag{3a}$$

which implies

$$f(\mathbf{r},t) = (1/2\pi)^3 \int_0^\infty \exp(-i\mathbf{k}\cdot\mathbf{r})\bar{f}(\mathbf{k},t)d^3k. \tag{3b}$$

Then, from Eq. (2), we find

$$\frac{\partial \bar{c}}{\partial t}(\mathbf{k},t) = \left[i\mathbf{k}\cdot\mathbf{u}(t) - \frac{1}{T(t)}\right]\bar{c}(\mathbf{k},t) + q(t)\exp(-k^2w^2/2). \tag{4}$$

This equation can be formally integrated if $\mathbf{u}(t)$ and $1/T(t)$ are integrable to yield

$$\bar{c}(\mathbf{k},t) = \bar{c}(\mathbf{k},0)\exp\left\{\int_0^t \left[i\mathbf{k}\cdot\mathbf{u}(s) - \frac{1}{T(s)}\right]ds\right\} + \int_0^t q(t')\exp\left(\int_{t'}^t \left[i\mathbf{k}\cdot\mathbf{u}(s) - \frac{1}{T(s)}\right]ds - \tfrac{1}{2}k^2w^2\right)dt'. \tag{5}$$

We assume for simplicity that $c(\mathbf{r},0) = 0$, all $\mathbf{r}$. Then the first term in Eq. (5) vanishes. We can now examine the statistical properties of the stochastic variable $c(\mathbf{r},t)$. We define $E[f(\mathbf{r},t)]$, the expectation value of the function $f(\mathbf{r},t)$; $\mathrm{cov}[f(\mathbf{r},t),g(\mathbf{r}',t')]$, the covariance of the functions $f(\mathbf{r},t)$ and $g(\mathbf{r}',t')$; and $\mathrm{var}[f(\mathbf{r},t)]$, the variance of $f(\mathbf{r},t)$, where

$$\mathrm{cov}[f(\mathbf{r},t),g(\mathbf{r}',t')] \equiv E[f(\mathbf{r},t)g(\mathbf{r}',t')] - E[f(\mathbf{r},t)]E[g(\mathbf{r}',t')]$$

and

$$\mathrm{var}[f(\mathbf{r},t)] \equiv \mathrm{cov}[f(\mathbf{r},t),f(\mathbf{r},t)].$$

From Eq. (5) we have

$$E[c(\mathbf{r},t)] = \left(\frac{1}{2\pi}\right)^3 \int_0^\infty \exp(-i\mathbf{k}\cdot\mathbf{r})E[\bar{c}(\mathbf{k},t)]d^3k$$
$$= \left(\frac{1}{2\pi}\right)^3 \int_0^\infty d^3k\,\exp(-i\mathbf{k}\cdot\mathbf{r}) \int_0 dt' E\left(q(t')\exp\left\{\int_{t'}^t \left[i\mathbf{k}\cdot\mathbf{u}(s) - \frac{1}{T(s)}\right]ds\right\}\right) \times \exp(-k^2w^2/2). \tag{6}$$

Similarly,

$$\mathrm{cov}[c(\mathbf{r},t),c(\mathbf{r}',t')] = \left(\frac{1}{2\pi}\right)^6 \int_0^\infty d^3k\,\exp(-i\mathbf{k}\cdot\mathbf{r}) \int_0^\infty d^3k' \times \exp(-i\mathbf{k}'\cdot\mathbf{r}')\,\mathrm{cov}[\bar{c}(\mathbf{k},t),\bar{c}(\mathbf{k}',t')], \tag{7}$$

etc.

The averaging process represented by the $E(\ )$ notation is that performed by a measuring instrument whose spatial resolution d^3r is large enough to be crossed by many independent air parcels. The material in each parcel follows a trajectory corresponding to a single time series of $\mathbf{u}(t)$, and thus its concentration at $\mathbf{r}$ is the random variable $c(\mathbf{r},t)$. The measuring instrument averages over many of these parcels.

In the following sections we will examine some simple models for the fluctuating quantities $\mathbf{u}(t)$, $q(t)$ and $T(t)$ in terms of which Eqs. (6) and (7) can be explicitly solved.

3. Multivariate Gaussian distribution for $\mathbf{u}(t)$

In this section we present a model for the random variable $\mathbf{u}(t)$, in terms of which we will then attempt to solve Eq. (2) for $c(\mathbf{r},t)$.

Let us suppose that $E[\mathbf{u}(t)] \equiv \bar{\mathbf{u}} = \bar{u}\hat{\mathbf{i}}$, where $\hat{\mathbf{i}}$ is the unit vector along the x axis and $\bar{u}$ is independent of time. Then we can write $\mathbf{u}(t) = \bar{u}\hat{\mathbf{i}} + \mathbf{u}'(t)$. We now assume that the components of $\mathbf{u}'(t)$ are mutually independent and have identical statistics. The values at different times of each component $u_\alpha'(t)$ are jointly normally distributed so that

$$[u_\alpha'(t_i)u_\alpha'(t_i)] = \sigma^2, \quad \text{all } i, \tag{8}$$

where $u_\alpha'(t)$ is the x, y or z component of $\mathbf{u}'(t)$ and

$$E[u_\alpha'(t_i)u_\alpha'(t_j)] = \sigma^2 R(|t_i - t_j|). \tag{9}$$

Furthermore, since the $u_\alpha'(t_i)$ are jointly normally distributed,

$$E\{[u_\alpha'(t)]^n\} = \begin{cases} 0, & n \text{ odd} \\ (1\cdot3\cdot5\cdot7 \cdots 2m-1)\sigma^{2m}, & n = 2m. \end{cases} \tag{10}$$

All the effects of meteorological conditions on $\mathbf{u}(t)$ are parameterized by the amplitude σ of the fluctuations and the autocorrelation function of the velocity $R(t)$. The exact form of $R(t)$ is not necessary here. By definition $R(0) = 1$. Physical considerations require that $R(t) \to 0$ as $t \to \infty$ and that $\partial R(t)/\partial t|_{t=0} \to 0$.

4. Solution of the simplified continuity equation for velocity-independent sources and sinks

With the above definition of the distribution of $\mathbf{u}(t)$ we solve for $c(\mathbf{r},t)$. From Eq. (6), temporarily assuming $q(t) = Q$ and $T(t) = T$, both constants, we find

$$E[\bar{c}(\mathbf{k},t)] = Q \int_0^t E\left\{\exp\left[i\mathbf{k}\cdot \int_{t'}^t \mathbf{u}(s)ds\right]\right\} \times \exp[-(t-t')T^{-1} - \tfrac{1}{2}k^2w^2]dt'. \quad (12)$$

The averaged quantity is

$$E\left\{\exp\left[i\mathbf{k} \int_{t'}^t \cdot\mathbf{u}(s)ds\right]\right\} = \exp[ik_x\bar{u}(t-t')] \times \prod_{\alpha=1}^{3} E\left\{\sum_{m=0}^{\infty} \frac{(ik_\alpha)^m}{m!}\left[\int_{t'}^t u_\alpha'(s)ds\right]^m\right\}. \quad (13)$$

From Eqs. (11) and (12) this is

$$\exp[ik_x\bar{u}(t-t')] \times \prod_{\alpha=1}^{3} \sum_{m=0}^{\infty} \frac{(-k_\alpha^2)^m}{(2m)!}\left[\sigma^2\int_{t'}^t ds_1 \int_{t'}^t ds_2 R(|s_1-s_2|)\right]^m (1\cdot 3\cdot 5 \cdots 2m-1);$$

thus

$$E\left\{\exp\left[i\mathbf{k}\cdot \int_{t'}^t \mathbf{u}(s)ds\right]\right\} = \exp[ik_x\bar{u}(t-t') - \tfrac{1}{2}k^2v_{t-t'}], \quad (14)$$

where

$$v_{t-t'} \equiv \mathrm{var}\left[\int_{t'}^t u_\alpha'(s)ds\right] = \sigma^2 \int_{t'}^t ds_1 \int_{t'}^t ds_2 R(|s_1-s_2|). \quad (15)$$

Eq. (14) could have been derived more simply by noticing that if $u_\alpha'(s)$ is normally distributed, then the integral $\int_{t'}^t u_\alpha'(s)ds$ is also normally distributed and (14) then follows from the general properties of normal distributions. We have explicitly expanded the exponential, because this method will be useful in further derivations.

From Eqs. (6), (13) and (14) we have

$$E[c(\mathbf{r},t)] = \frac{Q}{(2\pi)^3}\int_0^t dt' \int_{-\infty}^{\infty} d^3k \exp[ik_x\bar{u}(t-t') - i\mathbf{k}\cdot\mathbf{r} - \tfrac{1}{2}k^2(v_{t-t'} + w^2 - T^{-1}(t-t')]$$

$$= \frac{Q}{(2\pi)^{3/2}}\int_0^t \frac{dt' \exp\left\{-\left[\dfrac{[x-\bar{u}(t-t')]^2 + y^2 + z^2}{2(v_{t-t'} + w^2)} + \dfrac{(t-t')}{T}\right]\right\}}{(w^2 + v_{t-t'})^{3/2}}. \quad (16)$$

Eq. (16) shows that the material emitted in a time element dt' finds itself at time t in a region centered at $x = \bar{u}(t-t')$ with width $(v_{t-t'} + w^2)^{1/2}$. This result follows from the assumed Gaussian character of the temporal fluctuations in $\mathbf{u}(t)$ and has been derived or assumed in statistical treatments of turbulence (Tennekes and Lumley, 1972, Chap. 6; Monin and Yaglom, 1971, Chap. 10; Frenkiel, 1952) on the basis of the Central Limit Theorem for the distribution of $\int \mathbf{u}'(t)dt$. The purpose of this work is to use the statistics of $\mathbf{u}'(t)$ to study the statistical properties of $c(\mathbf{r},t)$ for a variety of source-sink combinations.

It appears from (14) that fluctuations decrease all the amplitudes $\bar{c}(\mathbf{k},t)$. At first this may seem surprising, since mass is not destroyed by the variability in $\mathbf{u}$. Inspection of the Fourier integrals (16), however, shows that the decrease in mean amplitude due to fluctuations merely results from averaging over many trajectories. To see this, we notice from Eq. (16) that for $w \to 0$ without fluctuations, all the contributions emitted at $t = 0$ will exactly cancel except at $x = \bar{u}t$. However, this interference is destroyed if u is variable, and although $E[c(\mathbf{r},t)]$ still peaks at the same point, it is nonzero in a surrounding region of width v_t, and the mean mass integrated over all space is unchanged. We will examine some consequences of the relations between trajectories due to fluctuations in $\mathbf{u}(t)$ in subsequent sections.

In order to examine further the nature of the plume described by Eq. (16), let us assume temporarily that $R(t)$ has the particularly simple form

$$R(t) = \exp(-|t|/\tau). \quad (17)$$

While this representation cannot be accurate near $t = 0$, where $\partial R/\partial t$ must approach 0, Eq. (21) yields reasonable results for finite t with minimal computational effort. In this model, $u_\alpha'(t)$, called a Uhlenbeck-Ornstein process (Uhlenbeck and Ornstein, 1930) is the solution to the equation $du_\alpha' = -u_\alpha'\tau^{-1} \times dt + d\xi$, where $d\xi$ is Gaussian white noise with width Δ. With $\Delta^2/dt = \epsilon$, the energy dissipation/mass, this model has been used to describe vertical velocities in the inertial subrange in homogeneous, stationary turbulence (Jonas and Bartlett (1972).

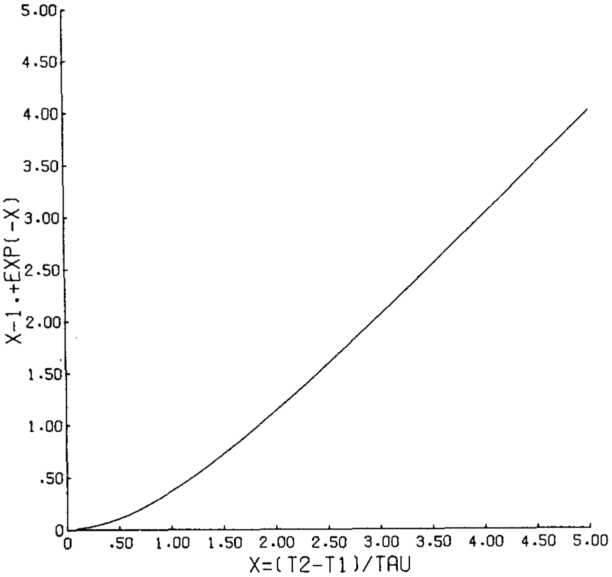

FIG. 1. The normalized variance of the time integral of the velocity, defined as $v_{t-t'}$, as a function of $x = |t - t'|/\tau$, assuming the velocity correlation function $R(t - t') = e^{-|t-t'|/\tau}$ [see Eq. (18)].

From Eq. (15), we find that

$$v_{t-t'} = \sigma^2 \int_{t'}^{t} ds_1 \int_{t'}^{t} ds_2 \exp[-(s_1 - s_2)/\tau]$$
$$= 2\sigma^2\tau[t - t' - \tau\{1 - \exp[-(t - t')/\tau]\}]. \quad (18)$$

In Fig. 1 we show the function $(v_{t-t'}/2\sigma^2\tau^2)$ as a function of the parameter $x = |t - t'|/\tau$.

We now consider the two limiting cases: $|t - t'|/\tau \to 0$ and $|t - t'|/\tau \to \infty$.

a. $|t - t'|/\tau \to 0$

For long persistence time the perturbation u' is essentially constant over the propagation interval (t',t). We have

$$\lim v_{t-t'} \xrightarrow[|t - t'|/\tau \to 0]{} \sigma^2(t - t')^2; \quad (19)$$

thus the average concentration distribution is a blob whose center is moving at velocity $\bar{u}$ along the x axis and whose width at time $t \sim \sigma t$.

It is to be noted that this case corresponds to multiple trajectories, each of which obeys Eq. (4) for constant u, where the probability density of u is

$$p(u) = \frac{\exp[-(u - \bar{u})^2/2\sigma^2]}{\sigma\sqrt{2\pi}}. \quad (20)$$

Thus the result in (16) for the limit could also be found by solving (4) for $\bar{c}(\mathbf{k},t)$ with $\mathbf{u}$ as a constant parameter, thus deriving a "box" model solution, and then averaging that solution over the distribution given in (20).

The concentration distribution $c(\mathbf{r},t)$ can be visualized by considering it to consist of a collection of "puffs" of material put out at times $t' < t$ by the source. The distribution of initial propagation velocities for the puffs is independent of τ, and determined by Eqs. (8)–(10). These initial velocities persist for times on the order of τ, so that subsequent fluctuations in propagation velocities produce a set of relatively narrow almost discrete puffs of small amplitude distributed over a wide region.

b. $|t - t|/\tau \to \infty$

In this case we find

$$\lim v_{t-t'} \xrightarrow[(t - t')/\tau \to \infty]{} 2\sigma^2(t - t') \quad (21)$$

and substituting this expression in (16) shows that in this limit $E[c(\mathbf{r},t)]$ is the solution to a diffusion equation with effective Brownian diffusion constant $D = \sigma^2\tau$.

In this case the concentration distribution at large times t is the result of many changes in propagation velocity of the puffs emitted at earlier times t'. Each of these puffs spreads and the sum becomes a relatively smooth cloud whose maximum propagates with the mean velocity and the density has large amplitude only in a region surrounding the maximum. The dependence of these results on the ratio $|t - t'|/\tau$ provides limits on the validity of "diffusivity" approximations (see, e.g., Slade, 1968) in which (2) is replaced by a deterministic equation of the form

$$\frac{\partial}{\partial t} c_K(\mathbf{r},t) = g(\mathbf{r})q(t) - c_K(\mathbf{r},t)\frac{a(\mathbf{r})}{T(t)} + K(t)c_K(\mathbf{r},t), \quad (22)$$

where we have used a subscript K to distinguish this solution from that in (16). We now examine this approximation.

5. Comparison with the diffusion equation approximation

Initially we put $q(t) = Q$, $a(\mathbf{r})/T(t) = 1/T$. Then the solution to Eq. (22) is

$$c_K(\mathbf{r},t) = \frac{Q}{(2\pi)^{3/2}} \int_0^t \frac{dt' \exp\left[-\left(\dfrac{x - \bar{u}(t - t')^2 + y^2 + z^2}{2w^2 + 4\int_{t'}^{t} K(s)ds} + \dfrac{(t - t')}{T}\right)\right]}{\left[w^2 + 2\int_{t'}^{t} K(s)ds\right]^{3/2}}, \quad (22')$$

where we have again assumed a Gaussian source distribution.

Comparison of Eqs. (16) and (22′) shows that $c_K(\mathbf{r},t) = E[c(\mathbf{r},t)]$ for

$$K(t) = \frac{1}{2}\frac{\partial}{\partial t} v_{t-t'} = E\left[u_\alpha'(t)\int_{t'}^{t} u_\alpha'(s)ds\right]. \quad (23)$$

Clearly, the expression is a function of both t and t' unless the time interval $t - t'$ is large compared with the fluctuation correlation time τ of the wind velocity. It is then only in the limit that the time scales $T/\tau \to \infty$ (assuming constant T) that the diffusivity formulation is applicable. Moreover, this condition is modulated by the time-dependence of the source strength since multiplication of the integrands in (16) and (22′) by identical functions $q(t')$ may change the resulting integrals significantly.

We now show that under the assumptions we have made about the statistics of $\mathbf{u}(t)$, Eq. (23) is equivalent to the mixing-length approximation for $K(t)$, i.e.,

$$K(t) \equiv -\frac{\text{cov}[u_\alpha(t),c(\mathbf{r},t)]}{\dfrac{\partial}{\partial x_\alpha}E\{c(\mathbf{r},t)\}};\ \alpha = 1, 2 \text{ or } 3. \quad (24)$$

The covariance function is

$$\text{cov}[u_\alpha(t),c(\mathbf{r},t)] = \frac{Q}{2\pi}\int_0^t dt' \int_{-\infty}^{\infty} d^3k \exp\left[\left(ik_x\bar{u} - \frac{1}{T}\right)(t-t') - \tfrac{1}{2}k^2w^2 - i\mathbf{k}\cdot\mathbf{r}\right]\cdot\text{cov}\left[u_\alpha(t), \exp\left(i\mathbf{k}\cdot\int_{t'}^{t}\mathbf{u}'(s)ds\right)\right].$$

There are several ways to calculate this quantity. For our purposes, the most instructive (although certainly not the most elegant) method again involves expanding the exponential. Since the u_α are independent of one another we have

$$\text{cov}\left[u_\alpha(t), \exp\left(i\mathbf{k}\cdot\int_{t'}^{t}\mathbf{u}'(s)ds\right)\right] = \text{cov}\left[u_\alpha(t), \sum_{\beta=1}^{3}\sum_{m=0}^{\infty}\frac{(ik_\beta)^m}{m!}\left(\int_{t'}^{t}u_\beta'(s)ds\right)^m\right]$$
$$= ik_\alpha E\left(u_\alpha'(t)\int_{t'}^{t}u_\alpha'(s)ds\right) \times \exp[-k^2\tfrac{1}{2}(v_{t-t'} + w^2) + ik_x\bar{u}(t-t') - (t-t')/T] \quad (25)$$

or

$$\text{cov}[u_\alpha(t),c(\mathbf{r},t)] = \frac{Q}{(2\pi)^3}\int_0^t dt'\int_{-\infty}^{\infty}d^3k_\alpha ik_\alpha E\left(u_\alpha'(t)\int_{t'}^{t}u_\alpha'(s)ds\right) \times (ik_x\bar{u} - T^{-1})(t-t') - i\mathbf{k}\cdot\mathbf{r} - \tfrac{1}{2}k^2(v_{t-t'} + w^2)]. \quad (26)$$

On the other hand, from Eq. (16) we find that

$$\frac{\partial}{\partial x_\alpha}E(c(\mathbf{r},t)) = -\frac{Q}{(2\pi)^3}\int_0^t dt'\int_0^{\infty} d^3k\, ik_\alpha \times \exp[(ik_x\bar{u} - T^{-1})(t-t') - i\mathbf{k}\cdot\mathbf{r} - \tfrac{1}{2}k^2(v_{t-t'} + w^2).$$

Comparison of Eqs. (23), (25) and (26) confirms the result in (24).

Thus if the sinks and sources are spatially independent and $t/\tau \to \infty$, Eqs. (2) and (22) are equivalent with $K(t)$ defined by (24). However, if either of these conditions is violated the situation becomes more complicated.

We first examine the accuracy of the diffusivity formulation as a function of the time-dependence of the source and sink terms. In Figs. 2a–2c we compare the solution $c_K(\mathbf{r},t)$, Eq. (22′), assuming $K = \sigma^2\tau$, with $E[c(\mathbf{r},t)]$, Eq. (16), for different time-dependent sources. For simplicity in all cases we have put $1/T \to 0$, and $w = y = z = 0$. In Fig. 2a we see both solutions for $qt = Q$, a constant, for $K = 0.1\sigma^2$ and $\tau = 0.1$, 0.5 and 1.0 in arbitrary units. The exact solution, Eq. (16), is always less on the x axis than $c_K(\mathbf{r},t)$ [which approaches the theoretical value $(2\pi\sigma^2\tau x)^{-1}$ for large t] because for finite τ the cloud spreads more. In Figs. 2b and 2c we repeat the calculation for $q(t) = Q[\sin(\omega t) + 1]$. For finite τ the diffusivity formulation and the exact formulation can be quite different, and there are effects at large time scales of the coherence in the velocity fluctuations at shorter scales.

Similar limitations hold in which the sources and/or sinks are spatially dependent. For concreteness, let us suppose only $1/T$ is a function of $\mathbf{r}$. In this case the continuity equations cannot be solved in closed form. We have,

$$E\{\bar{c}(\mathbf{k},t)\} = \int_0^t\int_{-\infty}^{\infty}E\{\exp[i\mathbf{k}\cdot\mathbf{u}(t'-t)]\bar{c}(\mathbf{k}-\mathbf{k}')\} \times \exp[T'(t'-t)](k')d^3\mathbf{k}'dt', \quad (27)$$

and defining $\bar{c}_K(\mathbf{k},t)$ as the Fourier transform of $c_K(\mathbf{r},t)$,

$$\bar{c}_K(\mathbf{k},t) = \int_0^t \bar{c}_K(\mathbf{k}-\mathbf{k}')\exp[+K\mathbf{k}^2(t'-t)] \times \exp[T'(t'-t)](k')d^3\mathbf{k}'dt'.$$

The Fourier transformed concentrations found by the two methods will be exactly equal only for $\bar{c}(\mathbf{k}-\mathbf{k}')$ independent of $\exp[i\mathbf{k}\cdot\mathbf{u}(t-t')]$. If there is any interference between wavenumbers the equality will be destroyed and each successive term in an iteration series solution to (24) will be more strongly affected by the interference.

Thus the range of applicability of a diffusivity model depends very strongly on the spatial and temporal dependence of the sink function, which limits the use of a value of the diffusivity observed for one substance to infer the diffusion of another with a very different source/sink structure.

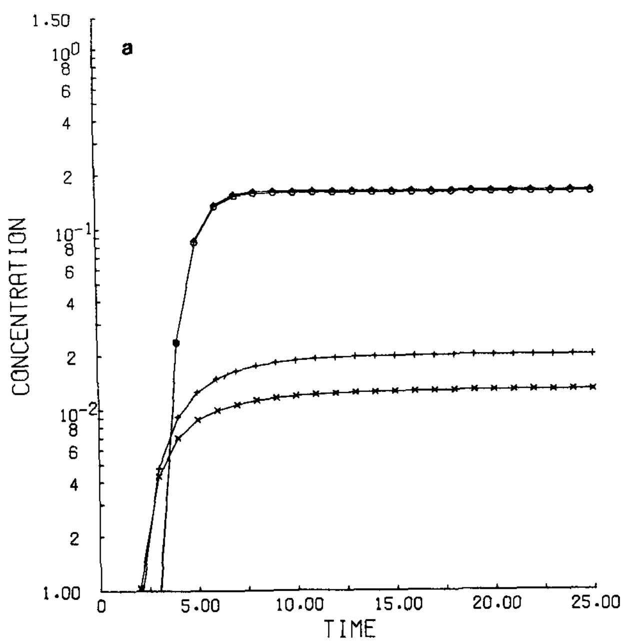

FIG. 2a. The solutions $E[c(\mathbf{r},t)]$ and $c_K(\mathbf{r},t)$ [see Eqs. (16) and (22')] for $\mathbf{r} = (5,0,0)$, $\bar{u} = 1$, $\sigma = 1$, assuming $K = 0.1\sigma^2$ and $v_{t-t'}$ is given by Eq. (18). ○ ($\tau = 0.1$), + ($\tau = 1.0$), × ($\tau = 2.0$). Constant source and no sinks.

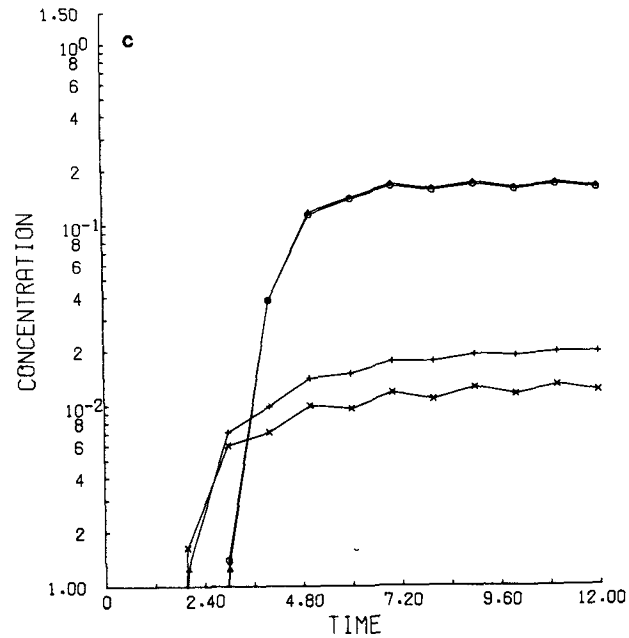

FIG. 2c. As in Fig. 2a except $\omega = 3.0$.

6. The statistical properties of the steady-state concentration distribution

By means of algebra similar to that leading to (25) and (26), we can also find $\mathrm{cov}[u(t_1),c(\mathbf{r},t_2)]$ for any values of t_1 and t_2. The shape of the curve depends on $q(t)$.

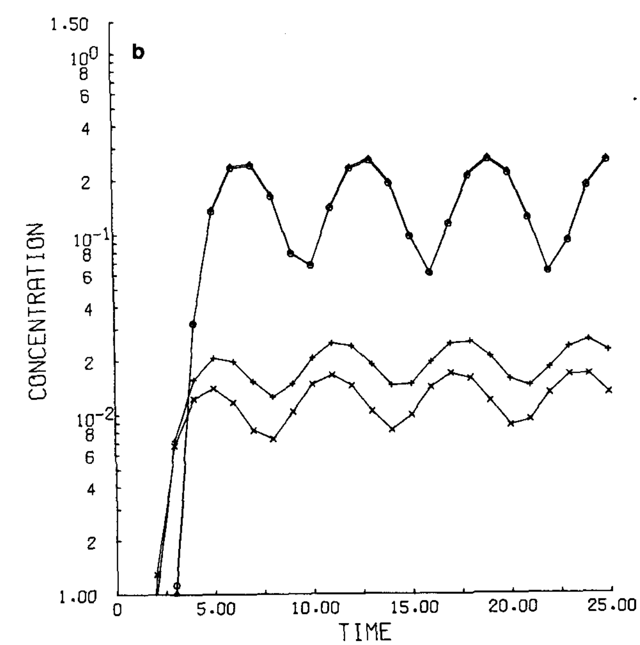

FIG. 2b. As in Fig. 2a except source $q(t) = 1 + \sin\omega t$, $\omega = 1.0$.

In Fig. 3 this function is displayed for a constant source and for several values of τ, assuming $v_{t-t'}$ is of the form given in Eq. (18), $w = 0$, and t_2 is large enough that $c(\mathbf{r},t_2)$ is essentially constant so that the covariance depends only on the difference $(t_1 - t_2)$. For τ very small there is a negative peak at

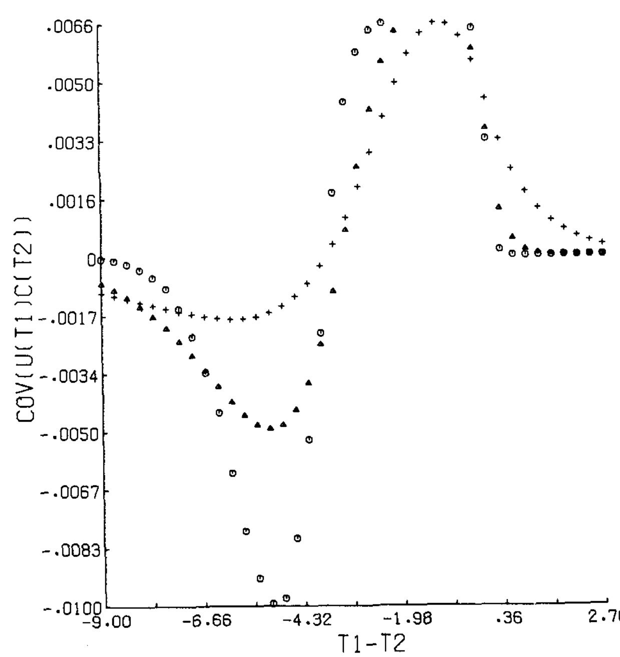

FIG. 3. $\mathrm{cov}[u(t_1),c(\mathbf{r},t_2)]$ as a function of $t_1 - t_2$ for constant source and no sinks, assuming the velocity correlation function is given by Eq. (17). $\mathbf{r} = (5,0,0)$, $\bar{u} = 1$, $\sigma = 1$, $t_2 = 20$. ⊙ ($\tau = 0.1$), △ ($\tau = 0.3$), + ($\tau = 1.0$).

$t_1 - t_2 = -x/\bar{u}$ for which the trailing edge of the block of material affected by the fluctuation $u'(t_1)$ just passes **r** at time t_2. This negative, sharp peak is flattened and smeared out for longer τ because the material which has passed **r** is less coherent and more spaced out. For all τ there is a positive peak at $t_1 = t_2$, when the covariance function is directly related to the local diffusivity. The width of this peak is approximately independent of τ.

We now calculate the covariance function of $c(\mathbf{r},t)$. We have, for example,

$$\text{cov}[c(\mathbf{r},t),c(\mathbf{r}, t + h)] = E[c(\mathbf{r},t)c(\mathbf{r}, t + h)] - E[c(\mathbf{r},t)]E[c(\mathbf{r}, t + h)].$$

For $q(t) = Q$, neglecting sinks and transients, we have

$$E[c(r,t)c(\mathbf{r}, t + h)] = \frac{Q^2}{(2\pi)^3}\int_0^t dt_1 \int_0^{t+h} dt_2 L^{-3} \times \exp\{-(1/2L^2)[D_1{}^2(v_{t+h-t_2} + w^2) + D_2{}^2(v_{t-t_1} + w^2)] - 2\rho\mathbf{D}_1\cdot\mathbf{D}_2(v_{t-t_1}v_{t+h-t_2})^{1/2}\}, \quad (28)$$

where we define a correlation function

$$\rho \equiv \frac{\int_{t_1}^{t} ds_1 \int_{t_2}^{t+h} ds_2 R(|s_1 - s_2|)}{(v_{t-t_1}v_{t+h-t_2})^{1/2}},$$

and

$$\mathbf{D}_1 \equiv \mathbf{r} - \bar{\mathbf{u}}(t - t_1),$$

$$\mathbf{D}_2 \equiv \mathbf{r} - \bar{\mathbf{u}}(t + h - t_2),$$

$$L^2 \equiv (v_{t+h-t_2} + w^2)(v_{t-t_1} + w^2) - \rho^2 v_{t-t_1}v_{t+h-t_2}.$$

Inspection of Eq. (28) shows that for computation of the higher moments of the convective distribution it is essential that w be kept finite, for when $h \to 0$, $\rho \to 1$ and if $w = 0$ then $L = 0$; the integrand becomes infinite. In this case, the integral in Eq. (28) does not converge. This behavior reflects the fact that for $w = 0$ an infinite concentration is needed at the source in order to have a finite mass emission rate. In a stochastic calculation there is a nonvanishing probability that a puff of infinite concentration will stay intact for finite values of x and t. This probability is sufficiently small that the first moment of the concentration distribution remains finite as $w \to 0$ [see Eq. (16)], but the tail of the distribution as $c(\mathbf{r},t) \to \infty$ makes the second moment infinite unless w is finite. To approximate the effects of a confined source on the concentration at **r** we take $w/|\mathbf{r}|$ to be small.

In Fig. 4 we present curves of the correlation function

$$\text{corr}[c(\mathbf{r},t),c(\mathbf{r}, t + h)] \equiv \mathcal{R}(h) \equiv \frac{\text{cov}[c(\mathbf{r},t),c(\mathbf{r}, t + h)]}{\{\text{var}[c(\mathbf{r},t)]\ \text{var}[c(\mathbf{r}, t + h)]\}^{1/2}}$$

for $y = z = 1/T = 0$, $q(t) = Q$, $w/x = 1/25$ and for sufficiently large t that the function is independent of t. The covariance decreases with h but its dependence on τ is complex. To understand the behavior we examine the numerator and denominator of the correlation function separately.

We first consider the variance. As τ increases $\sigma^2[c(\mathbf{r},t)]$ decreases, as does $E[c(\mathbf{r},t)]$ because the cloud is broadened by those trajectories which take material far from the source. However, this decrease is mitigated by the fact that the material does not "mix" as efficiently for large τ as for smaller τ, so that a snapshot of the spatial distribution shows "lumps" and holes, and therefore the variance may be somewhat increased over that which would be expected due to the broadening effect alone.

Examination of the numerator in $\mathcal{R}(h)$ shows the same behavior. If we imagine the snapshot of $c(\mathbf{r},t)$ replaced by a moving picture of $c(\mathbf{r},t)$ at one spatial point, we again find that as τ increases the broadening of the cloud lessens $\text{cov}[c(\mathbf{r},t)c(\mathbf{r}, t + h)]$ while the persistence of individual clumps tends to increase it, because there is less of a chance that $c(\mathbf{r},t)$ will change in an interval h as τ increases, but if it does change (because a lump has passed over the point **r**, for example) the effect may be relatively large. The net effect of these two tendencies is to decrease the covariance with increasing τ. The decay rate of the covariance appears (for the cases tested) to be fast with respect to the fluctuations in **u**, although since it is (not unexpectedly) not a simple exponential decay it is not possible to assign it a single time constant.

We see, finally, that both $\text{cov}[c(\mathbf{r},t),c(\mathbf{r}, t + h)]$ and $\text{var}[c(\mathbf{r},t)]$ decrease with increasing τ but at different rates, leading to the somewhat surprising result shown in Fig. 4, *viz.*, that the correlation function $\mathcal{R}(h)$ first decreases and then increases as the velocity fluctuations become more persistent.

In this section we have computed some of the lower moments of the steady-state concentration distribution. The other moments can be calculated by analogous methods. We now apply the model to the study of effects inducted by velocity-dependent sources and sinks.

7. Velocity-dependent sources

Eqs. (6) and (7) give prescriptions for calculating the first two moments of the concentration distribution for time-dependent sources and sinks. If the source and sink are fluctuating functions of time, but stochastically independent of the wind velocity

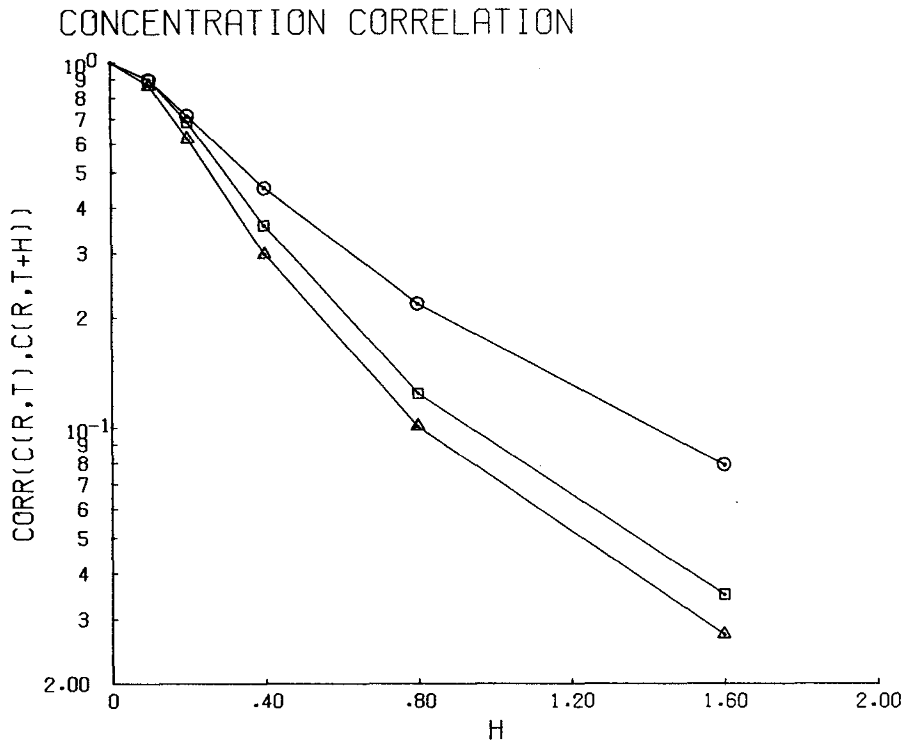

FIG. 4. corr[$c(\mathbf{r},t),c(\mathbf{r},t+h)$] for constant source and no sinks assuming the velocity correlation function is given by Eq. (17). $\mathbf{r}=(5,0,0)$, $\bar{u}=1$, $\sigma=1$, $t=20$. ⊙ ($\tau=0.1$), △ ($\tau=1.0$), ⊡ ($\tau=4.0$).

u, then in Eq. (6)

$$E\left\{q(t')\exp\left[\int_{t'}^{t} i\mathbf{k}\cdot\mathbf{u}(s)-(1/T(s))ds\right]\right\}$$
$$=E[q(t')]\exp[-ik_x\bar{u}(t-t')-\tfrac{1}{2}k^2v_{tt'}$$
$$\times E\left\{\exp\left[-\int_{t'}^{t} ds/T(s)\right]\right\}$$

and calculation of the integrals is straightforward. We shall now briefly examine some simple cases in which the fluctuations in sources are, instead, correlated with those in the wind velocity in order to present the method of calculating the quantities in Eqs. (6) and (7).

We put $q(t)=q[\mathbf{u}(t)]$ with $T=$ constant. This case might correspond, for example, to transport of wind-blown dust or sand, where erosion of surfaces due to sandblasting by the fluctuating boundary layer winds is itself the source of the particles. We then have

$$E\left\{q(t')\exp\left[\int_{t'}^{t} i\mathbf{k}\cdot\mathbf{u}(s)ds\right]\right\}$$
$$=\prod_{\alpha=1}^{3}E\left\{q(\mathbf{u}(t'))\sum_{m=0}^{\infty}\frac{(ik_\alpha)^m}{m!}\left(\int_{t'}^{t}u_\alpha'(s)ds\right)^m\right\}$$
$$\times\exp[ik_x\bar{u}(t-t')].$$

The upward particle flux in the boundary layer during erosion is found to be proportional to $E\{u_z^2\}$ where $E\{u_z^2\}\sim u_*^2$, the square of the effective friction velocity (Gillette *et al.*, 1974). Let us suppose for concreteness that the source itself has the same dependence, i.e., $q(u)\sim Au_z^2\delta(z)$ [kg m^{-2} s^{-1}], independent of x and y. Thus in the source area the continuity equation (2) becomes one-dimensional. We have

$$\frac{\partial c}{\partial t}(z,t)=-(u_z-V_s)\frac{\partial c}{\partial z}(z,t)+Au_z^2\delta(z),\quad (30)$$

where c now represents the airborne mass per horizontal area and V_s [m s^{-1}] is the sedimentation velocity. This equation yields

$$E(\tilde{c}(k_z,t))$$
$$=A\int_0^t dt'E\left\{u_z^2(t')\sum_{m=0}^{\infty}\frac{(ik_z)^m}{m!}\left(\int_{t'}^{t}u_z'(s)ds\right)^m\right\}$$
$$\times\exp[-ik_zV_z(t-t')]$$
$$=A\int_0^t dt'\left[\sigma^2-k_z^2E\left(u_z'(t')\int_{t'}^{t}u_z'(s)ds\right)^2\right]$$
$$\times\exp[-\tfrac{1}{2}k_z^2v_{t-t'}-ik_zV_s(t-t')].$$

From Eq. (6), then, over the source area we have

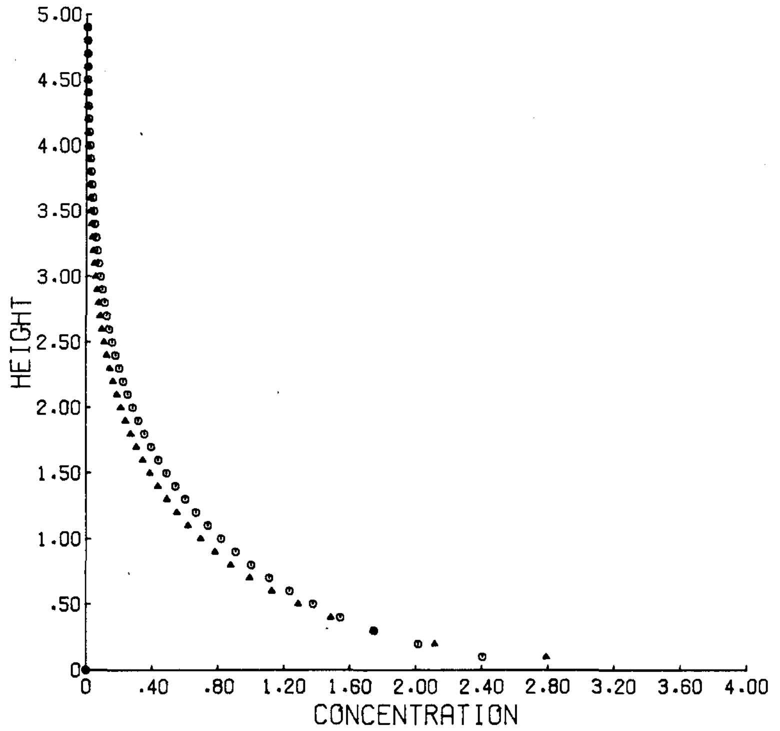

FIG. 5. Triangles: $E[c(z,t)]$ from Eq. (31) assuming the velocity correlation function is given by Eq. (17). $A = 1$, $V_s = 0.2u_*$, $\sigma^2 = u_*^2$, $u_* = 0.5$, $t = 10$. $\tau = 0.1$. Circles: contribution of first term in Eq. (31).

TABLE 1. Symbols.

Symbol	Units	Definition	Introduced in equation
$c(\mathbf{r},t)$	kg m^{-3}	tracer concentration	(1)
$\mathbf{u}(t)$	m s^{-1}	wind velocity	(1)
$g(\mathbf{r})q(t)$	kg m^{-3} s^{-1}	source rate	(2)
$T(t)/a(\mathbf{r})$	s	sink time constant	(2)
w	m	effective width of Gaussian source distribution	(2′)
$\tilde{f}(\mathbf{k},t)$		three-dimensional Fourier transform of the function $f(r,t)$	(3a)
$E[f(\mathbf{r},t)]$		expectation value of $f(\mathbf{r},t)$	(6)
$\mathrm{cov}[f(\mathbf{r},t),g(\mathbf{r}',t')]$		$E[f(\mathbf{r},t)g(\mathbf{r}',t')] - E[f(\mathbf{r},t)]E[g(\mathbf{r}',t')]$	(7)
σ	m s^{-1}	width of normal distribution of wind speed	(8)
$R(t)$		Normalized autocorrelation function of wind speed	(9)
$\mathrm{var}[f(\mathbf{r},t)]$		$E[f^2(\mathbf{r},t)] - \{E[f(\mathbf{r},t)]\}^2$	(15)
$v_{t-t'}$	m^2	variance of the quantity $[= \int_{t'}^{t} u_\alpha'(s)ds]$	(15)
K	m^2 s^{-1}	turbulent diffusion coefficient	(22)
V_s	m s^{-1}	sedimentation velocity	(30)
A	kg s^{-1} m^{-1}	constant defining dust erosion source	(30)

$$E\{c(z,t)\} = \frac{A}{(2\pi)} \int_0^t dt' \left[\frac{\sigma^2}{v_{t-t'}{}^{1/2}} + E \frac{\left\{ u_z'(t') \int_{t'}^t u_z'(s)ds \right\}^2}{v_{t-t'}{}^{3/2}} \times (1 - Z^2/v_{t-t'}) \right] \exp\left[\frac{-Z^2}{2v_{t-t'}} \right], \quad (31)$$

where $Z \equiv z + V_s(t - t')$.

In Fig. 5 we show the profile of $Ec(z,t)$ for $\tau = 1.0$ under typical erosion conditions. The first term is shown to dominate in this case. A more complete treatment of the wind statistics, including vertical dependence of the parameters, is necessary for accurate prediction of the relationship between the concentration profile and the wind statistics.

We have now come to the conclusion of our examination of normally distributed wind velocities. The normal distribution model is seen to be quite useful in exploring certain kinds of source and sink combinations, and thus many different physical systems.

8. Conclusions

In our examination of the simple model for fluctuating wind velocities we have attempted to suggest the kinds of insight to be gained from stochastic interpretations of the continuity equations. The parameters of the fluctuating wind velocities are in principle directly measurable functions of meteorological conditions. Realistic wind spectra (e.g., Van der Hoven, 1957) may be roughly described by a sum of Uhlenbeck Ornstein processes, for example. Thus the outlined noise parameterization may yield estimates of dispersion characteristics which are one step more general, but mathematically not significantly more complex than the plume and diffusivity models in current use. The model allows estimation of errors introduced by the use of effective diffusivities computed from the concentration profile of one tracer to estimate mixing characteristics of another tracer [as, for example, is often done to investigate vertical mixing in the stratosphere (Wofsy and McElroy, 1973)].

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