An Analytical Solution for Three-Dimensional Stationary Flows in the Atmospheric Boundary Layer over Terrain

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ABSTRACT

An analytical solution to the Navier-Stokes equations for three-dimensional stationary flows of small Reynolds number in the atmospheric boundary layer over terrain is presented.

Analyses of the effects of topography, horizontal pressure gradient and Coriolis forces on the velocity distribution in the atmospheric boundary layer indicate that 1) the horizontal component of the velocity in the boundary layer turns right (left) with increasing height in the Northern (Southern) Hemisphere, 2) upward (downward) motion occurs on the windward (lee) side of the mountain, and 3) upward (downward) motion also occurs on the slope to the right (left) of the geostrophic wind in the Northern Hemisphere, whereas in the Southern Hemisphere downward (upward) motion occurs on the slope to the right (left) of the geostrophic wind.

1. Introduction

Velocity distribution in the atmospheric boundary layer over a flat surface was first studied by Ekman (1905) who assumed a constant eddy viscosity in the planetary boundary layer, and obtained an exact solution to the Navier-Stokes equations for the balance between Coriolis, pressure-gradient and viscous forces. In recent years, a large number of analytical and numerical models for the study of the mean and turbulent motions in the planetary boundary layer under various thermal stratifications has been constructed. However, most of these investigations have emphasized on flows in the boundary layer over flat surfaces.

Because of increasing concern about atmospheric pollution in many population centers, industrial and power plants, which are located in valleys and terrain, and since atmospheric motion is the mechanism for the transport and dispersion of pollutants, there is a growing interest in the atmospheric motion and pollution in these regions (Kao, 1976).

For flows in the planetary boundary layer over flat surfaces, the mean motion may be assumed to be homogeneous in the horizontal, therefore, the Navier-Stokes equations become linear (Ekman, 1905) and the motion is horizontal. However, when the inhomogeneity in the topographical configuration of the earth's surface is taken into account, the motion is three-dimensional and the equations of motion are no longer linear. The purposes of this note are to seek an analytical solution to the Navier-Stokes equations for three-dimensional stationary flows of small Reynolds numbers in the atmospheric boundary layer over mountain-terrain, and to analyze the effects of topography and Coriolis force on the velocity distribution in the boundary layer.

2. An analytic solution to the boundary-layer equations

Considering stationary flows of small Reynolds number in the planetary and surface boundary layers over terrain, let $h(x, y)$ be the topographical configuration of the terrain, h_s be the thickness of the surface boundary layer, and $G = u_g + iv_g$ be the geostrophic wind velocity. The Navier-Stokes equations, hydrostatic and continuity equations may be expressed as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = f(v - v_g) + K\frac{\partial^2 u}{\partial z^2}, \quad (1)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -f(u - u_g) + K\frac{\partial^2 v}{\partial z^2}, \quad (2)$$

$$\frac{\partial P}{\partial z} = -\rho g, \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (4)$$

where u, v, w are respectively the x, y, z components of the velocity, p is the pressure, g the gravity acceleration, ρ the density, f the Coriolis parameter, and K the coefficient of eddy diffusivity assumed constant as a first approximation.

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For stationary flow in the surface boundary layer, the equation of motion may be expressed as

$$\frac{\partial}{\partial z}|u + iv| = \frac{u_*}{k}\left(\frac{1}{z} + \frac{a}{L}\right), \quad (5)$$

where $i = (-1)^{1/2}$, $u_* = (\tau/\rho)^{1/2}$, a is a constant which has been experimentally determined to be 4.75, L the Monin-Obukhov length, and k the von Kármán's constant.

The boundary conditions for this model are that all components of the velocity are zero at the solid boundary, $z = h(x, y)$, the velocity tends to the geostrophic velocity as z tends to infinity, and that at the lower boundary of the planetary boundary layer the wind direction coincides with the wind stress, i.e.,

$$u = v = w = 0, \quad \text{at} \quad z = h(x, y), \quad (6)$$

$$u \to u_g,\ v \to v_g, \quad \text{as} \quad z \to \infty, \quad (7)$$

$$u + iv = A\frac{\partial}{\partial z}(u + iv), \quad \text{at} \quad z = h(x, y) + h_s, \quad (8)$$

where A is a constant.

Combining Eqs. (1) and (2), we obtain

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u + iv - G) = \left(K\frac{\partial^2}{\partial z^2} - if\right)(u + iv - G). \quad (9)$$

Differential equation (9) has the following solution which satisfies the boundary conditions (7):

$$(u + iv) = G + (B_r + iB_i)\exp\{-(1 + i)\nu[z - h(x, y) - h_s]\}, \quad (10)$$

$$w = (u_g + \exp\{-\nu[z - h(x, y) - h_s]\}\{B_r\cos\nu[z - h(x, y) - h_s] + B_i\sin\nu[z - h(x, y) - h_s]\})\frac{\partial h}{\partial x}$$
$$+ (v_g + \exp\{-\nu[z - h(x, y) - h_s]\}\{B_i\cos\nu[z - h(x, y) - h_s] - B_r\sin\nu[z - h(x, y) - h_s]\})\frac{\partial h}{\partial y}, \quad (11)$$

where B_r and B_i are constants, and $\nu = (f/2K)^{1/2}$.

Since wind in the surface boundary layer is unidirectional, wind velocity in the surface boundary layer may be expressed as

$$u + iv = |u + iv|e^{i\alpha}, \quad (12)$$

where α is the angle made between the wind in the surface boundary layer and the geostrophic wind.

At the lower boundary of the planetary boundary layer, Eq. (10) yields

$$(u + iv)_{z=h(x,y)+h_s} = G + (B_r + iB_i). \quad (13)$$

Elimination of the left-hand terms of Eqs. (12) and (13) gives

$$(B_r + iB_i) = |u + iv|_{z=h(x,y)+h_s}e^{i\alpha} - G. \quad (14)$$

Eq. (10) may then be written as

$$u + iv = G + \{|u + iv|_{z=h(x,y)+h_s}e^{i\alpha} - G\} \times \exp\{-(1 + i)\nu[z - h(x, y) - h_s]\} \quad (15)$$

for $z \geq h(x, y) + h_s$.

It can be shown by applying the boundary condition (8) to Eq. (15) that

$$|u + iv|_{z=h(x,y)+h_s}\{A\nu(\cos\alpha - \sin\alpha) + \cos\alpha\} - A\nu(u_g - v_g)' = 0, \quad (16)$$

$$|u + iv|_{z=h(x,y)+h_s}\{A\nu(\cos\alpha - \sin\alpha) + \sin\alpha\} - A\nu(u_g + v_g) = 0. \quad (17)$$

Solving for A and $|u + iv|_{z=h(x,y)+h_s}$ from Eqs. (16) and (17), we obtain

$$A = \frac{u_g(\cos\alpha - \sin\alpha) + v_g(\cos\alpha + \sin\alpha)}{2\nu(u_g\sin\alpha - v_g\cos\alpha)}, \quad (18)$$

$$|u + iv|_{z=h(x,y)+h_s} = u_g(\cos\alpha - \sin\alpha) + v_g(\cos\alpha + \sin\alpha). \quad (19)$$

Substitution of (19) into (15) gives

$$u + iv = (u_g + iv_g) - \exp\{-(1 + i)\nu[z - h(x, y) - h_s]\} \times \{(u_g + iv_g) - [u_g(\cos\alpha - \sin\alpha) + v_g(\cos\alpha + \sin\alpha)]e^{i\alpha}\}. \quad (20)$$

The real and imaginary parts of Eq. (20) can be shown to be, respectively,

$$u = u_g - \exp\{-\nu[z - h(x, y) - h_s]\} \times \{u_g\cos\nu[z - h(x, y) - h_s] + v_g\sin\nu[z - h(x, y) - h_s] - [u_g(\cos\alpha - \sin\alpha) + v_g(\cos\alpha + \sin\alpha)] \times \cos(\nu[z - h(x, y) - h_s] - \alpha)\}, \quad (21)$$

$$v = v_g - \exp\{-\nu[z - h(x, y) - h_s]\} \times \{-u_g\sin\nu[z - h(x, y) - h_s] + v_g\cos\nu[z - h(x, y) - h_s] + [u_g(\cos\alpha - \sin\alpha) + v_g(\cos\alpha + \sin\alpha)] \times \sin(\nu[z - h(x, y) - h_s] - \alpha)\}, \quad (22)$$

with

$$w = u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}. \tag{23}$$

To match the wind distribution in the surface boundary layer with the wind at the lower boundary of the planetary layer, let the wind distribution in the surface boundary layer be

$$|u+iv| = \frac{u_*}{k}\left\{\ln\left[\frac{z-h(x,y)+z_0}{z_0}\right] + a\left[\frac{z-h(x,y)}{L}\right]\right\}. \tag{24}$$

At the lower boundary of the planetary boundary layer, Eq. (5) becomes

$$|u+iv|_{z=h(x,y)+h_s} = \frac{u_*}{k}\left[\ln\left(\frac{h_s+z_0}{z_0}\right) + a\frac{h_s}{L}\right]. \tag{25}$$

Elimination of the right-hand side terms from (19) and (25) gives

$$u_* = \frac{k[u_g(\cos\alpha - \sin\alpha) + v_g(\cos\alpha + \sin\alpha)]}{\left[\ln\left(\frac{h_s+z_0}{z_0}\right) + a\frac{h_s}{L}\right]}. \tag{26}$$

It may be noticed that friction velocity u_* depends on G, L and α.

Substitution of (26) into (24) gives the velocity profiles in the surface boundary layer,

$$|u+iv| = \frac{[u_g(\cos\alpha - \sin\alpha) + v_g(\cos\alpha + \sin\alpha)]}{\left[\ln\left(\frac{h_s+z_0}{z_0}\right) + a\frac{h_s}{L}\right]} \times \left\{\ln\left[\frac{z-h(x,y)+z_0}{z_0}\right] + a\left[\frac{z-h(x,y)}{L}\right]\right\}, \tag{27}$$

$$w = |u+iv|\left(\cos\alpha\frac{\partial h}{\partial x} + \sin\alpha\frac{\partial h}{\partial y}\right),$$

for $h(x,y) \leqslant z \leqslant h(x,y) + h_s$.

We let the s and n axes be respectively oriented parallel and horizontally perpendicular to the geostrophic wind, and v_s and v_n be respectively the velocity components along the s and n axes. Therefore, G becomes a real number, α is the cross-isobaric angle of the wind at the surface, and Eq. (20) becomes

$$\begin{aligned} v_s + iv_n &= G + G\{(\cos\alpha - \sin\alpha)e^{i\alpha} - 1\} \\ &\quad \times \exp\{-(1+i)\nu[z - h(x,y) - h_s]\} \\ &= G + G\sin\alpha\{-(\sin\alpha + \cos\alpha) \\ &\quad + i(\cos\alpha - \sin\alpha)\}\exp\{-(1+i) \\ &\quad \times \nu[z - h)x,y) - h_s]\}. \end{aligned} \tag{28}$$

Since

$$e^{i[\alpha+(3/4\pi)]} = \cos[\alpha + (3/4\pi)] + i\sin[\alpha + (3/4\pi)], \tag{29}$$

Eq. (28) may be written as

$$\begin{aligned} v_s + iv_n &= G[1 + \sqrt{2}\sin\alpha\exp(-\nu[z - h(x,y) - h_s] \\ &\quad + i\{\alpha + (3/4\pi) - \nu[z - h(x,y) - h_s]\})]. \end{aligned} \tag{30}$$

Therefore,

$$v_s = G(1 + 2^{1/2}\exp\{-\nu[z - h(x,y) - h_s]\}\sin\alpha\cos\{\alpha + 3/4\pi - \nu[z - h(x,y) - h_s]\}), \tag{31}$$

$$v_n = 2^{1/2}G\exp\{-\nu[z - h(x,y) - h_s]\}\sin\alpha\sin\{\alpha + 3/4\pi - \nu[z - h(x,y) - h_s]\}, \tag{32}$$

$$\begin{aligned} w &= G(1 + 2^{1/2}\exp\{-\nu[z - h(x,y) - h_s]\}\sin\alpha\cos\{\alpha + 3/4\pi - \nu[z - h(x,y) - h_s]\})\frac{\partial h}{\partial x} \\ &\quad + 2^{1/2}G\exp\{-\nu[z - h(x,y) - h_s]\}\sin\alpha\sin\{\alpha + 3/4\pi - \nu[z - h(x,y) - h_s]\}\frac{\partial h}{\partial y}, \end{aligned} \tag{33}$$

for $z > h(x,y) + h_s$. Eq. (27) becomes

$$|v_s + iv_n| = \frac{G(\cos\alpha - \sin\alpha)}{\left[\ln\left(\frac{h_s+z_0}{z_0}\right) + a\frac{h_s}{L}\right]} \times \left\{\ln\left[\frac{z-h(x,y)+z_0}{z_0}\right] + a\left[\frac{z-h(x,y)}{L}\right]\right\},$$

$$w = |v_s + iv_n|\left(\cos\alpha\frac{\partial h}{\partial x} + \sin\alpha\frac{\partial h}{\partial y}\right), \tag{34}$$

for $h(x,y) \leqslant z \leqslant h(x,y) + h_s$.

The cross-isobaric angle α may be estimated from Eq. (32) by putting $v_n = 0$ at the geostrophic wind level, $z = H$. Thus,

$$\alpha = \left(\frac{\Omega\sin\phi}{K}\right)^{1/2}[H - h(x,y) - h_s] - 3/4\pi. \tag{35}$$

Eq. (35) indicates that the cross-isobaric angle is a function of the coefficient of eddy diffusivity, topographical configuration and latitude ϕ, and that α decreases with increasing height of the topography.

Substitutions of (35) into Eqs. (30), (31), (32) and (34) yield, respectively,

$$v_s = G(1 + 2^{1/2}\exp\{-\nu[z - h(x,y) - h_s]\} \times \sin\{\nu[H - h(x,y) - h_s] - \tfrac{3}{4}\pi\}\cos\nu(H - z)), \quad (36)$$

$$v_n = 2^{1/2}G\exp\{-\nu[z - h(x,y) - h_s]\} \times \sin\{\nu[H - h(x,y) - h_s] - \tfrac{3}{4}\pi\}\sin\nu(H - z), \quad (37)$$

$$w = G\frac{\partial h}{\partial x} + 2^{1/2}G\exp\{-\nu[z - h(x,y) - h_s]\} \times \sin\{\nu[H - h(x,y) - h_s] - \tfrac{3}{4}\pi\} \times \left[\cos\nu(H - z)\frac{\partial h}{\partial x} + \sin\nu(H - z)\frac{\partial h}{\partial y}\right], \quad (38)$$

for $z > h(x,y) + h_s$ and

$$|v_s + iv_n| = \frac{2^{1/2}G\cos\{\nu[H - h(x,y) - h_s] - \tfrac{1}{2}\pi\}}{\left[\ln\left(\dfrac{h_s + z_0}{z_0}\right) + a\dfrac{h_s}{L}\right]} \times \left\{\ln\left[\frac{z - h(x,y) + z_0}{z_0}\right] + a\left[\frac{z - h(x,y)}{L}\right]\right\}, \quad (39)$$

for $h(x,y) < z \leq h(x,y) + h_s$.

3. Flow characteristics in the atmospheric boundary layer over a circular mountain

To analyze the effects of topography, the Coriolis and pressure gradient forces on stationary flows of small Reynolds number in the planetary boundary layer, we consider a model mountain of which the height takes the form

$$h^*(x^*, y^*) = a\nu H\left\{\frac{1}{1 + \exp\{-b[(x^{*2} + y^{*2})^{1/2} + c]\}} + \frac{1}{1 + \exp\{b[(x^{*2} + y^{*2})^{1/2} - c]\}} - 1\right\}, \quad (40)$$

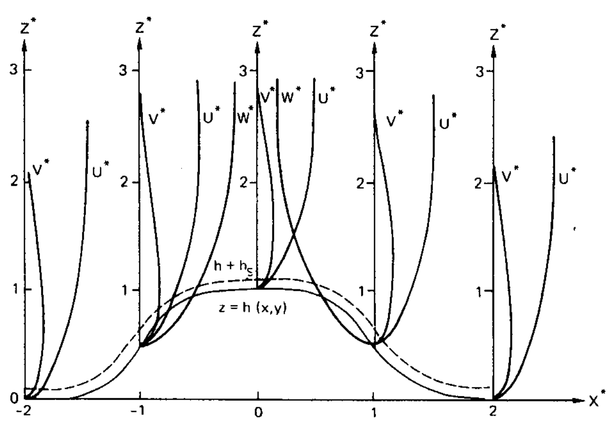

FIG. 1. Distribution of nondimensional velocity components in the vertical cross section passing through the mountain top, parallel to the geostrophic wind.

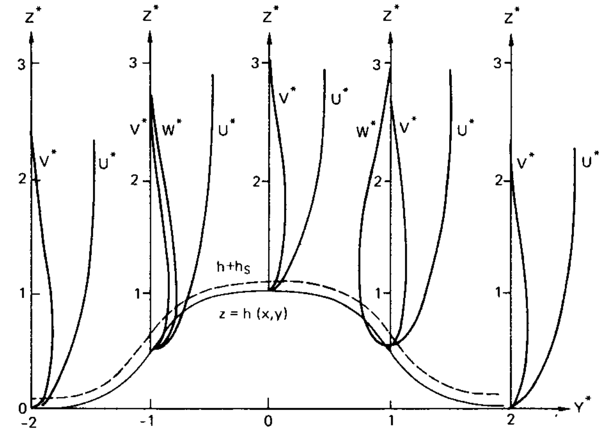

FIG. 2. As in Fig. 1 except the vertical cross section is perpendicular to the geostrophic wind.

where $x^* = \nu x$, $y^* = \nu y$, $h^* = \nu h$, $h_s^* = \nu h_s$ are the nondimensionalized coordinates x, y, height of the mountain, and the surface boundary thickness, respectively.

For $a = b = c = 1$, $\nu H = 1$, $h^*(0, 0) = 0.1$, $h_s^* = 0.1$, and $z_0^* = 0.0001$, we have computed the nondimensionalized velocity components, $U^* = v_s/G$, $V^* = v_n/G$ and $W^* = w/G$ with the use of Eqs. (36)–(39), and plotted the results on Figs. 1 and 2.

Fig. 1 shows the distribution of the nondimensional velocity components in the vertical cross section passing through the mountain top, parallel to the geostrophic wind. It is seen that $U^* = V^* = W^* = 0$ at the surface of the mountain, and that $U^* \to 1$, $V^* \to 0$ as $z \to \infty$. On the windward slope of the mountain, a horizontal convergence of U^* results in an upward motion, whereas on the lee side of the mountain a downward motion occurs as a consequence of a horizontal divergence of U^*.

Fig. 2 shows the distribution of the nondimensional velocity components in the vertical cross-section perpendicular to the geostrophic wind, passing through the mountain top. It is seen that $U^* = V^* = W^* = 0$ at the surface of the mountain, and that $U^* \to 1$, $V^* \to 0$ as $z \to \infty$. On the slope to the right of the mountain top, in the Northern (Southern) Hemisphere, a horizontal convergence (divergence) of V^* results in an upward (downward) motion, whereas on the slope to the left of the mountain top, in the Northern (Southern) Hemisphere, a horizontal divergence (convergence) of V^* contributes to a downward (upward) motion.

4. Conclusions

Analyses of the effects of topography, horizontal pressure gradient and Coriolis forces on the velocity distribution in the atmospheric boundary layer indicate that 1) the horizontal component of the velocity in the boundary layer turns right (left) with increasing height in the Northern (Southern) Hemisphere, 2) upward (downward) motion occurs on the windward (lee) side of the mountain, and 3) upward (downward) motion also occurs on the slope to the right (left) of the geostrophic wind in the Northern Hemisphere, whereas in the Southern Hemisphere downward (upward) motion occurs on the slope to the right (left) of the geostrophic wind. Therefore, there would be more chance of getting precipitation on the windward side and on the slope to the right (left) of the geostrophic wind in the Northern (Southern) Hemisphere.

It should be mentioned that the purpose of this paper is to seek basic characteristics of three-dimensional flow of small Reynolds number in the boundary layer over terrain in a rotating system. Because of the constraint of small Reynolds number, the solution presented here can only be applied to flow of small velocity over terrain of comparatively small height.

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