Effects of Aliasing on Spectral Moment Estimates Derived from the Complete Autocorrelation Function

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Estimation of Doppler spectral moments based on time-domain measurements of the complex autocorrelation function is common practice in radar meteorology (e.g., Zrnić, 1977). Using these efficient, time-domain algorithms, the signal power and the first and second moments of the Doppler spectrum can be estimated in real time for a large number of range cells with inexpensive hardware. These estimates are related to the radar reflectivity factor, the mean radial velocity, and the variance of the radial velocity of the scatterers in the radar sampling volume.

Autocorrelation algorithms for spectral moment estimation are usually based on a simple model of the autocorrelation function which is formulated in terms of the Doppler spectral moments. The accuracy of the moment estimates depends on the accuracy of the model and the accuracy of the autocorrelation estimates. Despite the common usage of time-domain estimators, the effects of velocity aliasing on the autocorrelation function may not be widely understood. This note demonstrates that while the argument of the autocorrelation function (which is related to the mean radial velocity) is biased by velocity aliasing, the magnitude (which is related to the spectral width) is not. In other words, estimates of the magnitude of the autocorrelation function are valid even when the Doppler spectrum is so broad that it is folded upon itself. Zrnić (1977) implies this in passing when he points out that "Aliasing from undersampling . . . does not bias the (spectral width) estimator. . . ." Here we present a simple proof of this and a brief discussion of its implications.

A pulsed Doppler radar provides uniformly spaced complex samples of the backscattered signal, that is, a sample function from a continuous complex random process. For a given pulse repetition frequency (PRF), the unambiguous Doppler frequency shift is

$$\omega_u = \frac{\pi}{\mathrm{PRF}} = \pi\tau_s, \qquad (1)$$

where τ_s is the sampling interval. The Nyquist interval is $[-\omega_u, \omega_u]$ which spans $2\omega_u$. The radar returns from a range cell form a discrete complex time series from which the power spectral density or Doppler spectrum can be computed. If a frequency component in the backscattered signal (such as the mean Doppler shift frequency) exceeds ω_u, it is aliased into the Nyquist interval by $\pm 2n\omega_u$, where n is an integer. If the bandwidth of the backscattered signal exceeds the Nyquist interval, then aliased frequency components will add to one another in the Nyquist interval. In this case, the Doppler spectrum is ambiguous. Intuitively, one would expect the autocorrelation and hence the spectral moment estimates to be ambiguous as well.

It is readily apparent that velocity aliasing has no effect on the computation of signal power. As long as the process is stationary, the estimate of signal power is unbiased regardless of the sampling rate or velocity aliasing. The variance of the estimate is, of course, determined by the number of independent samples used to compute the estimate. The signal power may be computed from the zero lag correlation or from the (ambiguous) Doppler spectrum. First-moment estimates, found from either the power spectrum or the phase of the autocorrelation, are ambiguous since the mean Doppler shift frequency can be aliased. Several algorithms have been developed for estimating the second moment of the Doppler spectrum (e.g., Srivastava *et al.*, 1979; Passarelli and Siggia, 1983). These are based on the magnitude of the autocorrelation function

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and it is not obvious that these algorithms are valid when the Doppler spectrum is ambiguous (i.e., is broader than the Nyquist interval). The proof given below shows that the measured magnitude of the autocorrelation function is not biased by a signal bandwidth constraint.

Consider the autocorrelation function R_k ($k = 0, 1, 2 \cdots$) of an infinite discrete time series. Its Fourier transform $S_\pi(\theta)$ is continuous on the nondimensional Nyquist frequency interval $[-\pi, \pi]$; i.e.,

$$S_\pi(\theta) = \sum_{k=0}^{\infty} R_k e^{-jk\theta}, \tag{2}$$

$$R_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_\pi(\theta) e^{jk\theta} d\theta. \tag{3}$$

Let $S_\infty(\theta)$ represent the underlying spectrum on $[-\infty, \infty]$ such that

$$S_\pi(\theta) = \sum_{n=-\infty}^{+\infty} S_\infty(\theta + 2\pi n). \tag{4}$$

The autocorrelation expressed in terms of S_∞ is then

$$R_k = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\int_{-\pi}^{\pi} S_\infty(\theta + 2\pi n) e^{jk\theta} d\theta, \tag{5}$$

or, letting $\phi = \theta + 2\pi n$,

$$R_k = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-j2\pi nk}\int_{-\pi+2\pi n}^{\pi+2\pi n} S_\infty(\phi) e^{jk\phi} d\phi. \tag{6}$$

Now $e^{-j2\pi nk} = 1$ for all n and k so that after expanding the sum, one obtains

$$R_k = \frac{1}{2\pi}\Big[\cdots + \int_{-5\pi}^{-3\pi} S_\infty(\phi) e^{jk\phi} d\phi + \int_{-3\pi}^{-\pi} (\cdots) d\phi + \int_{-\pi}^{\pi} (\cdots) d\phi + \int_{\pi}^{3\pi} (\cdots) d\phi + \int_{3\pi}^{5\pi} (\cdots) d\phi + \cdots\Big]. \tag{7}$$

However, the domains of integration are contiguous, covering the interval $[-\infty, \infty]$ so that

$$R_k = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_\infty(\phi) e^{jk\phi} d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_\pi(\theta) e^{jk\theta} d\theta. \tag{8}$$

Thus the autocorrelation of a folded spectrum on $[-\pi, \pi]$ is indistinguishable from that of its unfolded parent spectrum. Of course $\arg\{R_k\}$ can only be distinguished on the interval $[-\pi, \pi]$, but $|R_k|$ is unaffected by folding.

It is surprising that even for ambiguous spectra, one can use the unbiased measurements of $|R_k|$ in an autocorrelation algorithm and obtain a spectral width estimate since aliasing would preclude a direct frequency-domain computation of the spectral width. We know that the information contained in either domain is identical, so how is this contradiction resolved? The answer is that autocorrelation algorithms for spectral width assume a model for the autocorrelation function that has a corresponding spectral model. Under appropriate constraints on uniqueness, one could use the spectral model to invert (4) and determine the unfolded spectrum. Hence, the difference in information content is only apparent.

An implication is that time–domain spectral width estimates are valid, even for spectra that are so broad they are ambiguous. However, as discussed by Passarelli and Siggia (1983), the accuracy of a spectral width estimate is critically dependent on the accuracy of the underlying autocorrelation model, especially when the spectrum is very broad. This estimator bias can be reduced by constructing a more general autocorrelation model that employs measurements of R_k at a greater number of lags k. Another factor is the greater uncertainty in measurements of R_k for broad spectra. This can be countered by averaging over a greater number of pulses. Thus, given an accurate autocorrelation model and accurate estimates of R_k, autocorrelation-based algorithms will work for even ambiguous Doppler spectra.

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