Fractals, Raindrops and Resolution Dependence of Rain Measurements

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ABSTRACT

Large (128 × 128 cm) pieces of chemically treated blotting paper were exposed to rain and both the size and position of the drops were determined. Analyses were performed indicating that the spatial distribution is fractal. This implies that drops cluster over all the observed scales and, hence, that backscattered microwave radiation from weather radars will have a degree of coherent scattering and a systematic dependence on the measurement resolution not accounted for in the standard theory. This was quantified by two scaling exponents, and a scheme to correct radar measurements for these fractal effects was developed.

1. Introduction

Over forty years ago, Marshall and Palmer (1948) used chemically treated blotting paper to make the first measurements of the probability distribution of rain drop volumes in rain of various intensities. This distribution plays an important role both in cloud physics as well as in radar measurements of rain; in radar meteorology, the parameterized (exponential) form is called a "Marshall–Palmer" distribution. However, theories of drop formation and quantitative rain estimates require more than just the relative probabilities of drops of different sizes; they also require knowledge of the relevant spatial distributions. Usually, the latter are assumed homogeneous (the drops have Poisson statistics). When applied to radar "pulse volumes" (typically about 1 km^3), these assumptions (Marshall and Hitschfeld 1953; Wallace 1953) imply incoherent scattering. In this case, the variability in the observed "effective radar reflectivity factor" (Z_e) arises from two sources. The first is the natural variability of interest characterized by Z (the "radar reflectivity factor," proportional to the sum of the squares of the drop volumes), while the second (which could in principle be statistically removed), is due to the random positions (and hence phases) of each of the drops within the pulse volume. Under certain additional assumptions, Z can be related to the rain rate, volume of liquid water, or other parameters of interest. The determination of Z from the observed Z_e is therefore considered the basic "observer's problem" in radar meteorology.

However, rain is not homogeneous. Due to the action of cascade processes concentrating energy, water and other conserved fluxes into smaller and smaller regions of space, rain is highly variable, displaying scaling (multi)fractal structures over significant ranges in scale (Lovejoy 1981, 1982; Lovejoy and Mandelbrot 1985; Lovejoy et al. 1987; Gabriel et al. 1988; Lovejoy and Schertzer 1990a,b; Schertzer and Lovejoy 1987a,b, 1989, 1990). This subsensor inhomogeneity will lead to corrections in the standard theory. The corrections discussed here could be termed "monofractal"; they involve only two exponents and characterize the bias in the mean reflectivity factors. In another paper (Lovejoy and Schertzer 1990a), we investigate other corrections due to multifractal effects that will lead to range dependencies in the probability distribution of the reflectivity factor and that requires an entire exponent function for its specification.

2. Description of the experiment

To investigate the inhomogeneity, we followed Marshall and Palmer who dropped carefully calibrated drops down the four floors of the stairwell of the Macdonald physics building at McGill University and found that the colored stains on chemically treated blotting paper had radii (ρ) related (with little statistical scatter) to the original drop volume (V) as $V \propto \rho^2$. This is the relation expected if the penetration depth of the water into the blotting paper is constant. Our improvements with respect to Marshall and Palmer included 1) use of much larger pieces of paper (128

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$\times$ 128 cm compared to the original 16 $\times$ 24 cm size), 2) digitization of the results, and 3) recording of the positions (r_i) as well as the volumes of the drops. By exposing the blotting paper in rainfall for very short times ($\approx$1 s), we attempted to obtain a horizontal intersection (cross section) of the true (V_i, r_i) distribution in three-dimensional space. The apparatus used consisted of two square covers, one on either side of a square hole the size of the paper. This "shutter" was quickly pulled across the blotting paper during a moderately heavy stratiform rain in Montreal. The experiment was carried out by B. Miville and T. Pham as part of a third-year physics lab project. A total of three pieces of blotting paper were exposed, but owing to technical problems and limited time, only the one discussed here was digitized. The other two were manually analyzed in a different way (various multifractal characteristics were analyzed; see Lovejoy and Schertzer 1990b for more details). Obviously in order to establish the representativeness of the empirical parameter, much more work must be done; our intention here is to show that scaling of raindrops in space is not only plausible theoretically, but is also consistent with the first direct observations. All previous relevant studies of which we are aware (such as those of the fluctuating return of a radar signal) give only indirect information on the homogeneity/inhomogeneity of the drops in space, and are likely to be consistent with our findings.

One other experimental point is worth mentioning. Since the object was to investigate the instantaneous horizontal structure of rain, and given that rain drop fall speeds are typically 2–5 m s^{-1}, one second is not as short a time as might be hoped. To put the problem in context, consider very long exposures. In this case, [taking the rain as an (x, y, z, t) process], the blotting paper will record the *projection* of the rain on the x–y plane. However, the properties of projections and intersections are quite different. Any component of the multifractal rain measure with dimension $D \geqslant 2$ will lead to planar projections (i.e., the projection has dimension 2, and the blotting paper gets wet everywhere), whereas intersections will have dimensions less than two (see below).

Figure 1 shows the points corresponding to the centers of the circular blobs on the blotting paper; in this case there are 452 of them. The drop positions were digitized along with their radius (to an accuracy of 0.5 mm). The statistically most sensitive analysis method is to estimate the "correlation dimension" (D_2) of the (two dimensional) cross section. This is done by considering the function $\langle n(L) \rangle \propto L^{D2}$, which is the average number of other drops in a radius L around each drop. Since there are 452 $\times$ 451/2 = 101 926 drop pairs, this function contains a great deal of information about the drop clustering. Figure 2 shows that over the range 2 mm $\leqslant L \leqslant$ 40 cm, that $D_2 \approx$ 1.83. The large L behavior deviates below the line because many of these large circles go outside the blotting paper

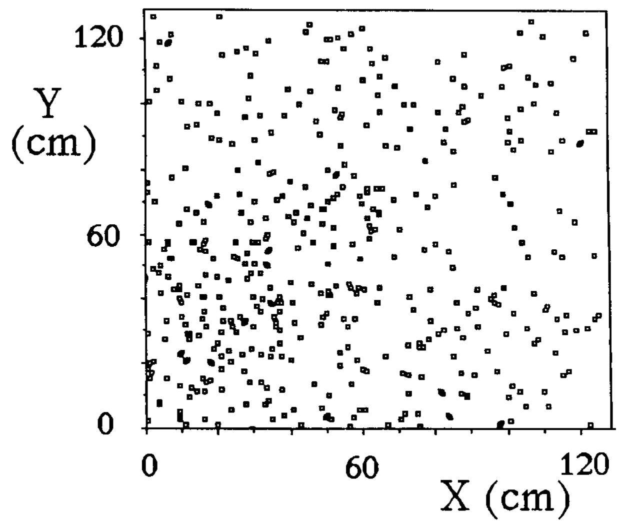

FIG. 1. Each point represented the center of a raindrop for the 128 $\times$ 128 cm piece of chemically treated blotting paper discussed in the text. There are 452 points, the exposure was about 1 s.

and the estimate of $\langle n(L) \rangle$ is, therefore, biased downwards. At the small scale end, a bias obtained due to the finite number of points; for example, clearly $\langle n(L) \rangle > 452^{-1}$. We therefore take this as evidence that rainfall is scaling over this range. Box counting can also be used to give an (less robust) estimate of D_2; on the manually digitized cases we obtained D_2 = 1.79 and D_2 = 1.93. Below, we continue to use the value 1.83, although clearly much more data must be analyzed for precise estimates.

3. Theoretical development and analysis

In order to extrapolate the $\langle n(L) \rangle$ result from the measured (horizontal) intersection to the full x, y, z space, the strong horizontal stratification of the rain process due to gravity must taken into account. Introducing the "codimensions" $C_3 = 3 - D_3$, $C_2 = 2 - D_2$ ($\approx$0.17 here) and using the formalism of "generalized scale invariance" (Schertzer and Lovejoy 1985a,b, 1987a,b), in x, y, z space, we expect $\langle n(L) \rangle \propto L^{D3} = L^{3-C3}$ with

$$C_3 = C_2 \frac{3}{d_{el}} \quad (1)$$

where d_{el} is the "elliptical" dimension of the rain process characterizing the stratification, estimated (Lovejoy et al. 1987) to have the value d_{el} = 2.22 $\pm$ 0.07 in rain (d_{el} would be three if the space was isotropic, and two if it was completely stratified into flat layers). Using Eq. 1, and expressing n in terms of the volume $v = L^3$ we obtain:

$$\langle n(v) \rangle \propto v^{(1-C_2/d_{el})} \quad (2)$$

Hence, using the above values of C_2, d_{el} in Eq. 2 the

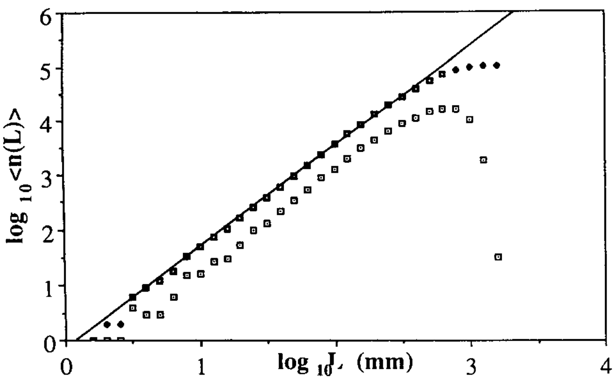

FIG. 2. This shows a log–log plot of the average number of drops in a circle size L surrounding each drop of Fig. 1 (solid squares), as well as the number in equally logarithmically spaced annuli (open squares—the sum of the latter from the smallest scale to L, gives the former). The straight-line has a slope of 1.83, and was fitted through the part of the graph (shown in squares) that was relatively unaffected by the finite number of drop fall-off at small L and the large L fall-off due to the finite size of the blotting paper.

drop density $\langle n(v)\rangle / v$ is no longer constant but decreases as $v^{-0.08}$.

The scaling nature of the drop distribution implies that the drops are (hierarchically) clustered over the range, and that when the microwaves scatter from the drops that there will be some degree of coherence. To quantify this, consider a radar at the origin that emits a pulse of electromagnetic waves that fills a volume $v = l \times r\theta \times r\theta$ where r is the range, θ the angular width of the radar beam, and l is the pulse length. The power received at the radar depends on various instrumental characteristics including the transmitter, antenna geometry etc. Putting these factors into a multiplicative constant (ignored below) and statistically averaging (indicated by angle brackets), the radar measures

$$\langle Z_e \rangle \propto \frac{\langle |A|^2 \rangle}{v} \qquad (3)$$

where

$$A = \sum_{j}^{n(v)} V_j e^{i\phi_j}. \qquad (4)$$

This formula expresses the fact that each drop has a cross section proportional to its volume (since water is a polar molecule). The phase $\phi_j = 2k \cdot r_j$ where k is the wave vector and the factor 2 arises because of the round-trip distance from the radar to the drop is $2r_j$. It is customary to introduce the "radar reflectivity factor" (usually measured in units of mm^6/m^3) whose ensemble average $\langle Z \rangle$ is defined by

$$\langle Z \rangle = \frac{\langle n(v) \rangle}{v} \langle V^2 \rangle. \qquad (5)$$

If the drops are uniformly randomly distributed (i.e., they have Poisson statistics), the ϕ_j are statistically independent. Considering the complex sum A as a random walk in phase space, as long as $\langle V^2 \rangle < \infty$ the central limit theorem applied to (4) implies $\langle |A|^2 \rangle = \langle n(v) \rangle \langle V^2 \rangle$ and, hence, the classical result $\langle Z_e \rangle = \langle Z \rangle$. However, if the drops are distributed over a fractal, we have partially coherent scattering and we expect drop correlations to yield an anomalous exponent:

$$\langle |A|^2 \rangle \propto \langle n(v) \rangle^{2H} \langle V^2 \rangle \qquad (6)$$

where $H = 1/2$ for completely incoherent scattering, and $H \neq 1/2$ when some degree of coherent scattering is present. Hence,

$$\langle Z_e \rangle \propto \langle Z \rangle \langle n(v) \rangle^{2H-1}. \qquad (7)$$

In order to evaluate H from the blotting paper we used the following procedure. First, in order to reduce statistical scatter, we take $|k|$ fixed and averaged over wavevectors in 19 equally spaced directions, adding more and more terms in the sum (A) by choosing drops at random from the 452 available. Figure 3 shows that convergence to a power law is obtained for $n \geqslant 16$. Varying $|k|$ in 10 equal logarithmic increments through the scaling region, from $2\pi/128\ cm^{-1}$ to $2\pi/1.28\ cm^{-1}$ (corresponding to distances of 1.28 to 128 cm), we obtained $2H = 1.24 \pm 0.09$ where the error is the standard deviation of the $2H$ values estimated from each of the values of $|k|$.

4. The correction exponents

We can now combine this result with our previous formula (2) for $n(v)$ to obtain the volume (and, hence, range) dependence of $\langle Z \rangle$, $\langle Z_e \rangle$. Recalling that $v = l\theta^2 r^2$ and keeping only the r dependence, combining

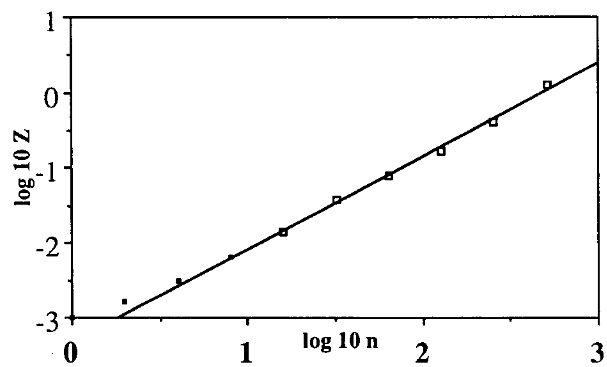

FIG. 3. The variation of the average (nondimensionalized) effective radar reflectivity of the distribution in Fig. 1 as a function of the number of drops (n). The curve is calculated as indicated in the text and involves averaging over 19 angles in Fourier space, and 10 logarithmically spaced wavelengths from 1–128 cm. The straightline shows the asymptotic power law behavior that is obtained for $n \geqslant 16$, with slope $= 2H = 1.24$ ($H = 0.62$). Note that white noise would yield $H = 1/2$. The increase implies some degree of coherent scattering.

Eqs. (2), (5), (7), and using the notation $\langle Z \rangle \propto r^{\xi}$, $\langle Z_e \rangle \propto r^{\xi_e}$ we obtain

$$\xi = -\frac{2C_2}{d_{el}}$$

$$\xi_e = 4H\left(1 - \frac{C_2}{d_{el}}\right) - 2. \qquad (8)$$

Taking $C_2 \approx 0.17$, $H = 0.62$, $d_{el} = 2.22$ yields $\xi = -0.15$, $\xi_e = 0.28$ (recall that the standard values are $C_2 = 0$, $H = 1/2$, $d_{el} = 3$, hence $\xi = \xi_e = 0$). To judge the overall magnitude of these effects, consider a weather radar such as the 10 cm wavelength radar at McGill, with minimum range (limited by ground echoes) of $\approx$10 km, and maximum range $\approx$ 240 km. Comparing near and far range, we obtain a variation in $\langle Z_e \rangle$ of $\approx 24^{0.28} \approx 2.4$, and a corresponding variation in $\langle Z \rangle$ of $24^{-0.15} = 0.6$. These effects are somewhat larger in magnitude than those due to absorption (by humidity, O_2, and by the drops themselves) and should be taken into account during radar calibration from rain gages.

5. Conclusions

We have argued that although inhomogeneity in rain is likely to extend down to millimeter scales, it can nevertheless be simply characterized by the scaling exponents C_2 (the codimension of the drop distribution in the horizontal plane) and H (the scaling exponent of the reflectivity factor with respect to the number of drops). The standard values, corresponding to perfectly uniform random distributions, yield $C_2 = 0$, $H = 1/2$, whereas we argue that $C_2 > 0$, $H \neq 1/2$. To support this idea, we report on the first direct empirical estimates of C_2, H obtained by exposing chemically treated blotting paper to rain, finding $C_2 \approx 0.17$, $H \approx 0.62$. Furthermore, the scaling leads to straightforward corrections to the standard theory in which the relationship between the radar measurements and the rain process involves a factor equal to the resolution of the sensor raised to a power whose value we estimate.

Resolution dependence is a general and basic problem in the remote sensing of geophysical fields since physical quantities should be independent of the characteristics of the sensors used to measure them, and calibration procedures typically involve comparing remote and in situ data averaged over very different time and space scales. Quantitative uses of remotely sensed data will require systematic development of resolution-independent measurement techniques probably based on the dimension function that provides a natural scale invariant description of both weak and intense regions of multifractal fields.

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