A General Formulation for Raindrop Size Distribution

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ABSTRACT

A general phenomenological formulation for drop size distribution (DSD), written down as a scaling law, is proposed. It accounts for all previous fitted DSDs. As a main implication of the expression proposed, the integral rainfall variables are related by power functions and agree with experimental evidence. Additional consequences are also analyzed. From this formulation there follows a general methodology for scaling all data in a unique plot, leading to more robust fits of the DSD. An illustrative example on real data is provided.

1. Introduction

Analytical formulations of the drop size distribution (DSD) can be used either to explicitly describe the distribution of the rainwater in the atmosphere for modeling purposes, or to derive relations between bulk rain variables (such as the total liquid water content per unit of air volume, the attenuation and scattering of electromagnetic waves propagating through the rain, the rain erosivity, etc.) and the rain intensity. The DSD is usually expressed using a distribution function $N(D)$ that gives $N(D)dD$ the mean number of drops per unit of air volume with diameters between D and $D + dD$.

In their pioneering study, Marshall and Palmer (1948, hereafter MP) proposed an exponential formulation for $N(D)$ according to the analysis of two datasets measured with the filter paper method. In spite of its attractive simplicity, this exponential DSD shows some inadequacies in describing other observed spectra, especially those with sampling time of one minute or less. It generally tends to overestimate the number of both the smallest drops (Waldvogel 1974) and the largest drops (Joss and Gori 1978). Therefore, more sophisticated formulations (e.g., generalized exponential, gamma, lognormal, or Weibull distribution functions) have been derived later, relying on complementary experimental data [see, e.g., the summaries proposed by Ulbrich (1983), Feingold and Levin (1986), and Zawadzki and De Agostinho Antonio (1988)].

All these studies formulate the DSD making the assumption that a reference bulk variable governs the parameters of the DSD. It is worth noting that the most common reference variable is the rain intensity R, probably because many DSD datasets were collected at ground level, where the rain flux is a natural reference. However, its role could be played by any other integral rainfall function, in general Ψ, without modifying the methodology. Thus, the DSD can be generally expressed as a distribution function of D and of the chosen integral variable Ψ—that is, $N(D, \Psi)$ [instead of $N(D)$].

The most common methodology, coming from MP's work, groups the measured realizations into classes of rainfall intensity in order to compute a mean spectrum for each one. Then, a distribution function with several parameters is fitted to each mean spectrum, and these parameters are related to the rain intensity through a power law.

An improvement of the methodology was proposed by Sekhon and Srivastava (1970, 1971) and used later by Willis (1984). They suggested a normalization of the diameters and of the drop size distribution in order to deal directly with the whole set of the spectra on a unique plot, where the parameters of the DSD could be fitted more robustly and independently of R. Using the median-volume diameter D_0 and the total liquid water content per unit of air volume W, Sekhon and Srivastava calculated dimensionless spectra and pro-

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posed a way of fitting a "*universal*" DSD (Willis 1984, p. 1650), which is more properly a "general" DSD since only the dependence on R, but not on other factors such as the kind of rainfall or the local characteristics, is lost. Nevertheless, this methodology requires setting the shape of the general DSD before the normalization procedure, and even if the parameters are fitted in a more robust way the shape is a priori chosen and does not follow from the data.

An additional method is proposed by Ulbrich (1983), who uses three convenient integral rainfall variables, written as moments of order 3, to fit the parameters of a gamma distribution function. In fact, the use of the moments of the DSD was first proposed by Waldvogel (1974) but it is more clearly stated as a method by Ulbrich. Again the use of classes of Ψ is no longer necessary, but the requirement of choosing the analytical function of the DSD a priori is anew a limitation. Therefore a more general alternative is still needed.

On the other hand, many studies have dealt with relating bulk rain variables without accounting explicitly for the DSD. The aim of these studies was to relate these properties to the reference variable, commonly the rainfall intensity R, and a power function was the favorite analytical form [see, e.g., Battan (1973) for the rainfall reflectivity $Z(R)$ and the total liquid water content per unit of air volume $W(R)$; Delrieu et al. (1991) for the rainfall attenuation $K(R)$; Sempere Torres et al. (1992) for the rainfall kinetic energy $KE(R)$ used in erosion studies; Laws and Parsons (1943), Atlas (1953), or Brandt (1989) for the median-volume diameter $D_0(R)$; and Steiner (1991) for the reflectivity-weighted vertical mean Doppler velocity $V_0(Z)$, just as an example where reflectivity is used as the reference variable]. These experimental relationships must be explained by any DSD model proposed.

This paper suggests (sections 2 and 3) that a general formulation of $N(D, \Psi)$ can account for these experimental relations as well as for all the previously used DSD models. Section 4 examines the implications of the choice of an integral rainfall variable Ψ deduced from the DSD as the reference variable. Finally, section 5 gives a methodology to fit the DSD model to experimental spectra.

2. A scaling law for the DSD

The DSD models used up to now in hydrometeorological studies [the exponential function proposed by MP, the Weibull distribution by Best (1951), the gamma distribution function by Ulbrich (1983) or Willis (1984), the lognormal function by Feingold and Levin (1986), etc.] are particular cases of a general expression where the number of drops per unit of air volume in the size range D to $D + \Delta D$, $N(D, \Psi)$, depends on D and on the reference variable Ψ as

$$N(D, \Psi) = \Psi^{\alpha_\Psi} g\left(\frac{D}{\Psi^{\beta_\Psi}}\right). \qquad (1)$$

In this general expression Ψ can be any integral rainfall variable although R has generally been used. For a given Ψ, α_Ψ and β_Ψ are constants (they do not have any functional dependence on Ψ), and g is a function that is independent of the value of Ψ and that will be called the general distribution function.

The expression (1) is in fact a scaling law like those applied in thermodynamics to the study of the critical phenomena (Stanley 1971), in polymer physics (De Gennes 1979), in nucleation processes, in percolation theory, and in surface growth (Vicsek 1992). In our context, the scaling law is a heuristic approach valid over a wide range of scales (i.e., over a wide range of intensities, or liquid water content).

Precipitation results from the interactions between raindrops, cloud droplets, water vapor, and ice particles. This dynamic process can be seen as a continuous competition between different mechanisms acting on the increase or decrease of drop sizes. As a result, the system responds by giving rise to the power-law relations experimentally founded, and thus to the scaling law for the DSD. The peculiarity here is that the scaling variable Ψ is an integral function of $N(D, \Psi)$.

This general expression allows the known DSD to be rewritten in a simpler way, as shown with selected examples:

(i) The exponential function proposed by MP was originally expressed as

$$N(D) = N_0 \exp(-\lambda D), \qquad (2)$$

where $N_0 = 0.08$ cm^{-4} and λ is in fact the inverse of the mean diameter, given by MP as

$$\lambda = 41 R^{-0.21}, \qquad (3)$$

with D in centimeters and R in millimeters per hour. It can be rewritten in the general way (1) as

$$N(D, R) = R^{\alpha} g\left(\frac{D}{R^{\beta}}\right), \qquad (4)$$

where Ψ is R, $\alpha_R = \alpha = 0$, $\beta_R = \beta = 0.21$ (the subindex is omitted in the case of R) and $g(x)$ is

$$g(x) = 0.08 \exp(-41x). \qquad (5)$$

(ii) The Sekhon and Srivastava version (1971) of the exponential function (2) with

$$N_0 = 0.07 R^{0.37} \qquad (6)$$

and

$$\lambda = 38 R^{-0.14} \qquad (7)$$

can be rewritten according to (1) with $\alpha = 0.37$, $\beta = 0.14$, and

$$g(x) = 0.07 \exp(-38x) \tag{8}$$

(the same units as in the MP expression).

(iii) The gamma distribution function given by Willis and Tattelman (1989) in the same units as

$$N(D) = N_0 D^{2.16} \exp\left(-5.59 \frac{D}{D_0}\right) \tag{9}$$

$$N_0 = \frac{512.9 \times 10^{-6} W}{D_0^4} \left(\frac{1}{D_0}\right)^{2.16} \tag{10}$$

$$D_0 = 0.157 W^{0.168} \tag{11}$$

(W in grams per cubic meter) can be rewritten as

$$N(D, D_0) = D_0^{\alpha_{D_0}} g\left(\frac{D}{D_0^{\beta_{D_0}}}\right), \tag{12}$$

with $\alpha_{D_0} = 1.95$, $\beta_{D_0} = 1$, and

$$g(x) = 31.0 x^{2.16} \exp(-5.6x). \tag{13}$$

Formula (12) can in turn be modified as

$$N(D, R) = R^{0.3} g\left(\frac{D}{R^{0.153}}\right) \tag{14}$$

and

$$g(x) = 50.6 x^{2.16} \exp(-56.8x) \tag{15}$$

using the relation between D_0 and R provided by Willis and Tattelman (1989):

$$D_0 = 0.0984 R^{0.153}. \tag{16}$$

(iv) The lognormal distribution proposed by Feingold and Levin (1986) as

$$N(D) = \frac{N_T}{(2\pi)^{1/2} \ln\sigma} \frac{1}{D} \exp\left[-\frac{\ln^2(D/D_g)}{2 \ln^2\sigma}\right] \tag{17}$$

$$N_T = 172 R^{0.22} \tag{18}$$

$$D_g = 0.72 R^{0.23}, \tag{19}$$

with $\sigma = 1.43$ [$N(D)$ in per meter per millimeter, R in millimeters per hour, and D in millimeters], can also be rewritten according to (4) in the general form with $\alpha = -0.01$, $\beta = 0.23$, and

$$g(x) = 191.85 x^{-1} \exp[-3.91 \ln^2(1.39x)]. \tag{20}$$

(In fact, the authors found a slight dependence of σ on R, $\sigma = 1.43 - 3 \times 10^{-4} R$ for $R > 5$ mm h^{-1}. However, they recognize that σ can be used as a constant over a wide range of rain rates.)

Table 1 summarizes the equivalences of some previously fitted $N(D)$ with the general expression proposed here.

3. Power relations between bulk variables

A number of implications follow from the formulation (1).

The first one is that the reference variable Ψ and the other bulk variables written as integral moments of order n of $N(D, \Psi)$ are powerly related.

Writing these bulk variables Ω_n as

$$\Omega_n = A_n \int_0^\infty dD N(D, \Psi) D^n \tag{21}$$

(where A_n is a constant), the liquid water content per unit of volume W is Ω_3, the radar reflectivity factor Z is Ω_6, and the mean diameter $\bar{D}$ is Ω_1/Ω_0. In what follows it is assumed that the range of variation of D is $(0, \infty)$. This mathematically useful assumption is frequently used in DSD studies and should be considered as a first approximation. Sekhon and Srivastava (1970, 1971), Ulbrich (1983), Willis (1984), and especially Ulbrich (1985) examined its consequences and proposed that the truncation effects (considering $D_{\min}$ and $D_{\max}$ instead of 0 and ∞) can be neglected in the most common types of rainfall, where $D_{\min}$ and $D_{\max}$ are sufficiently smaller and larger than the mean diameter. This essentially means that when D is bigger than the maximum drop diameter $D_{\max}$, $N(D, \Psi)$ is so small that the total contribution to Ω_n can be practically neglected.

Substituting expression (1) on the right-hand side of (21), and writing $x = D/\Psi^\beta$, one obtains

$$\Omega_n = a\Psi^b, \tag{22}$$

where

$$a = A_n \int_0^\infty dx\, g(x) x^n \tag{23}$$

and

$$b = \alpha_\Psi + (n + 1)\beta_\Psi. \tag{24}$$

Note that the analytical form of relation (22) and of exponent b (24) are independent of $g(x)$. In the particular case of taking D_0 as the reference variable and $g(x)$ as a gamma distribution function, the above equations lead to Eq. (7) given by Ulbrich (1983, p. 1766).

As an example, if Ψ is the rain intensity R, then

$$W = a_w R^{\alpha+4\beta} \tag{25}$$

$$Z = a_z R^{\alpha+7\beta} \tag{26}$$

and

$$\bar{D} = a_{\bar{D}} R^\beta. \tag{27}$$

Consequently, the proposed scaling law leads to power relations between integral rainfall parameters and R that agree with the experimental evidence. If they are considered as the best-suited ones, their derivation from the general formulation could be considered as an indicator of its consistency.

TABLE 1. Equivalence between the general formulation and the previous fitted functions in the literature: (a) R is the reference variable; (b) D_0 is the reference variable; (c) W is the reference variable.

Author			Function $g(x)$	Rainfall type
$\Psi = R$	α	β		
Marshall and Palmer (1948)	0.00	0.21	$g(x) = 0.08 \exp(-41x)$	widespread
Feingold and Levin (1986)	−0.01	0.23	$g(x) = 108.2x^{-1} \exp(-3.9 \ln^2 x)$	widespread
Delrieu et al. (1991)	0.025	0.215	$g(x) = 0.031 \exp(-35x)$	widespread
Jones (1956)	0.51	0.11	gamma*	widespread and stratiform rain
Jones (1956)	0.26	0.16	gamma*	widespread and stratiform rain
Atlas and Chmela (1957)	0.18	0.18	gamma*	widespread and stratiform rain
Fujiwara (1965)	0.04	0.21	gamma*	widespread and stratiform rain
Joss and Waldvogel (1968)	0.00	0.21	$g(x) = 0.07 \exp(-41x)$	widespread
Joss and Waldvogel (1968)	0.00	0.21	$g(x) = 0.30 \exp(-50x)$	drizzle
Wexler (1948)	−0.06	0.23	gamma*	orographic rain
Blanchard (1953)	−0.43	0.31	gamma*	orographic rain
Ramana et al. (1959)	−0.29	0.28	gamma*	orographic rain
Higgs (1952)	−2.74	0.80	gamma*	shower
Jones (1956)	0.50	0.10	gamma*	shower
Imai (1960)	0.00	0.22	gamma*	shower
Muchnik (1961)	−0.40	0.30	gamma*	shower
Fujiwara (1965)	0.26	0.16	gamma*	shower
Foote (1966)	−0.63	0.35	gamma*	shower
Blanchard (1953)	0.18	0.18	gamma*	thunderstorm rain
Sivaramakrishnan (1961)	0.18	0.18	gamma*	thunderstorm rain
Fujiwara (1965)	0.08	0.20	gamma*	thunderstorm rain
Joss and Waldvogel (1968)	0.00	0.21	$g(x) = 0.014 \exp(-30x)$	thunderstorm
Sekhon and Srivastava (1971)	0.37	0.14	$g(x) = 0.07 \exp(-38x)$	thunderstorm
Willis (1984)	0.31	0.158	$g(x) = 139.6x^{2.5} \exp(-57.4x)$	hurricane
Willis and Tattelman (1989)	0.30	0.153	$g(x) = 50.6x^{2.16} \exp(-56.8x)$	hurricane
$\Psi = D_0$	α_{D_0}	β_{D_0}		
Atlas (1964)	0.0	1	$g(x) = 0.08 \exp(-3.67x)$	widespread
Willis (1984)	1.95	1	$g(x) = 38.9x^{2.5} \exp(-5.6x)$	hurricane
Willis and Tattelman (1989)	1.95	1	$g(x) = 31.0x^{2.16} \exp(-5.6x)$	hurricane
$\Psi = W$	α_w	β_w		
Best (1951)	−1.24	0.56	$g(x) = 295.67x^{-0.73} \exp(-4.73 \times 10^7 x^{3.27})$	cloud droplets
Sekhon and Srivastava (1971)	0.40	0.15	$g(x) = 0.23 \exp(-25x)$	thunderstorm
Sempere Torres et al. (this paper)	0.087	0.228	$g(x) = 0.33x^{3.54} \exp(-4.01x)$	widespread

* Exponents calculated by Ulbrich (1983) fitting a gamma distribution to the original $Z(R)$ relations given by Battan (1973). As they are not studies on DSD, it is not possible to write the expression for $g(x)$.

4. Self-consistency relationships

A second implication comes from taking an integral rainfall variable as the reference variable used in (1). This leads to constraints on the exponents α_Ψ and β_Ψ and the general function g. These constraints are called self-consistency relationships (Bennet et al. 1984).

a. Ψ is a simple moment of the DSD

If we take Ω_n as the reference variable used in (1), (21) is written as

$$\Omega_n = A_n \Omega_n^{\alpha_{\Omega_n}+(n+1)\beta_{\Omega_n}} \int_0^\infty dx\, g(x) x^n. \quad (28)$$

Since α_{Ω_n} and β_{Ω_n} are constants, and the integral is independent of the value of Ω_n, the only way to verify this equation irrespectively of the value of Ω_n is for Eqs. (23) and (24) to read as

$$A_n \int_0^\infty dx\, g(x) x^n = 1 \quad (29)$$

$$\alpha_{\Omega_n} + (n+1)\beta_{\Omega_n} = 1. \quad (30)$$

Thus, the two exponents of (1) are not independent, and the general function $g(x)$ is constrained.

Taking, for example, $\Psi = W$, then the relation between exponents reads

$$\alpha_w + 4\beta_w = 1, \quad (31)$$

and the general function g must satisfy

$$\int_0^\infty dx\, g(x) x^3 = \frac{1}{A_3}. \quad (32)$$

Of special interest is the case in which Ψ is an averaged diameter, for instance, the mean diameter $\bar{D} = \Omega_1/\Omega_0$, the mass-weighted averaged diameter $D_m = \Omega_4/\Omega_3$, or the median-volume diameter D_0. In this case, the self-consistency relation leads the exponent β_D to equal 1 (α_D disappears due to the quotient of moments), and there is consequently only one exponent free, δ. The DSD thus takes the form

$$N(D, \bar{D}) = \bar{D}^{\delta} g\left(\frac{D}{\bar{D}}\right), \quad (33)$$

where $\bar{D}$ could also be D_0 or D_m. Note that this expression means that $N(D, \bar{D})$ is a homogeneous function

$$N(\lambda D, \lambda \bar{D}) = \lambda^{\delta} N(D, \bar{D}). \quad (34)$$

This is an alternative way of writing a scaling law.

b. Ψ *is a weighted moment of the DSD*

Rainfall intensity and reflectivity-weighted vertical Doppler velocity can be considered as weighted moments of the DSD, the moment orders being 3 and 6 and the terminal velocity of drops $V(D)$ being the weighting function. The same is true for the kinetic energy ($n = 3$ and $V^2(D)$ the weighting function), and for the microwave attenuation [$n = 0$ and the microwave total attenuation cross section $Q_t(D)$ the weighting function]. In general, these weighted moments can be written as

$$\Pi_n = B_n \int_0^{\infty} dD\, h(D) N(D, \Psi) D^n, \quad (35)$$

where $h(D)$ is the weighting function. If Π_n is taken as the reference variable Ψ in (1), the same procedure as in the previous section leads to

$$1 = B_n \int_0^{\infty} dx\, g(x) x^n h(x \Pi_n^{\beta_{\Pi_n}}) \Pi_n^{\alpha_{\Pi_n} + (n+1)\beta_{\Pi_n} - 1}. \quad (36)$$

For a general expression of $h(D)$, the right-hand side of (36) will depend on Π_n, and thus it will not be verified for any value of Π_n breaking the self-consistency. However, if a power relation between h and D is assumed, that is,

$$h(D) = kD^p, \quad (37)$$

then (36) reads as

$$1 = B_n k \Pi_n^{\alpha_{\Pi_n} + (n+p+1)\beta_{\Pi_n} - 1} \int_0^{\infty} dx\, g(x) x^{n+p}, \quad (38)$$

and solving this self-consistency equation for any value of Π_n leads to the constraints

$$\int_0^{\infty} dx\, g(x) x^{n+p} = \frac{1}{B_n k} \quad (39)$$

$$\alpha_{\Pi_n} + (n + p + 1)\beta_{\Pi_n} = 1. \quad (40)$$

The case of the rainfall intensity R, the most common reference variable, is particularly interesting. In this case (36) reads as

$$1 = B_3 \int_0^{\infty} dx\, g(x) x^3 h(xR^{\beta}) R^{\alpha + 4\beta - 1}. \quad (41)$$

(Notice that for $\Psi = R$, α and β are used instead of α_R and β_R.)

Again, to verify this equation for any value of R is impossible if $V(D)$ is a general expression. Only by assuming a power relation between V and D, that is,

$$V(D) = cD^v, \quad (42)$$

is the self-consistency verified. This power relation has been used often (e.g., Spilhaus 1948; Atlas and Ulbrich 1977; Kessler 1969). Therefore, (39) and (40) give

$$\int_0^{\infty} dx\, g(x) x^{3+v} = \frac{1}{B_3 c} \quad (43)$$

$$\beta = \frac{1}{4 + v}(1 - \alpha). \quad (44)$$

Again, these relations imply that there is only one free exponent and that there is a constraint for $g(x)$.

Using the expression provided by Atlas and Ulbrich (1977), where $c = 17.67$ and $v = 0.67$ (D in centimeters and V in meters per second), (44) reads

$$\beta = 0.214(1 - \alpha). \quad (45)$$

We can use previous experimental studies on DSD to analyze whether they verify this relation. Table 1 summarizes various works of the literature. Experimental α and β are plotted in Fig. 1 together with the linear relation given by (45). The agreement is fairly good and the exact position of points is determined by the different nature of data, kind of rainfall, location, device, and altitude of measurement.

It can be argued that this power law for the terminal drop velocity in stagnant air has been shown not to be the best approximation (Beard 1976). However, any other $V(D)$ relation (e.g., Upplinger 1981; Beard 1976; Atlas et al. 1973; Best 1950) leads to inconsistency in (41), and therefore to the general formulation. In other words, all previous works cited in this paper assumed the existence of a power law for the terminal velocity, even if they did not use it explicitly. Moreover, as Fig. 1 shows, if all these works are well based the *effective* $V(D)$ should be a power law.

This result gives rise to many questions leading to the need of additional studies based on measurements of real rainfall terminal velocities complementary to those of Gunn and Kinzer (1949) and Foote and du Toit (1969) (the most generally used). A supplementary issue is that the suitability of R as the reference variable of a DSD per unit of air volume can therefore be questioned. However, it is worth noting that when dealing with disdrometer data an alternative idea could

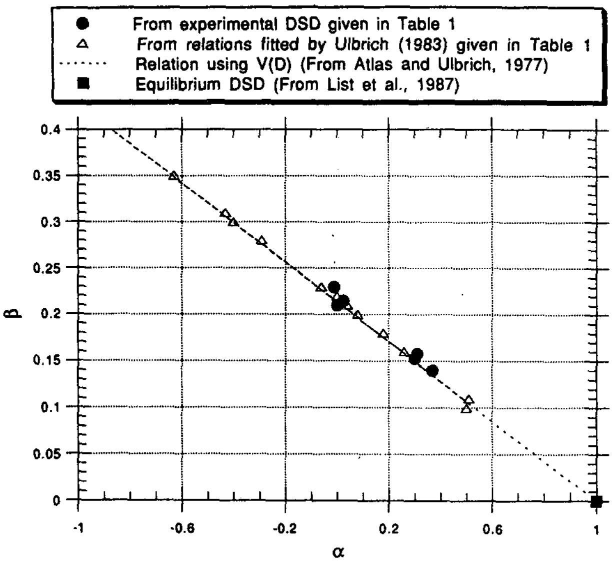

FIG. 1. Relation between exponents α and β: (i) from experimental $N(D)$ of Table 1 when $\Psi = R$ (dots); (ii) from fitted relations provided by Ulbrich (1983), in Table 1 (triangles) [the Higgs (1952) point is not shown to make the plot clearer, but it is easy to verify that it is exactly on the line]; (iii) linear relation (45) using $V(D)$ from Atlas and Ulbrich (1977) (dashed line); (iv) for the equilibrium DSD proposed by List et al. (1987) (square).

be to identify the DSD in terms of flux instead of concentration, which would be more consistent with available data.

On the other hand, an additional α and β can be obtained from the theoretical studies on the temporal evolution of DSD carried out by List et al. (1987). Using the mechanisms of coalescence and breakup of drops, they obtain an equilibrium DSD, which does not evolve any more in time [List 1987, Eq. (14), p. 364]. This equilibrium DSD satisfies (1) with $\alpha = 1$ and $\beta = 0$, independently of which integral variable is chosen as Ψ. In this particular case, the exponent b in (22) becomes 1, and the relation between any bulk variable and the reference variable is linear instead of being a power function. Moreover, the averaged diameters—for example, D_m, D_0—become constants. This equilibrium DSD shape has been observed in persistent intense rain in the Tropics by Zawadzki and De Agostinho Antonio (1988).

The inclusion of this point leads to a complementary interpretation of the scaling law exponents. We have seen that α_Ψ and β_Ψ are related and that only one of them is free, let us say α_Ψ. This exponent can be interpreted as a measure of how far a particular drop population is from the equilibrium state determined by $\alpha_\Psi = 1$.

Finally, it is worth noting that these implications are implicitly assumed by all previous DSDs, irrespective of the chosen shape of the distribution function. The formulation proposed provides only a general framework for clarifying and interpreting them.

5. A methodology to fit DSD from experimental spectra

A methodology to fit the exponents α_Ψ and β_Ψ, and the general function g, in two different steps can be derived from the general DSD formulation proposed in (1). In the following discussion the reference variable Ψ is assumed to be selected. As said before, any integral rainfall variable can play this role; the choice should depend on the proposed use of the model. Up to now rainfall intensity was generally preferred, but further work should analyze which specific Ψ is the most appropriate for each different application.

a. *Identification of the exponents α_Ψ and β_Ψ*

From the main implications of formula (1) examined in sections 3 and 4, the exponents α_Ψ and β_Ψ appear to be related

(i) to the exponent of the power relationship between integral rainfall variables,

(ii) by the self-consistency relation due to the selection of an integral variable as the reference variable Ψ.

As the general function g does not interfere in these relations, they allow α_Ψ and β_Ψ to be determined independently.

Thus, the first relation between the parameters is found by fitting a power function between any couple of integral variables, Ψ_1, Ψ_2:

$$\Psi_1 = c\Psi_2^{\gamma}. \qquad (46)$$

It is easy to show from (1) that the exponent of this power function γ depends only on α_Ψ and β_Ψ. In the particular case where Ψ_2 is the reference variable, one obtains (24). The second relationship can come from a third integral variable or from one of the self-consistency formulas (30) or (40), depending on the nature of the moment of DSD taken as reference variable Ψ. An example of this methodology is discussed below.

An alternative graphical method, generally used in the field of dynamic scaling laws, can also be used to identify the exponents α_Ψ and β_Ψ. For different classes of Ψ the mean spectra are plotted on a $\log[N(D, \Psi)]$ versus $\log(D)$ system. They will look like different curves, one for each Ψ. According to formula (1), the contribution of Ψ to these graphs can be separated in terms of vertical and horizontal displacements of the origin, one for each graph [$\alpha \log(\Psi)$ and $\beta \log(\Psi)$, respectively]. Thus a simple visual matching of the different scattergrams gives estimates of α and β. However, the numerical methodology proposed earlier seems to be more reliable since the parameters are identified with the complete set of data.

b. Identification of the general function g

Once α_Ψ and β_Ψ have been identified, an experimental function g is obtained by plotting the whole set of measured spectra on a graph $N(D, \Psi)/\Psi^{\alpha_\Psi}$ versus D/Ψ^{β_Ψ}. The use of these coordinates has a scaling effect on the spectra, making them comparable independently of the value of Ψ. The choice and fit of an authorized analytical model to the experimental function g is then performed. The model is said to be authorized since it must satisfy the self-consistency relationship in terms of (29) or (39), depending on the nature of Ψ.

c. An illustrative example

Let the total water content per unit of air volume W be the reference variable. For any integral variable, and in particular for any integral moment Ω_n, (46) reads

$$\Omega_n = a_n W^{b_n} \tag{47}$$

with b_n given by (24) as

$$b_n = \alpha_w + \beta_w(n + 1). \tag{48}$$

Note that just choosing two different integral moments, (47) and (48) determine α_w and β_w. However, we can propose to use several integral moments to obtain a more robust estimation.

The data from the Cevennes-86 Experiment (Andrieu et al. 1989) presented in Delrieu et al. (1991) are used in this example. Some 3000 1-min. spectra of drop sizes recorded in 25 bins using a Joss and Waldvogel disdrometer are available. Table 2 shows the exponents b_n of the first six moments, excluding that related to the reference variable ($n = 3$), obtained from the experimental data. The fit of a linear regression between b_n and $n + 1$ determines the exponents α_w and β_w (see Fig. 2) as

$$\alpha_w = 0.087 \tag{49}$$

$$\beta_w = 0.228, \tag{50}$$

and therefore

$$\alpha_w + 4\beta_w = 0.999. \tag{51}$$

The experimental results lead to a good agreement with (48), but also verify in a remarkable way the self-consistency relation (31). Note that the point $n = 3$ is not used in the α_w and β_w determination. Therefore, the self-consistency agreement of (51) is an implicit property of the experimental data.

TABLE 2. Experimental exponents b_n, from Eq. (47), fitted on the Cevennes-86 data.

Moment order (n)	Exponent b_n
0	0.340
1	0.517
2	0.747
4	1.251
5	1.476
6	1.660

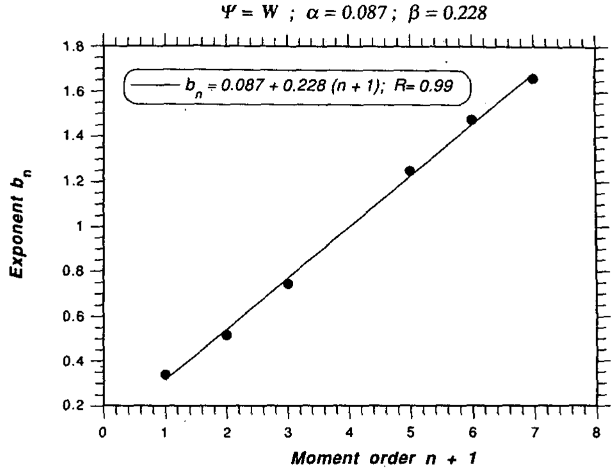

FIG. 2. Fit of Eq. (48) on the experimental exponents b_n given by Table 2 for the Cevennes-86 data.

Once α_w and β_w are estimated, the scaled scattergram can be used to choose and fit the general function g. Figure 3 shows the 2711 scaled spectra. There is of course some dispersion, but the scaling process leads to a clear representation of the experimental g.

This figure is similar to those obtained by a normalization procedure [see, e.g., Fig. 5 in Sekhon and Srivastava (1971), Figs. 3 and 7 in Willis (1984), or Fig. 4 in Willis and Tattelman (1989)], but in our case the scaling process has been performed without imposing a specific shape to g. Thus, any appropriate model of distribution function for g can be proposed and its parameters fitted according to the scaled scattergram, which involves the use of the complete set of spectra.

Finally, it is worth noting that this suggested two-step identification methodology merges the main advantages of previous works: the use of the DSD moments and of a normalization procedure. The use of the DSD moments to fit its parameters comes from Waldvogel (1974), who identified the parameters N_0 and λ of an exponential distribution by using Z and W (moments of order 6 and 3). From a statistical viewpoint, the identification of parameters through a moment method is known to be reliable even if the choice of the moments has a noticeable influence on the results [the influence of this choice is highlighted, e.g., by the comments of Ulbrich (1983)]. However, the use of several moments to fit α_Ψ and β_Ψ from (48) guarantee the results. The scaling procedure applied to the spectra is a generalization of the normalization proposed by Sekhon and Srivastava (1970, 1971). This scaling procedure allows the identification of g to use the complete set of spectra and, thus, it turns out to

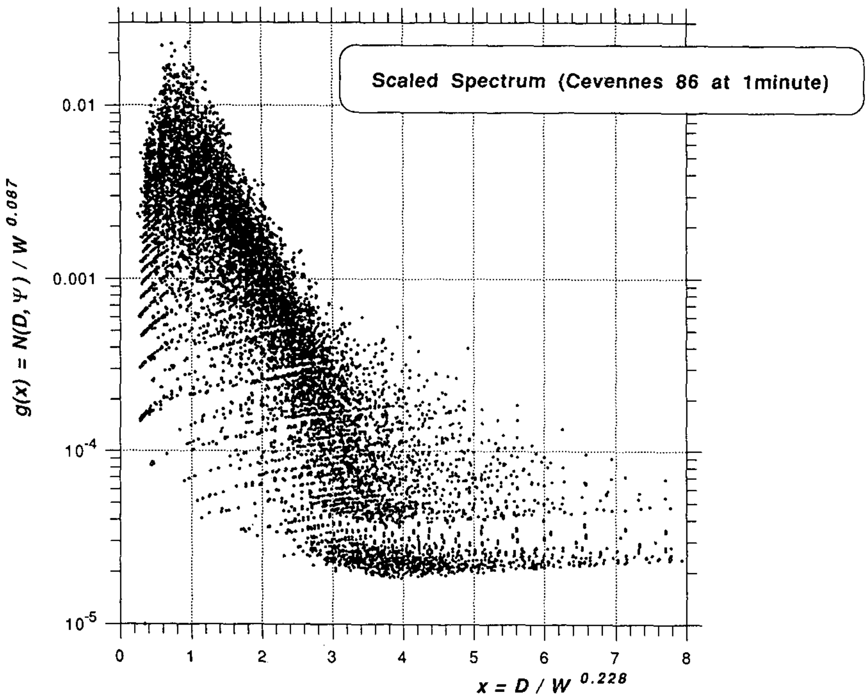

FIG. 3. Scaled scattergram for the Cevennes-86 data. All spectra are plotted together to give the experimental plot of $g(x)$.

be a statistically more robust way to identify the parameters of g than the procedure based on a separate identification for different classes of Ψ values. [In the example, 2711 spectra are used. In the case of using Ψ classes, some of them could have been fitted using less than 100 spectra (Delrieu et al. 1991).]

In addition, some of the drawbacks are avoided. The essential one is not to be obliged to make an a priori choice of the analytical form of the general function g. Moreover, the constraints linked to the self-consistency relation are more easily accounted for. These advantages come from the general formulation (1), which also implies the clear separation between shape identification and parameter estimation.

On the other hand, there is an additional point usually omitted in classical works and advanced by Waldvogel (1974): the possible variation of the DSD parameters with time. Classically a rainfall event, or a kind of rainfall event, is considered to be characterized by the same DSD, and its parameters have no explicit dependence on time. Note, however, that this dependence is indeed introduced by the variation of the reference variable in time (e.g., the quantity of liquid water content available in the atmosphere), and this variation drives the variation of the DSD's properties. Nevertheless, the possibility of dealing with an event merging two or more types of rainfall (two or more sets of parameters) should be contemplated.

Finally, further investigation is needed to find the relation between the fitted general function and the kind of rainfall, and to find out the minimum number of parameters required. It is worth mentioning that if function g depended only on one parameter, it would be "universal" in the sense that the self-consistency constraint would lead it to not have any free parameters, and g would consequently be independent of that kind of rainfall. Therefore, all the dependence on the kind of rainfall, location, device, etc., will be summarized in the exponent α_Ψ. Also, if a relation between the parameters of g and α_Ψ existed, it would imply that all reference to the kind of rainfall and other factors would again be contained in the exponent α_Ψ. The brilliant work of Ulbrich (1983) supports this point.

6. Conclusions

1) We propose a general formulation for drop size distribution (DSD) in terms of a scaling law that is able to reproduce and interpret all previous studies dealing with DSDs.

2) This formulation provides a general framework where the classical role of the rain intensity as the reference variable can be played by any integral rainfall variable without the generality of the expression being modified.

3) In the scaling-law function, the general distribution function and its parameters are independent of the reference variable values.

4) In this framework the empirical power relations between the rainfall bulk variables are a consequence of the proposed scaling law.

5) The exponents of this scaling law are related since an integral rainfall parameter is used as the reference variable of the general formulation. As a consequence, there is only one free exponent in the scaling law, and the general distribution function must verify an integral equation that reduces its free parameters by one. In the particular case of using the rain intensity as the reference variable, this self-consistency relation requires a power relation between terminal velocity and diameters.

6) A general methodology to fit the distribution function determining the DSD follows from the general formulation proposed here. This methodology does not involve setting an a priori shape to this function and it allows the fit to use all the realizations of the measured drop spectrums, regardless of the associated rainfall intensity.

These conclusions are in fact contained in all the previous works mentioned, they are particular cases of the proposed scaling law. This general framework should be interpreted as a powerful way to sum them up, and a tool to point out both the usually accepted hypothesis as well as its limitations. Further work should be carried out to test the adequacy of the implicit assumptions on real data, particularly the need for a power dependence of V on D, and to analyze the best choice of Ψ and g for each case.

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