Application of Optimum Interpolation to the Analysis of Precipitation in Complex Terrain

MEENU BHARGAVA AND MAURICE DANARD

Department of Computer Science, University of Victoria, Victoria, British Columbia, Canada

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ABSTRACT

Optimum interpolation is a procedure that allows the combination of observations with preliminary trial fields of the same quantities in order to produce an updated field in which the error variance is minimized. In this paper, an operational method is described to analyze observed precipitation amounts, based on optimum interpolation. Since the area dealt with is topographically complex, this factor has been included in the operational method. The trial fields are provided by a three-dimensional numerical weather prediction model. This paper presents an estimation of the covariances of observational and trial field errors. Two mathematical assumptions are made: 1) trial field errors and observational errors are not correlated with each other; 2) observational errors and the deviations of the trial field values from the observations are uncorrelated. The first assumption is customarily made in any application of optimum interpolation. The second assumption is specific to this paper. These two statements together imply that observational errors are uncorrelated. A technique is derived to determine which observations influence a given grid point and their respective weights. The selection of influencing observations is done by calculating the spatial dependence of τ, the trial field error covariance. A cutoff point is determined on the smoothed curve where the τ value is a small fraction of the τ value at the origin. The procedure is applied to the heavy rainstorm of 11–13 July 1983 in the upper Columbia River watershed in southeastern British Columbia. Certain practical problems do arise in the implementation. The noncoincidence of model day and climate day tends to introduce systematic errors within the observations. This result conflicts with the assumption that observational errors are uncorrelated. Additionally, the observing system is not designed to make allowance for topographical detail. Errors are thus introduced in the observations from a variety of sources.

1. Introduction

Optimum interpolation procedures were introduced to meteorology by Eliassen (1954) and Gandin (1963). Suppose one wishes to find the best estimate of a meteorological field at a point from observations in its vicinity. Optimum interpolation uses past history about the structure of the atmosphere to determine the weights to be applied to the observations. The assumption is made that the observations are spatially correlated. This implies that observations that are close together are highly correlated and that as the observations get farther apart, the correlation decreases.

The optimum interpolation method for the objective analysis of meteorological fields produces the best solution in the sense that the interpolation error is, on average, minimized. The method allows for the extraction of as much useful information as the observations provide. Nevertheless, this complex scheme was not put into immediate use after it was initially proposed by Eliassen (1954). The reason for this delay is twofold:

1) The scheme requires a knowledge of covariances. These are often not known and thus an estimate is required. Establishing such an estimate is often fraught with difficulty as a host of local factors are involved.
2) Essentially, one must determine a priori which observations are significantly correlated with the value at the point of interpolation (i.e., one must determine a region of influence around the point).

The main aim of this paper is the application of optimum interpolation to the analysis of 1- and 3-day precipitation amounts over complex terrain. Such applications are relatively rare.

Tanguay and Robert (1990) apply optimum interpolation to the analysis of precipitation over North America. There are several important differences between their study and the present one, however:

1) Tanguay and Robert have a zero trial or background field. On the other hand, in the present study, the trial field is the precipitation predicted by a three-dimensional initial value model in which small-scale topographical effects are included.
2) Tanguay and Robert employ a grid size of 50 km. The present study has a grid size of about 20 km so small-scale influences of topography are more important.

Corresponding author address: Prof. Maurice Danard, Department of Computer Science, University of Victoria, P.O. Box 3055, Victoria, B.C., Canada, V8W 3P6.

3) Tanguay and Robert approximate the autocorrelation function by the second- and fourth-order Taylor series expansion of the Gaussian profile. This results in a significant reduction in the effort required to calculate the station weights. Such an approximation is not made here.

Finkelstein (1984) and Venkatram (1988) apply kriging to analyze the acidity of precipitation. Kriging is a special case of optimum interpolation with a zero trial field [as in Tanguay and Robert (1990)] and the constraint that the sum of the weights applied to the surrounding observations is unity. Bigg (1991) used kriging to analyze spatial variability of monthly, seasonal, annual, and extreme event precipitation in England.

2. Optimum interpolation theory

The following derivations [Eqs. (1)–(9)] have been adapted from Gustafsson (1981). Equations (10)–(15) are new, however.

Let P be some meteorological field (1- and 3-day precipitation amounts in the present context). Consider a point g with n observations P_i^O around it, which will be used to calculate the analyzed value P_g^N at the point g. We determine P_g^N as the sum of a trial value P_g^T at point g plus a linear combination of the deviations of the observed values from the trial values ($P_i^O - P_i^T$). That is,

$$P_g^N = P_g^T + \sum_{i=1}^{n} \lambda_i (P_i^O - P_i^T). \quad (1)$$

The λ_i's are the interpolation weights. Intuitively, one would expect the λ_i's to be positive and decrease monotonically with increasing distance from the point g. No a priori conditions will be imposed on the λ_i's, however. These are so determined so that the analysis error of (1) is minimized. The mean square error of (1) is

$$\overline{E^2} = \overline{(P_g^N - P_g)^2}, \quad (2)$$

where P_g is the unknown true value at point g and the overbar is the ensemble average (approximated here as the average over a large number of cases). The weights λ_i are chosen so that (2) is minimized. Substitute (1) in (2). Furthermore, assume that trial field errors and observational errors are uncorrelated. That is,

$$\overline{(P_i^T - P_i)(P_j^O - P_j)} = 0, \quad (3)$$

where P_i is the true value at point i. Then Gustafsson's (1981) Eq. (4.2.2.5) is obtained:

$$\overline{E^2} = \sigma_g^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j (\tau_{ij} + \eta_{ij}) - 2\left(\sum_{i=1}^{n} \lambda_i \tau_{gi}\right), \quad (4)$$

where

$$\sigma_g^2 = \overline{(P_g^T - P_g)^2} \quad (5)$$

is the error variance of the trial field at point g,

$$\tau_{ij} = \overline{(P_i^T - P_i)(P_j^T - P_j)} \quad (6)$$

is the covariance of the trial field errors, and

$$\eta_{ij} = \overline{(P_i^O - P_i)(P_j^O - P_j)} \quad (7)$$

is the covariance of the observational errors. Differentiate (4) with respect to λ_i to obtain Gustafsson's Eq. (4.2.2.7):

$$\sum_{i=1}^{n} \lambda_i (\tau_{ik} + \eta_{ik}) = \tau_{kg} \quad (8)$$

for $k = 1, \cdots, n$. Equation (8) is a set of n linear equations in the n unknowns λ_i's. The minimized analysis error variance is

$$E_m^2 = \sigma_g^2 - \sum_{i=1}^{n} \lambda_i \tau_{ig}, \quad (9)$$

which is Gustafsson's Eq. (4.2.2.9).

The difficulty with using (6)–(9) as they stand is that the true values must be known. To render the problem more tractable, (6)–(9) will be transformed into expressions involving measured rather than true values. Subtract (3) from (6) to obtain

$$\tau_{ij} = \overline{(P_i^T - P_i)(P_j^T - P_j^O)}. \quad (10)$$

So far, the only assumption employed is (3), which is customarily made in any application of optimum interpolation. Now assume that the observational errors are uncorrelated with the deviations of the trial field values from the observed values. That is,

$$\overline{(P_i^O - P_i)(P_j^T - P_j^O)} = 0. \quad (11)$$

Subtract (11) from (10) to give

$$\tau_{ij} = \overline{(P_i^T - P_i^O)(P_j^T - P_j^O)}, \quad (12)$$

in which all the variables are known. Add (7) and (11) to obtain

$$\eta_{ij} = \overline{(P_i^O - P_i)(P_j^T - P_j)}. \quad (13)$$

From (3), however, (13) reduces to

$$\eta_{ij} = 0. \quad (14)$$

Thus, assumption (11) together with the customary assumption (3) implies that the observational errors are uncorrelated. The implication of this corollary will be discussed later in section 4a. Equation (8) then reduces to

$$\sum_{i=1}^{n} \lambda_i \tau_{ik} = \tau_{kg} \quad (15)$$

for $k = 1, \cdots, n$ and where τ is given by (12).

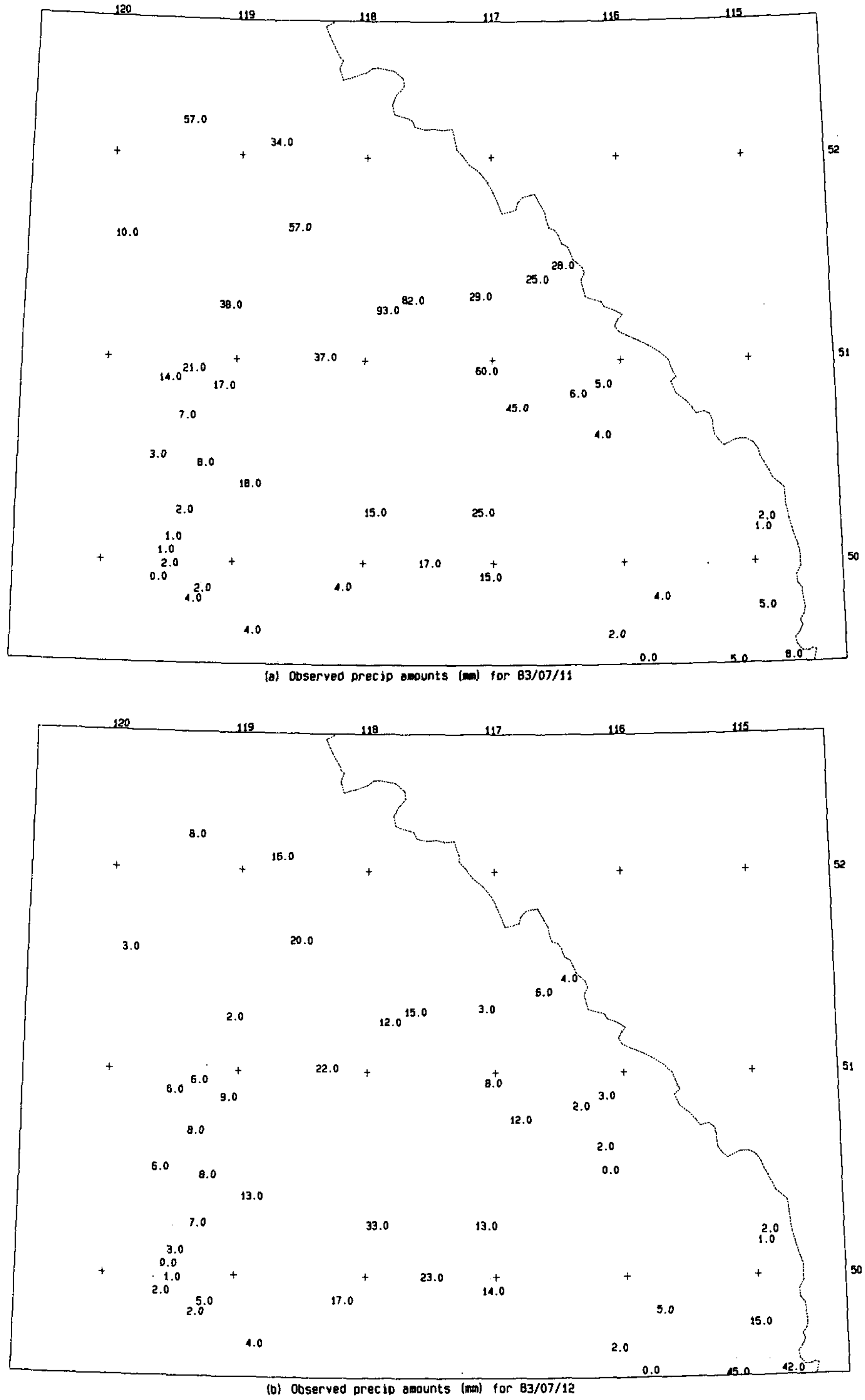
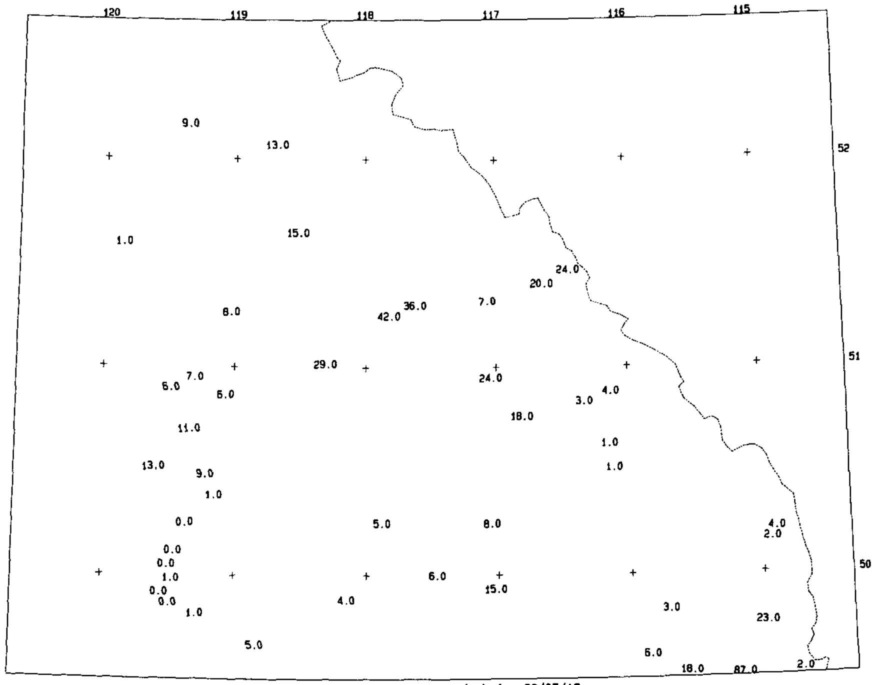


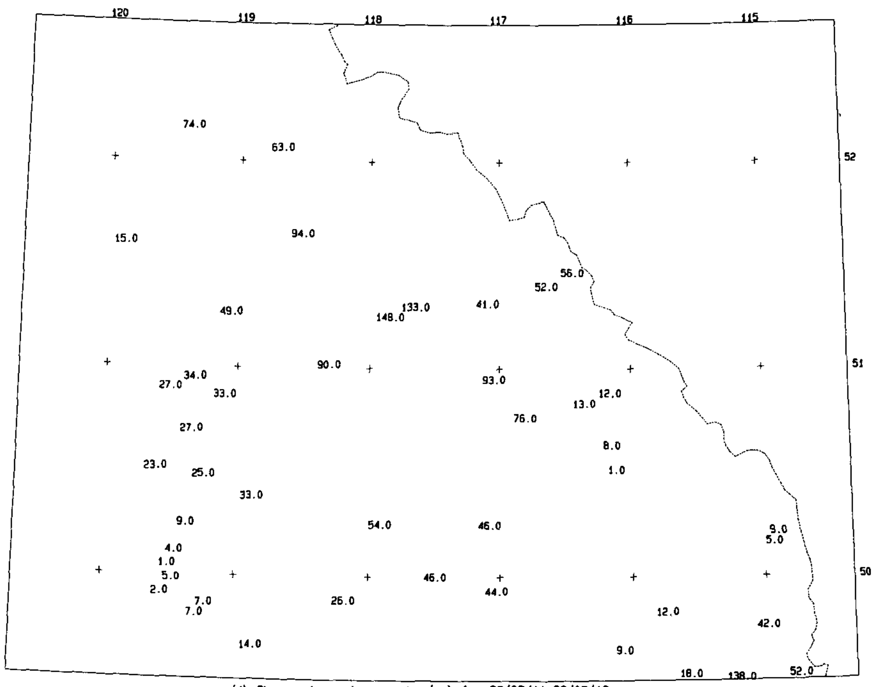
FIG. 1. Observed precipitation amounts (mm) for (a) 11 July 1983, (b) 12 July 1983, (c) 13 July 1983, and (d) 3-day total. The stations are located at the decimal points. The irregular diagonal line is the British Columbia–Alberta border.

3. Analysis

In order to determine the spatial structure of the covariances, the τ's are computed for four different datasets. These are the precipitation amounts for the three days 11 July 1983, 12 July 1983, and 13 July 1983 and the 3-day total. These are shown in Figs. 1a–d. These datasets are from the Columbia River basin in southeastern British Columbia. Initial trial fields



(c) Observed precip amounts (mm) for 83/07/13



(d) Observed precip amounts (mm) for 83/07/11-83/07/13

FIG. 1. (*Continued*)

of the precipitation over an 18 × 18 grid covering the region are obtained from a three-dimensional initial value model. This model incorporates topographical effects. Figures 2a–d show the trial fields. The τ_{ij}'s for each precipitation interval are calculated using (13) for observation points i and j. The distance between points i and j determined the bin in which τ_{ij} would be included. When observation points i and j are coincidental, the τ_{ij} is included in the zero bin. The remaining bins have intervals of 50 km. Thus, for any two distinct points that lie within 50 km of each other, the τ would go in that bin. Points that lie from 50 to

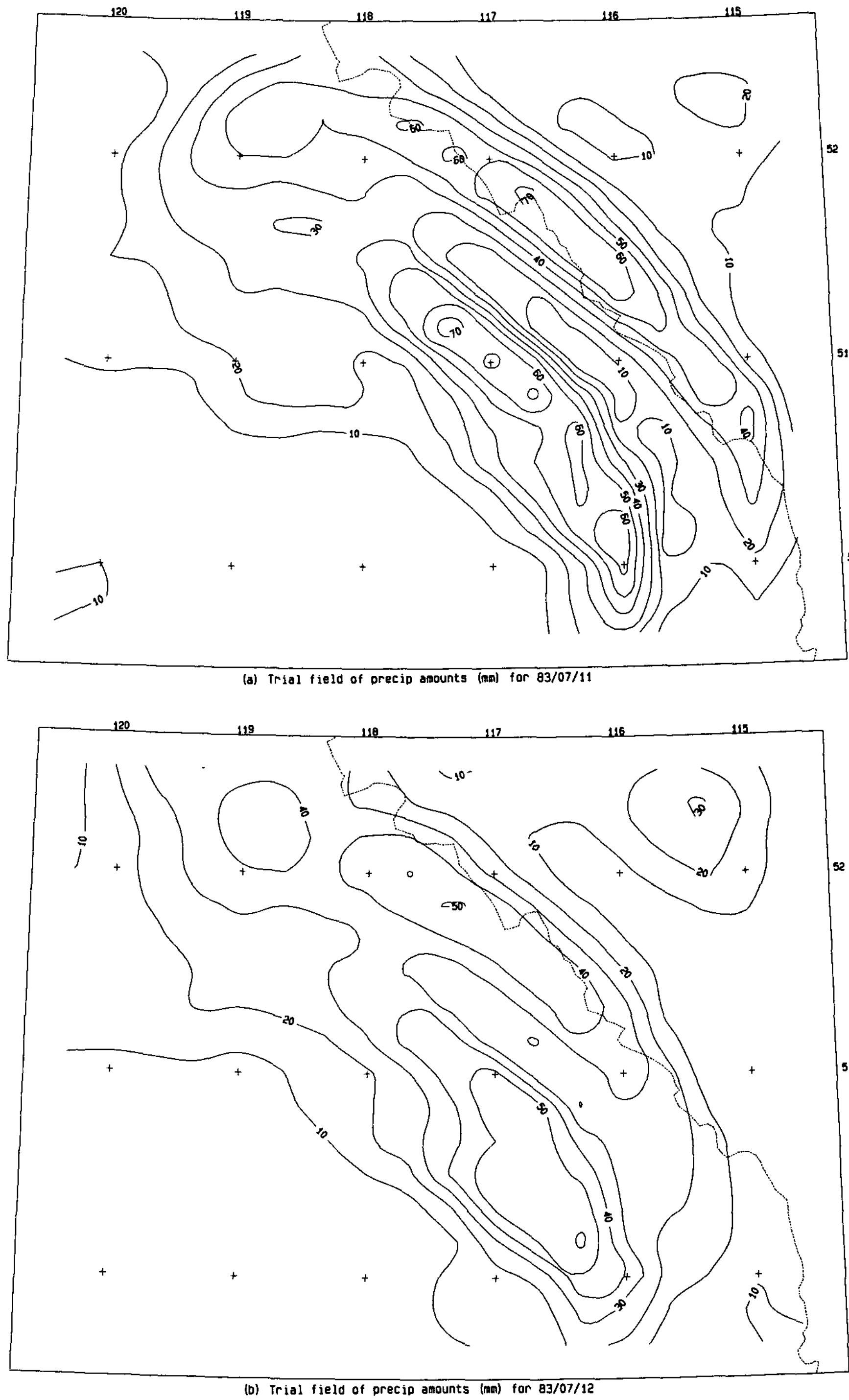
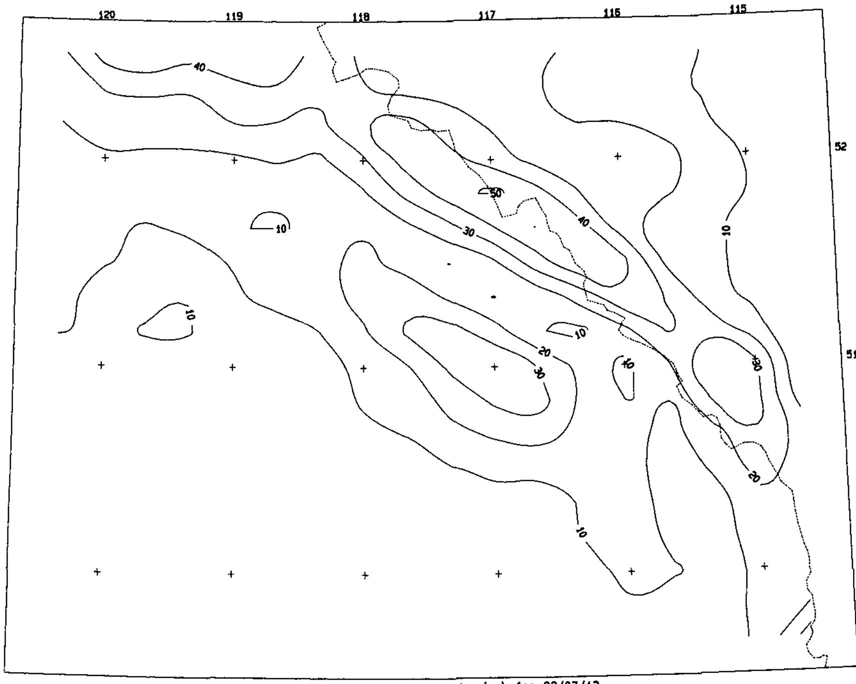
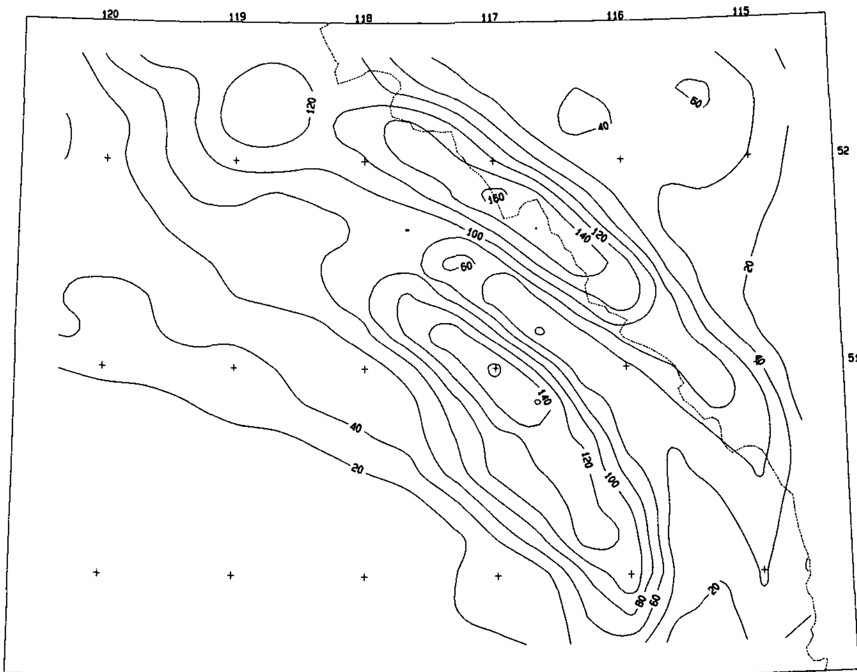


FIG. 2. Trial fields of precipitation amounts (mm) for (a) 11 July 1983, (b) 12 July 1983, (c) 13 July 1983, and (d) 3-day total.

100 km of each other are included in the next bin, etc. For each bin, an approximation to the ensemble average is obtained by averaging the contents of that bin. Since $\tau_{ij} = \tau_{ji}$, care is taken so that each τ_{ij} is included in the appropriate bin only once. The ensemble averages are then plotted against the midpoints of the different bins. These plots are shown in Fig. 3. Due to the nonmonotonicity of these raw curves, interpolating between any two points would give rise to a singular matrix of τ values. To obtain a more "nonsingular"



(c) Trial field of precip amounts (mm) for 83/07/13



(d) Trial field of precip amounts (mm) for the 3-day total

FIG. 2. (*Continued*)

matrix of τ values, the graphs in Fig. 3 are approximated by cubic polynomials shown in Fig. 4. The approximation is a least-squares fit of a linear combination of Chebychev polynomials. Note that in all cases, the value of the covariance at 100 km is a small fraction of the maximum covariance value. It is decided to define the radius of influence as 100 km. Thus, any observation point that lay within 100 km of the point in question is taken as an influencing observation, and all observations beyond this distance are ignored. The τ values are determined using the fitted curve. The matrix of τ values so obtained is positive definite and

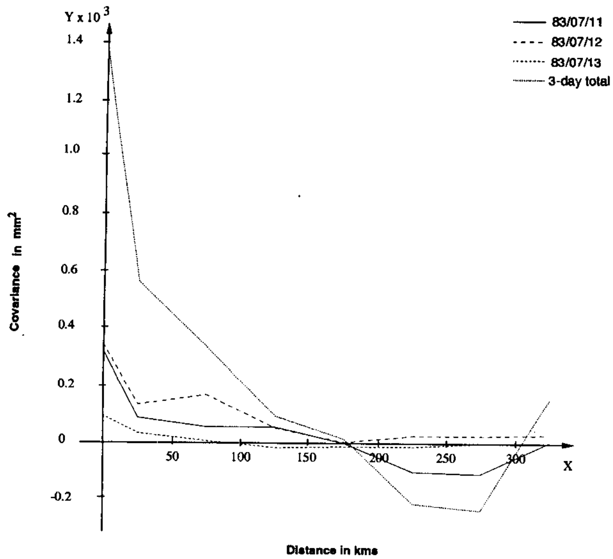


FIG. 3. Spatial structure of covariances for 11, 12, and 13 July 1983 and the 3-day total.

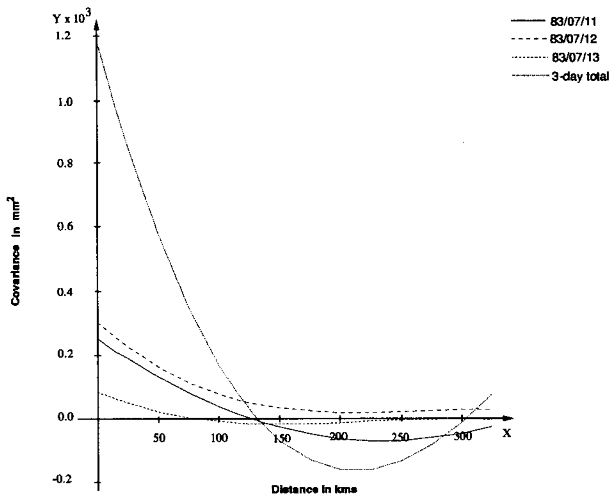


FIG. 4. Chebychev approximation to the spatial structure of the covariances for 11, 12, and 13 July 1983 and the 3-day total.

well conditioned. The resulting system of linear equations is solved using Cholesky factorization.

Estimating the interpolation error. The minimized analysis error variance (9) was arrived at in section 1. Clearly, σ_g^2 must be estimated. Neglecting interpolation error in P_g^T in order to simplify the estimation gives

$$(P_g^O - P_g^T) = (P_g^O - P_g) + (P_g - P_g^T). \quad (16)$$

Assuming that the point in question, g, coincides with any of the observation points allows for the dropping of the subscript g. Squaring the whole expression, taking the ensemble average and rearranging yields

$$\overline{(P - P^T)^2} = \overline{(P^O - P^T)^2} - \overline{(P^O - P)^2},$$

$$\sigma_g^2 = \delta^2 - \epsilon^2, \quad (17)$$

where δ^2 is the mean square of the differences between the observed and trial precipitation and ϵ^2 is the mean square of the observational errors. In (17), the cross products are eliminated due to assumption (3). Available data allows for the evaluation of δ^2. Now

$$\epsilon^2 = \overline{(P^O - P)^2}. \quad (18)$$

Assuming that $[\overline{(P^O - P)^2}]^{1/2}$ is proportional to P^O then

$$\epsilon^2 = r^2\overline{(P^O)^2}, \quad (19)$$

where r is the constant of proportionality that reflects the reliability of the observations. Given data allows for the ascertainment of $\overline{(P^O)^2}$. What is then required is an indication of the accuracy of the observations in order to evaluate ϵ^2. A value of 0.5 was assigned to r. This effectively says that the standard error of the observations is assumed to be 50% of the root-mean-square observed precipitation. One way to determine a value for r would be to have a high-resolution network of accurate precipitation measurements. In the absence of such a network, a value of 0.5 assigned to r does not seem unreasonable. Clearly, its exact value is open to question. This will be discussed again at the end of section 4a.

By substituting (18) in (17), σ_g^2 may be computed. The minimum analysis error variance can then be evaluated using (9).

4. Discussion

Table 1 gives sample values of the error variance decrease for grid points. Note that the decrease is quite small. This is mainly due to data sparsity and observational errors that are discussed in the next subsection. Figs. 5a–d show the final analyzed precipitation fields. It is observed from Figs. 5a–d that the final analysis reflects the general pattern of the trial fields (Figs. 2a–d). Furthermore, the maximum amounts in Figs. 1a–d appear to be accurately analyzed.

The question arises of whether optimum interpolation is a good objective analysis technique. In theory, it is, but the success of its practical application is greatly influenced by many factors. The discontinuous nature of precipitation, along with the sparse

TABLE 1. Examples of reductions in error variance ($\sigma_g^2 - E_{\min}/\sigma_g^2$).

Grid point	Period	Reduction in error variance
49.66°N, 119.00°W	11 July 1993	15.11%
52.16°N, 117.66°W	12 July 1993	2.11%
51.66°N, 117.00°W	13 July 1993	2.53%
51.50°N, 117.66°W	3-day total	0.56%

network of observations incorporating important local variations, significantly affects the practicality of the scheme. Most practical models accept certain assumptions based on empirical observations. The assumptions are the fulfillment of isotropy and homogeneity by the correlation fields of the empirical observations. We define isotropy as a characteristic of the field when the covariance is independent of a rotation in the field around the middle point on the line between the two positions. Equivalently, when the variable under analysis exhibits the same properties in different directions, the field is said to be isotropic. Homogeneity in reference to the covariance describes the independence of the covariance with respect to a translation of the two positions. In an attempt to achieve isotropy and homogeneity, we actually analyze the difference between observed amounts and the trial values that include topographical effects.

a. Sources of errors

As mentioned above, the success of optimum interpolation is dependent on many factors, one of them being observational error. This observational error can be thought of as three components, all of which arise from different circumstances. These components most often are indistinguishable from one another. This often makes the error analysis complex. These three components are 1) local errors; 2) subgrid-scale errors (spatial errors); and 3) temporal errors.

Of the three components, the best-known component has been the local error. Local errors arise due to the local aerodynamics above the mouth of the raingage. Ideal situations would imply that observed precipitation equals actual precipitation. In practicality, the inclusion of the observing system affects the wind pattern over the area. Thus, local aerodynamics tend to add noise to the measurements. Moreover, systematic errors are introduced if the gauge is not level. Measures are taken to reduce these two components of local errors. Precipitation measurements are frequently underestimated. Louie and Goodison (1985) give average underestimates of 0.4%–2.5% for three locations in Canada.

Subgrid-scale errors or spatial errors arise due to the displacement of observation sites by small amounts. Since the grid distance used is approximately 20 km, a slight displacement of the observation sites, say by 10 m, can be considered negligible. Any difference in observed amounts over such small distances is regarded as a local error. Nevertheless, over complex terrain, due to the nature of the topography, a displacement in the observation site, which is small compared to the grid size, often produces large discrepancies in the observations of the two different measuring sites. However, this spatial characteristic of the observations is often ignored when collecting data and is thus a source of noise in the data.

Temporal errors are not exactly observational errors in the conventional sense of the term. They arise mostly due to the difference in the model-defined day and the actual observation day. Most models operate under UTC. Thus a model day starts at 1200 UTC (0400 PST) and ends at 1200 UTC the next day, that is, a duration of 24 h. However, observations are not taken in a model day but over the climate day, which usually ends on or about 0800 PST. Thus the climate day usually starts and ends about 4 h later than the model day. This can lead to systematic errors within the observations. These errors are not in the observations themselves but are the "difference" between observed and model-predicted values. Thus the assumption that the observational errors are uncorrelated made in section 1 is strictly not true. It may be remarked that for locations in the Atlantic time zone, where 1200 UTC is 0800 local standard time, the assumption of uncorrelated errors is more likely to be correct.

Because the terrain is so complex and the grid size is large, probably the greatest contribution to measurement error in this application comes from subgrid-scale variations (E. Coatta 1991, personal communication). The only way to assess this effect is to have a high-resolution network of accurate precipitation measurements. To simulate such a network, precipitation was abstracted for a 4×4 grid of points $5'$ apart superimposed on a $15' \times 15'$ area from an analysis of observed 5-day precipitation in Colorado [U.S. Department of Commerce (1988), Fig. 2.22, p. 45]. The ratio of the standard deviation to the mean precipitation was around 0.3. The value of 0.5 assigned to r in (19) attempts to account for all three types of errors discussed above.

b. Sensitivity of the solution to perturbations in data

To determine how sensitive the solution, that is, the weights obtained, was to the initial data, it was decided to perturb the original precipitation data slightly. This perturbation was done by adding "small" random numbers to the observations. The random numbers were generated from a uniform distribution over the interval $(-1.5, 1.5)$. The system was solved as in the nonperturbed case. It was observed that the weights are generally insensitive to small perturbations in the data. Table 2 gives the weights obtained for both the unperturbed and perturbed systems. It may be prudent to note at this time that the perturbations added to the observations were all small. In section 4a, the different sources of errors that occur in the observations are discussed. The order of magnitude of these errors is much greater than the perturbations applied.

5. Concluding remarks

In this paper, optimum interpolation has been applied to the analysis of precipitation in complex terrain.

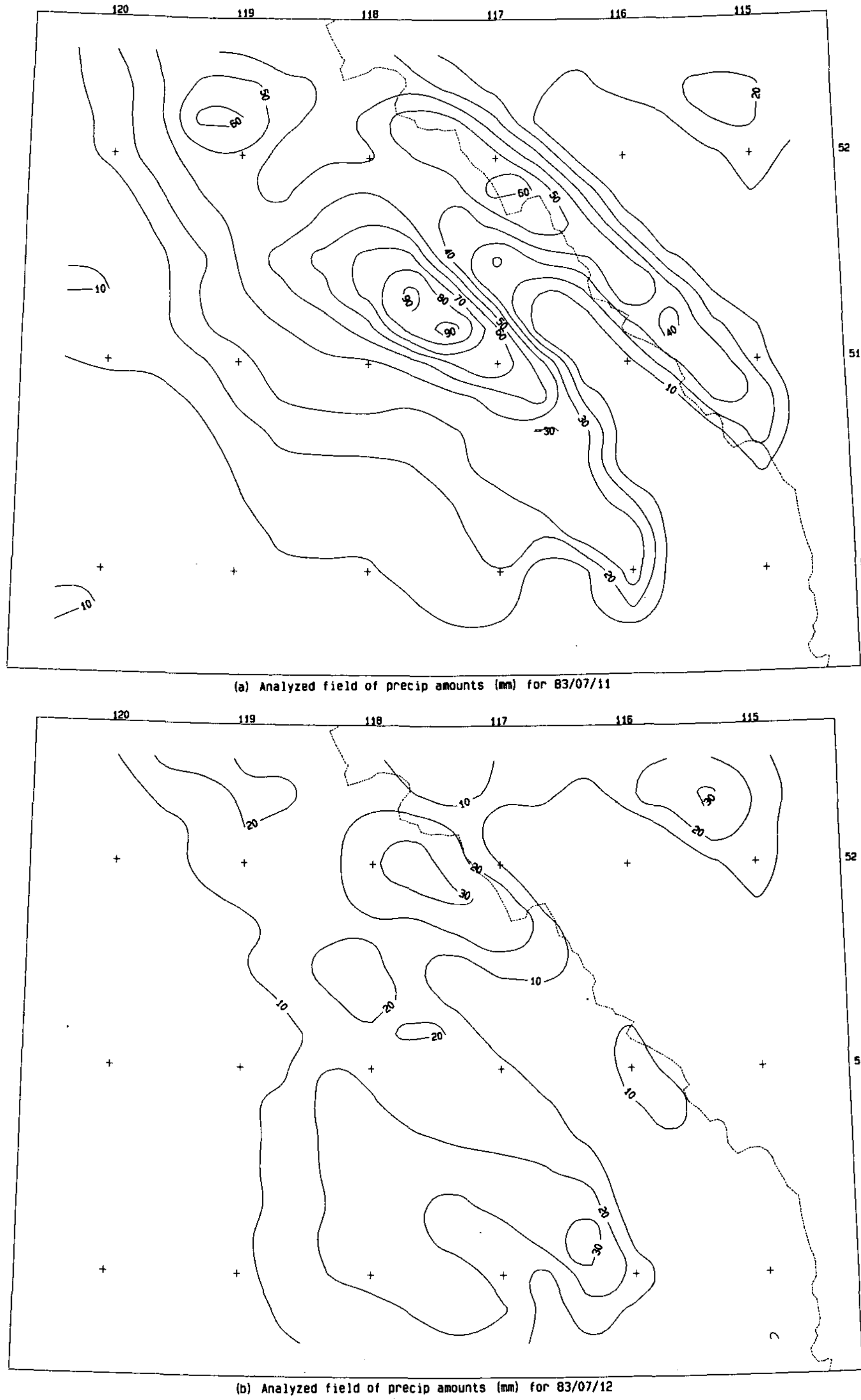
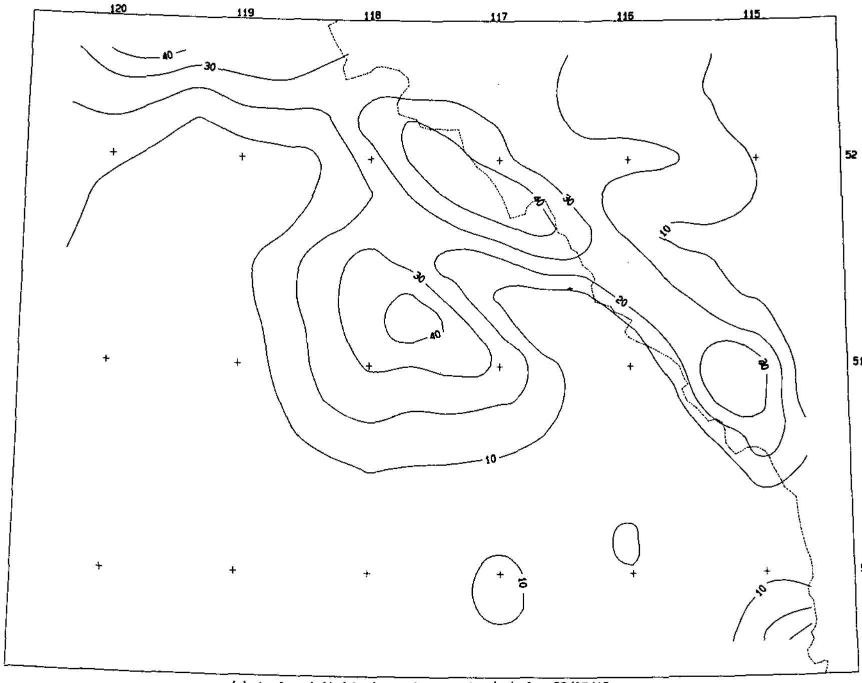


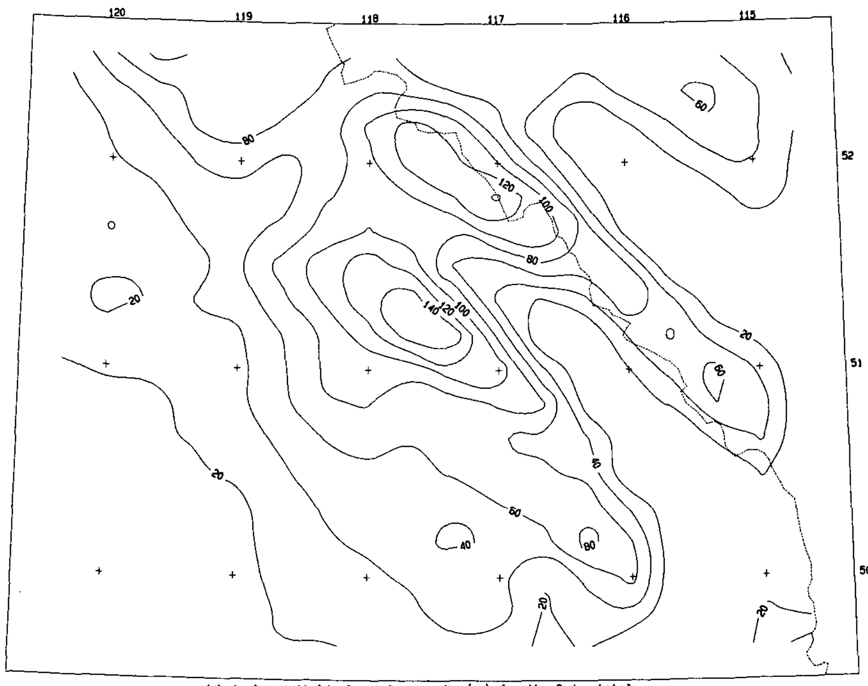
FIG. 5. Analyzed fields of precipitation amounts (mm) for (a) 11 July 1983, (b) 12 July 1983, (c) 13 July 1983, and (d) 3-day total.

Topography is an influential factor and it has been incorporated in the trial fields employed in the application. To the authors' knowledge, the use of optimum interpolation in this specific context is unique.

This work has made use of two mathematical assumptions. The first assumption, routinely made in any application of optimum interpolation, is that trial field errors and observational errors are independent of each other. The second assumption, which is specific to this work, states that there is no correlation between the observational errors and the deviations of trial field values from the observations. These two assumptions



(c) Analyzed field of precip amounts (mm) for 83/07/13



(d) Analyzed field of precip amounts (mm) for the 3-day total

FIG. 5. (*Continued*)

together imply that observational errors are uncorrelated with each other. This allows, then, for an establishment of a formula for covariances that utilizes available measurements. The assumption of uncorrelated observational errors is strictly not valid when observations have systematic errors. Future work may want to look at ways to eliminate systematic errors from observations or to develop a methodology that accounts for the presence of the same. Certain refinements, such as more accurate ways of determining values of r, calculation of the ensemble as a temporal rather than spatial average, etc., can also be undertaken.

TABLE 2. Comparison of weights obtained for grid point at 52.16°N, 117.66°W for 12 July 1983.

Observation point	Weights from original system	Weights from perturbed system
52.05°N, 118.58°W	0.3573	0.3677
51.63°N, 118.41°W	0.1049	0.1022
41.28°N, 117.51°W	0.1491	0.1524

In the application of (15), the radius of influence was chosen so that all terms on the right-hand side (i.e., all the τ_{kg}'s) are nonnegative. That is, all the observations used are positively correlated with the precipitation at point g. However, some of the λ_i's turn out to be negative, which seems inconsistent with the positive covariances. This apparent paradox has not yet been adequately resolved as of this writing.

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REFERENCES

Bigg, G. R., 1991: Kriging and intraregional rainfall variability in England. *Int. J. Climatol.*, **11**, 663–675.

Danard, M. B., and J. Galbraith, 1992: Prediction of the rainstorm of 11–13 July 1983 in the upper Columbia River basin using a 3-dimensional initial-value model. Report prepared by Atmospheric Dynamics Corps, Victoria, B. C., for B. C. Hydro Hydroelectric Engineering Division, 54 pp. [Available from B. C. Hydro Hydroelectric Engineering Div. 6911 Southpoint Drive, Podium A02, Burnaby, B. C. V3N 4X8.]

Eliassen, A., 1954: Provisional report on spatial covariance and autocorrelation of the pressure field. Inst. Weather and Climate Res., Acad. Sci. Oslo, Rept. No. 5. See also *Dynamic Meteorology, Data Assimilation Methods,* (1981), L. Bengtsson, M. Ghil, E. Källen, Eds., Springer-Verlag, 319–330.

Finkelstein, P. L., 1984: The spatial analysis of acid precipitation data. *J. Climate Appl. Meteor.*, **23**, 52–62.

Gandin, L. S., 1963: *Objective Analysis of Meteorological Fields.* Hydromet Press, 242 pp.

Gustafsson, N., 1981: A review of methods for objective analysis, *Dynamic Meteorology, Data Assimilation Methods,* L. Bengtsson, M. Ghil, E. Kallen, Eds., Springer-Verlag, 17–76.

Louie, P. Y., and B. E. Goodison, 1985: Canadian methods for precipitation measurements and corrections. *Workshop on the Correction of Precipitation Measurements,* Zurich, Switzerland.

Matheron, G., 1971: The theory of regionalized variables and its applications, *Cahiers Cent. Morphologie Mathematique, No. 5,* Centre du Geostatisque, Fontainbleau, France.

Middleton, W. E., and A. F. Spilhaus, 1953: *Meteorological Instruments,* University of Toronto Press, 286 pp.

Tanguay, M., and A. Robert, 1990: An efficient optimum interpolation analysis scheme, *Atmos.–Ocean,* **28,** 365–377.

U.S. Department of Commerce, 1988: *Hydrometeorological Report No. 55, Probable maximum precipitation estimates in the United States between the continental divide and the 103rd meridian.* 242 pp.

Venkatram, A., 1988: On the use of kriging in the spatial analysis of acid precipitation data. *Atmos. Environ.*, **22,** 1963–1975.