Numerical Study on Flow Pass of a Three-Dimensional Obstacle under a Strong Stratification Condition

W. SHA AND K. NAKABAYASHI

Department of Mechanical Engineering, Nagoya Institute of Technology, Nagoya, Japan

H. UEEDA

Research Institute for Applied Mechanics, Kyushu University, Kyushu, Japan

(Manuscript received 5 November 1996, in final form 3 February 1997)

ABSTRACT

A three-dimensional, nonhydrostatic, numerical turbulent model was used to study the flow pass of a three-dimensional obstacle under a strong stratification condition. The numerical results clarify the behavior of the flow at a low Froude number, showing the relative importance of the stratification effects on the flow splitting, wave breaking, and lee vortices phenomena.

A vertical vorticity budget study shows that the tilting and friction terms are important to the formation of the lee vortices. On the other hand, the advection and stretching terms are responsible for carrying the vorticity to the lee side. The baroclinicity term can be ignored.

1. Introduction

Flow structure over or around a three-dimensional obstacle changes drastically with stratification, which can be characterized by a Froude number (Fr = U/Nh0, where U is the flow speed, h0 is the height of the obstacle, and N is the Brunt–Väisälä frequency). At low Froude numbers, two phenomena are recognized: the onset of flow around the obstacle instead of that over the obstacle (i.e., flow splitting phenomenon) and an onset of wave breaking above the obstacle (i.e., wave breaking phenomenon). Arguments from a linear theory (Smith 1989), experiments (Hunt and Snyder 1980), and numerical simulations (Smolarkiewicz and Rotunno 1989; Crook et al. 1990; Miranda and James 1992) suggest the occurrence of the flow splitting and wave breaking phenomena at sufficiently low Froude numbers.

A pair of vortices (lee vortices) is formed on the lee side of the obstacle at a lower Froude number (Hunt and Snyder 1980; Smolarkiewicz and Rotunno 1989; Crook et al. 1990; Miranda and James 1992). In the atmosphere the air flows in approximately horizontal planes around topography as the stratification is strong enough. Despite their practical importance to aeronautics and air pollution dispersion, the distinctions among these phenomena are still little known. Moreover, the effects of stratification on flow splitting and wave breaking are also far from clear.

As for the lee vortices, there is still a debate on its formation mechanism. That is, interaction between the fluid and boundary is the only one mechanism of vortex formation under neutral stratification condition, while baroclinicity may be another candidate for vortex formation. There is still an argument about the contribution of the later mechanism in a real atmosphere. The present work tries to investigate the importance of the stratification effect on the flow splitting and wave breaking phenomena by a numerical model. The vertical vorticity equation with its budget calculation is also provided to understand which terms in the equation are important for the formation of the lee vortices. Section 2 describes the numerical model. Section 3 presents the numerical results and discussions. In Section 4 we give the conclusions.

2. Model equation

The numerical model used in this study was the same as in Sha et al. (1996). Only a brief description of the model features will be given here.

The model equations are based on the atmospheric primitive equations simplified by adoption of the anelastic and Boussinesq approximations. A terrain-following coordinate system is used. After transformation from the usual Cartesian coordinates (x, y, z) to the terrain-following coordinate (x, y, η) with
Fig. 1. Isentropic displacement for $z/N = (a) \pi/8, (b) \pi/4, (c) \pi/2, \text{ and (d) } \pi$. Contour interval is 0.25.

$$\eta = z - h \quad \frac{H - h}{h}$$

where $h(x, y)$ is the height of the topography and $H$ is the height of model domain, the governing equations of the three-dimensional, nonhydrostatic, and dry numerical mode are

$$\begin{align*}
\frac{du}{dt} &= f(v - u_x) - \frac{1}{p} \frac{\partial p'}{\partial x} - \frac{1}{p} \eta - 1 \frac{\partial h}{\partial x} + \frac{\partial}{\partial x} \left( K_{hm} \frac{\partial u}{\partial y} \right) \\
+ &\frac{\partial}{\partial y} \left( K_{hm} \frac{\partial u}{\partial x} \right) + \frac{1}{(H-h)^2} \frac{\partial}{\partial \eta} \left( K_{km} \frac{\partial u}{\partial \eta} \right), \\
\frac{dv}{dt} &= -f(u - u_x) - \frac{1}{p} \frac{\partial p'}{\partial y} - \frac{1}{p} \eta - 1 \frac{\partial h}{\partial y} + \frac{\partial}{\partial y} \left( K_{hm} \frac{\partial v}{\partial x} \right) \\
+ &\frac{\partial}{\partial x} \left( K_{hm} \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial \eta} \left( K_{km} \frac{\partial v}{\partial \eta} \right) \\
+ &\frac{1}{(H-h)^2} \frac{\partial}{\partial \eta} \left( K_{km} \frac{\partial v}{\partial \eta} \right). \\
\end{align*}$$

$$\begin{align*}
\frac{dw}{dt} &= -\frac{1}{p} \frac{\eta - 1}{\partial h \partial \eta} + \frac{\theta'}{\partial \eta} + \frac{1}{H-h} \frac{\partial}{\partial \eta} \left( K_{km} \frac{\partial u}{\partial \eta} \right) \\
+ &\frac{1}{H-h} \frac{\partial}{\partial \eta} \left( K_{km} \frac{\partial v}{\partial \eta} \right) + \frac{1}{(H-h)^2} \frac{\partial}{\partial \eta} \left( K_{km} \frac{\partial w}{\partial \eta} \right), \\
\end{align*}$$

and

$$\begin{align*}
\frac{d\theta}{dt} &= \frac{\partial}{\partial x} \left( K_{sh} \frac{\partial \theta}{\partial x} \right) + \frac{\partial}{\partial y} \left( K_{sh} \frac{\partial \theta}{\partial y} \right) + \frac{1}{(H-h)^2} \frac{\partial}{\partial \eta} \left( K_{eh} \frac{\partial \theta}{\partial \eta} \right), \\
\frac{\partial}{\partial x} [\bar{p}(H-h)u] + \frac{\partial}{\partial y} [\bar{p}(H-h)v] + \frac{\partial}{\partial \eta} [\bar{p}(H-h)\omega] &= 0,
\end{align*}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial \eta}.$$
Besides the usual symbols in the aforementioned equations, \( \rho' \) and \( \theta' \) are the mesoscale deviations of pressure and potential temperature from the basic state.

In the turbulence modeling, the turbulent eddy diffusivity for momentum \( K_{\text{mm}} \) is evaluated from the turbulent kinetic energy \( E \) and the turbulent dissipation rate \( \varepsilon \) (Sha et al. 1991, 1996) as follows:

\[
K_{\text{mm}} = c_m \frac{E^2}{\varepsilon}.
\]

Here \( c_m \) is 0.14. The vertical eddy diffusivity for heat \( K_{\text{vh}} \) is represented by

\[
K_{\text{vh}} = \frac{K_{\text{mm}}}{\sigma_T},
\]

where \( \sigma_T \) is the turbulent Prandtl number. The horizontal eddy diffusivity for momentum \( K_{\text{hm}} \) in a nonlinear three-dimensional form (Physick 1988) defined as

\[
K_{\text{hm}} = \beta (\Delta x \Delta y) \left[ \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right]^{1/2}
\]

was used, where \( \beta \) is a constant determined by numerical experiment and \( \Delta x \) and \( \Delta y \) are the grid spacings.

### a. Numerical aspects and computational domain

The prognostic equations are integrated forward explicitly in time with the time step chosen to satisfy the CFL criterion. Centered difference approximations are used in space; the advective terms are evaluated by the upstream difference with a spline technique. To increase the accuracy of the finite difference approximations, a staggered grid is used, both horizontally and vertically. In the horizontal directions, a grid interval of \( \Delta x = \Delta y = 1000 \) m is adopted, while in the vertical direction an expanding grid, with the greater resolution \( \Delta z = 20.2 \) m near the ground, is used.

Numerical calculation is performed on a domain of which lateral boundaries are located at 100 km both in \( x \) and \( y \) directions, while the top boundary is located at \( z = 8.6 \) km. The domain consists of \( 101 \times 101 \times 45 \) grid points. The mesoscale pressure \( p' \) is computed by solving a three-dimensional discrete Poisson equation with a Neumann boundary condition. The equation is solved directly by a Gaussian elimination method in the vertical direction, and by eigenfunction decomposition.
All experiments use the same mountain shape $h(z)$ as Smith (1980),

$$h(z) = h_0\left[1 + \left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{b}\right)^2\right]^{-3/2},$$

where $h_0$ is the height of the mountain, and $a$ and $b$ are the half-width of the mountain in the $x$, $y$ directions centered at $x_0, y_0$. In these numerical experiments, we chose $h_0 = 2$ km and $a = b = 5$ km.

### b. Boundary and initial conditions

A no-slip lower boundary condition $u = v = 0$ is imposed. The surface temperature is constant. The Monin–Obukhov similarity law is used to determine the surface heat and momentum fluxes and $u$, $v$, and $\theta$ at the nearest grids above the lower boundary. At the upper boundary all variables are fixed at their initial values. In the upper levels of the model, an absorbing layer is employed using Rayleigh damping to suppress downward reflection of energy. At the inflow lateral boundary, $u$, $v$, and $w$ are fixed at their initial values, and an absorbing region similar to that in the upper level is installed upstream. This region serves to maintain the integrity of the upstream profiles throughout the integration. At the outflow lateral boundaries a radiation condition is used (Sha et al. 1991). The linear vertical profile of potential temperature and uniform profile of velocity are used to initialize the model by assuming the profiles horizontally homogeneous in the computational domain.

### 3. Results and discussions

#### a. Quasi-linear gravity waves

The displacement of the isentropic surfaces for the case of $Fr = 1.5$ at four height levels is shown in Fig. 1. It gives a quantitatively accurate representation of the hydrostatic lee waves. A comparison with the linear analysis of Smith (1980, Fig. 1) and numerical work of Miranda and James (1992, Fig. 3) shows a very close similarity. Thus, the model is confirmed to be used to simulate the flows at low Froude numbers.

#### b. Splitting flow and breaking gravity waves

Figure 2 shows the wind $(u, w)$, the potential temperature $(\theta)$ on the central $y$ plane, and the surface wind $(u, v)$ for the case of $Fr = 0.22$. The vertical wind field $(u, w)$ and the potential temperature distribution show an evidence of very reduced gravity wave activity in the cross sections. The horizontal surface flow is characterized by strong splitting around the mountain and a clearly defined vortex pair in the mountain wake. The vortices have axes in the vertical direction. After being
created above the leeside slope, they detach from the slope and flow downstream. Figure 3 is the same as Fig. 2 but for Fr = 0.44. The overall patterns are quite similar to the ones in Fig. 2, but the details are different. The flow is in a wave breaking regime with a large decline of the isotherm immediately after the mountain peak. The lee vortices are smaller and weaker than that in the case of Fr = 0.22. The wind \((u, w)\), the potential temperature \((\theta)\) on the central \(y\) plane, and the surface wind \((u, v)\) are shown in Fig. 4 for the case of Fr = 0.66. An intense and broad hydraulic jump is induced, and strong downslope winds are formed just above the leeside slope of the mountain. Moreover, it makes the isotherm adhering to the leeside slope of the mountain from the mountaintop to the mountain base. No lee vortices can be found in this case. These facts suggest the strong downslope winds minimize the mountain wake region and destruct the lee vortices.

c. Budget study of the vertical vorticity

The vertical vorticity equation in the \((x, y, z)\) coordinates can be expressed as

\[
\frac{\partial \zeta}{\partial t} = \left( \frac{\partial \zeta}{\partial x} - \frac{\partial \zeta}{\partial y} \right) - \zeta \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \nabla^2 \zeta
\]

where \(\zeta\) is the vertical vorticity. Term A is the local change of the vertical vorticity, B is the advection term, C is the stretching term, D is the tilting term, E is the baroclinicity term, and F is residual term mainly including the turbulent friction. Now, let us examine the vertical budget of the vorticity equation to determine which terms are important to the formation of the lee vortices.

After the stationary condition is attained, long-term averaging was made for each term. Then, the time-averaged value of the local change of the vertical vorticity, that is, term A, is assumed to be zero. Figures 5a–f are the surface distributions of the vertical vorticity itself, and terms of advection, stretching, tilting, baroclinicity, and friction for the case of Fr = 0.22. Figure 5a shows, respectively, the lee vortices on the lee side of the slope. The advection and stretching terms (Figs. 5b,c) are one order smaller than the tilting term (Fig. 5d) or friction term (Fig. 5f), but they are responsible for the increase or decrease of the vertical vorticity. These terms are

---

**Fig. 4.** Same as Fig. 2 but for Fr = 0.66.
Fig. 5. Surface distribution of (a) vertical vorticity, (b) advection term, (c) stretching term, (d) tilting term, (e) baroclinicity term, and (f) friction term on the lowest level ($z = 20.2$ m) for case of $Fr = 0.22$. 
FIG. 6. Same as Fig. 5 but for Fr = 0.44.
considered to be responsible for carrying the vortices to the lee side. Figures 5d and 5f show that the tilting and friction terms are most important for maintaining the vertical vorticity. It is also found in Fig. 5e that the baroclinicity term is much smaller than the other terms and can be ignored. The same is also true in horizontal planes at upper heights.

Figures 6a–f are the same as Figs. 5a–f but for the case of \( \text{Fr} = 0.44 \), although the order of each term is different. Therefore, it can be concluded that tilting and friction are responsible for the maintenance of the vertical vorticity, and the contribution of baroclinicity can be ignored.

4. Conclusions
Numerical simulations have been done to study the flow pass of a three-dimensional obstacle under stable stratified conditions. A three-dimensional, nonhydrostatic, numerical turbulent model was used. The numerical results clarify the different behavior of the flow at low Froude numbers, showing the relative importance of the stratification effects on the flow splitting, wave breaking, and lee vortices phenomena and the possibility of nighttime high atmospheric pollution at a local mountainous area. For \( \text{Fr} = 0.22 \), the flow is characterized by a strong streamline splitting with the formation of a very defined vortex pair (lee vortices). For \( \text{Fr} = 0.44 \), the flow is in the wave breaking regime and the lee vortices still exist. For \( \text{Fr} = 0.66 \), the flow is characterized by wave breaking with a strong hydraulic jump and no lee vortices can be found.

A vertical vorticity budget calculation shows that the tilting and friction terms are important to the maintenance of the lee vortices. On the other hand, the advection and stretching terms are responsible for carrying the vortices to the lee side, while the baroclinicity term can be ignored.

Acknowledgments. Computations were carried out at the computing center of the National Institute for Environmental Studies, Japan.

REFERENCES