Radar–Rain Gauge Comparisons under Observational Uncertainties

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ABSTRACT

A simple, analytically tractable model of the radar–rain gauge rainfall observational process, including measurement errors, is presented. The model is applied to study properties of different reflectivity–rainfall (*Z–R*) relationships estimated from radar and rain gauge data. Three common *Z–R* adjustment schemes are considered: direct and reverse nonlinear regression, and the probability matching method. The three techniques result in quite different formulas for the estimated *Z–R* relationships. All three also are different from the intrinsic *Z–R* of the model and depend strongly on the assumed observational uncertainties. The results explain, to a degree, the diversity of *Z–R* relationships encountered in the literature. They also suggest that development of new tools that account for the uncertainties is necessary to separate the observational and natural causes of the *Z–R* variability.

1. Introduction

To apply radar reflectivity data in hydrometeorology, one needs to determine a Z_m–R_p relationship to convert radar-measured reflectivities Z_m into the radar-predicted rainfall intensities R_p. The process of adjusting this conversion based on a data sample usually is referred to as calibration (estimation, tuning) of a *Z–R* relationship. As a result of numerous data analyses, an abundance of quite different reflectivity–rainfall relationships has been obtained. In a comprehensive monograph by Battan (1973), a summary of the power-law *Z–R* relationships encountered in many studies is presented. For example, this summary reports estimated *Z–R* exponent values in a broad range from 0.75 up to 3.0. The diversity of the obtained *Z–R* relationships is attributed typically to large and systematic climatologic variability of the microphysics of precipitation systems. The differences that could originate from other reasons, including different estimation schemes, observational uncertainties, and/or *Z–R* estimation objectives, still are not understood fully. These differences are, in a limited scope, the main focus of this study. A large impact of radar characteristics and data processing on selected statistics of rain fields has been demonstrated by Krajewski et al. (1996). In the present study, we continue their pursuit to try to get more insight into the reflectivity to rain-rate conversion aspect of the larger problem.

Two goals of joint radar and rain gauge rainfall data analysis can be distinguished. The first goal is to produce reliable radar-based predictions of rainfall intensities. Radar products represent different approximations of this quantity because their derivation can be subject to different user-oriented criteria. The second goal is to determine possible physical dependency (i.e., relationship) between surface reflectivity and rainfall intensity in a specific precipitation system. This dependency should be an objective property of the precipitation regime and, thus, it should be independent of the observational uncertainties and of the scheme applied to its identification. So far, not much attention has been paid to this distinction between the two goals, and usually they are considered as one task of ‘‘*Z–R* relationship estimation.’’ In this study, we propose a simple nonlinear model of joint radar–rain gauge measurements to show that these two objectives are not equivalent. We apply it to investigate and to conceptualize the possible impact of large measurement errors of both reflectivity and rainfall sensors on our inference based on the joint radar–rain gauge data.

Specifically, we compare results of three popular *Z–R* estimation methods that often are applied in research and/or operational practice of radar hydrometeorology. The schemes analyzed here are two approaches based on nonlinear regression, and the probability matching method (PMM). We show that the three techniques produce different parameters of the estimated *Z–R* relationships. The estimated parameters are three different functions of the assumed measurement errors and the intrinsic *Z–R* relationship of the model. These results suggest that the diversity of *Z–R* relationships encountered in the literature perhaps can be attributed not only to their climatological variability but also to observational uncertainties and different estimation methods.

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The model presented here is intended to provide insight into the impact of those factors. We also use it to discuss briefly some possible methods that could account for the uncertainties and help to separate the observational and natural causes of the *Z–R* variability.

The model that we define in the next section is idealized and is not designed to mimic accurately all aspects of the real radar–rain gauge measurements. An advantage of the specific framework assumed here is that the model is tractable analytically and the pursued solutions can be derived in the form of closed mathematical formulas. They explicitly demonstrate the behavior of our rainfall observation system in a general and synthetic way. However, their direct practical applicability might be limited due to the assumed simplifications of the model. Also, no discussion of the superiority or inferiority of any of the three estimation methods is attempted here. This discussion is a matter of using specific application-oriented criteria and is beyond the scope of this study.

This paper is organized as follows. In the next section we define and discuss the model framework. In section 3, solutions for the three *Z–R* estimation methods are derived. In section 4, the solutions are compared and discussed. A summary and conclusions section closes the study.

2. Definition of the model

The model developed in this study consists of simple parameterizations of the radar and rain gauge observation errors and of a fixed one-to-one dependency between the surface reflectivity and rainfall intensity. We assume that this physical dependency can exist only between quantities that are associated with the same spatial domain. The domain is determined by the radar resolution and can be defined as a single radar grid or a projection of a radar volume bin onto the surface. The differences between the surface reflectivity averaged over this grid area and the actually measured radar reflectivity can originate from numerous sources (Battan 1973; Zawadzki 1984). In our model they all are treated synthetically and are described as one error factor. The second error component applies to the differences between the rainfall intensity averaged over the same grid and its rain gauge sampling. Large differences of the spatial resolution between the quasi–point rain gauge sampling and the grid-averaged radar measurements result from extreme spatial variability of rainfall fields. They prevent any straightforward interpretation of the radar–rain gauge comparisons (Zawadzki 1975; Kitchen and Blackall 1992). In fact, there is evidence that the area–point effects might even dominate the comparisons (Ciach and Krajewski 1997, 1999). This possibility is not surprising once it is realized that the difference between radar grid area and rain gauge collection area may be as big as eight orders of magnitude (this scale difference is comparable to the difference between a radar grid and a continent). Again, in our model the area–point differences and other rain gauge errors are described synthetically as one error factor.

Now we proceed with mathematical definition of the model. The underlying physics of the rainfall system is described by a fixed nonlinear dependency between the true grid-averaged rainfall intensity R_a and its corresponding surface reflectivity Z_a averaged over the same area. They are associated with each other through a commonly known power-law relationship:

$$Z_a = AR_a^b, \tag{1}$$

where parameter A traditionally is called the multiplier of the *Z–R* relationship and parameter b is called its exponent. The observables associated with the two physical quantities are corrupted with large measurement errors. In this model, the errors are assumed to be multiplicative and independent of the truth. Thus, the rain gauge–measured rainfall intensity R_m and the radar-measured reflectivity Z_m are linked with their corresponding physical quantities defined in (1) through the following observation equations:

$$R_m = R_a E_R, \tag{2a}$$

and

$$Z_m = Z_a E_Z, \tag{2b}$$

where E_R and E_Z are the measurement errors for the rainfall and reflectivity, respectively. As already mentioned, E_R represents the difference between rainfall intensity averaged over a radar grid and its quasi–point rain gauge sampling. The factor E_Z describes the difference between surface reflectivity averaged over the same area and the actually measured radar reflectivity.

Finally, we assume statistical independence and lognormal probability distribution of the true grid-averaged rainfall intensity and of the error factors. These random variables are parameterized in the following way:

$$R_a \sim \mathrm{LN}(\mu_R, \mathrm{CV}_R), \tag{3a}$$

$$E_R \sim \mathrm{LN}(1, \mathrm{CV}_{E_R}), \tag{3b}$$

and

$$E_Z \sim \mathrm{LN}(1, \mathrm{CV}_{E_Z}), \tag{3c}$$

where the random variable R_a is defined by its mean μ_R and coefficient of variation CV_R, and CV_{E_R} and CV_{E_Z} are the coefficients of variation of the corresponding errors. Because of the assumed functional dependency expressed in (1), the parameters of the Z_a distribution are determined by the model parameters that already have been defined. Equations (1), (2), and (3) completely define the mathematics of our model. Now, we will briefly discuss its assumptions.

We assumed lognormal probability distributions of all the random variables involved, and the power-law and multiplicative forms of the nonlinear interactions between them. This makes the model framework mathematically consistent because the lognormality is invariant only to the power-law and multiplication operations. Also, each set of the true and measured values is an independent sample and the model does not account for the temporal structure of the rainfall process. Similarly, the spatial dependencies in the rain fields are not addressed in an explicit way. Instead, differences between the true grid-averaged rainfall intensity and its quasi–point rain gauge sampling are synthetically described as a multiplicative, lognormal measurement error E_R. All these structural features make the model analytically tractable, as shown in the next section. We need to reiterate that the model describes a highly idealized picture of the radar–rain gauge reality and only some of its assumptions can be supported by the published results. Other features have been chosen so that they do not contradict the existing experimental evidence, are intuitively sensible, and are plausible.

Another assumption is the fixed Z_a–R_a relationship [(1)] between the true physical variables. Although a real precipitation system is usually a combination of different regimes, many researchers support the idea of fixed (or almost fixed) Z–Rs, at least for specific rainfall regimes (Atlas and Chmela 1957; Joss and Waldvogel 1970; Rosenfeld et al. 1995). Thus, a physical meaning of this assumption may be that the model applies to such a specific situation.

The lognormality of the rainfall intensities has been proposed and tested by Kedem et al. (1994, 1997). These results, however, are based on rain gauge data only. There is no experimental evidence as to whether they also can apply to the rainfall intensities averaged over a radar grid area, which is about a hundred million times larger than the rain gauge collection area. In regard to the measurement errors as they are defined in this study, there is virtually no data on their functional and statistical structure because relevant experiments have not even been designed yet. The multiplicative form [used also by Krajewski and Georgakakos (1985)] reflects a general feature of positively defined physical quantities: the fact that the absolute measurement errors tend to be bigger for large values of the measured variables than for small values. The lognormality of the errors is assumed first of all for the mathematical consistency of the model. It is unlikely that real errors have exactly this distribution, although we can expect that they are positively defined and highly variable because of the extreme variability of the factors involved. These arguments suggest that our lognormal multiplicative error model is at least a physically sensible approximation.

Last, one should note that the expected values of the errors in (3b) and (3c) are assumed to be equal to 1.0, which is equivalent to the lack of overall measurement biases in our model. The main reason for this lack of bias is our choice of the focus of this work, which concentrates on random errors and their impact on the estimated Z–R relationship. Of course, systematic biases also are important and could be analyzed within our model, but to keep this study concise we decided that they are beyond its scope.

3. Solutions for three *Z–R* estimation schemes

In this section, we explore three methods that are commonly used to adjust a Z_m–R_p relationship, which converts radar-measured reflectivities into radar-predicted rainfall intensities, using a specific radar–rain gauge data sample. The three schemes are direct nonlinear regression (from Z to R), reverse nonlinear regression (from R to Z), and PMM. They will be defined rigorously below. In our model, because of the lognormality of all random variables and the fact that only power-law transformations preserve it, the Z_m–R_p relationship will be also in the class of power-law functions. There are two mathematically equivalent forms of this relationship:

$$Z_m = \alpha R_p^{\beta}, \tag{4a}$$

and

$$R_p = \alpha^{-1/\beta} Z_m^{1/\beta} = c Z_m^{d}, \tag{4b}$$

where the values of the parameters α and β in (4a), or $c = \alpha^{-1/\beta}$ and $d = 1/\beta$ in (4b), need to be estimated based on available measurement data. Equation (4a) defines the relationship in a traditional way, after Marshall and Palmer (1948), as reflectivity being a function of rainfall. For the purpose of radar rainfall prediction, an inverse [(4b)] of this form is used. We will refer to the traditional form [(4a)] to compare directly the estimated parameters with the parameters of the assumed intrinsic Z–R defined by (1).

In this study, we are interested not in the sampling properties of the parameter estimates but only in their asymptotic forms in the limit of a large sample. This approach allows for analytical derivations of the estimator formulas. Also, to keep this study concise, we limit the discussion to the estimates of exponent β only and skip derivation of the multiplier α in (4). From a physical point of view, the multiplier can be treated as an adjustment that removes the long-term overall bias of radar rainfall predictions. It can be determined uniquely by β and the model parameters through the following comparison of the R_p and R_m ensemble means:

$$E\{R_m\} = E\{R_p\} = E\{\alpha^{-1/\beta} Z_m^{1/\beta}\}, \tag{5a}$$

and

$$\alpha = \left(\frac{E\{R_m\}}{E\{Z_m^{1/\beta}\}}\right)^{-\beta} = \left(\frac{\mu_R}{E\{Z_m^{1/\beta}\}}\right)^{-\beta}. \tag{5b}$$

Last, only an outline of the derivations is presented below. The details are described thoroughly in Ciach (1997).

a. Direct nonlinear regression

In general, this nonlinear regression tool determines the shape of a Z_m–R_p relationship as conditional expectation of the measured rainfall, conditioned on a fixed value of the measured reflectivity:

$$R_{p1}(z_m) = E\{R_m | Z_m = z_m\} = c_1 z_m^{d_1} = \alpha_1^{-1/\beta_1} z_m^{1/\beta_1} \quad (6)$$

where the subscript ''1'' denotes the first Z_m–R_p estimation method considered in this study. Hereinafter, according to common convention in statistics, we use capital letters to denote random variables and corresponding lowercase letters for their measured values. It has been proven rigorously (Karr 1993) that conditional expectation is a unique and most general solution that minimizes the mean square prediction error.

To derive this conditional expectation for our model, first note that the rain gauge error cancels out from (6) because of its independence, and its expectation being equal to 1.0:

$$\begin{aligned} R_{p1}(z_m) &= E\{R_m | Z_m = z_m\} = E\{R_a E_R | Z_m = z_m\} \\ &= E\{R_a | Z_m = z_m\} E\{E_R\} \\ &= E\{R_a | Z_m = z_m\}. \end{aligned} \quad (7)$$

Let us transform the problem temporarily into logarithmic space. The logarithms of R_a and Z_m have a bivariate Gaussian distribution characterized by five parameters:

$$(\ln R_a, \ln Z_m) \sim N(\mu_{\ln R}, \mu_{\ln Z_m}, \sigma^2_{\ln R}, \sigma^2_{\ln Z_m}, r_{\ln R;\ln Z_m}), \quad (8)$$

where $\mu_{\ln R}$ and $\sigma^2_{\ln R}$ are the mean and variance of the logarithm of the true rainfall intensity R_a, $\mu_{\ln Z_m}$ and $\sigma^2_{\ln Z_m}$ are the mean and variance of the logarithm of the measured radar reflectivity Z_m, and $r_{\ln R;\ln Z_m}$ is the correlation coefficient of $\ln R_a$ and $\ln Z_m$. The parameters in this distribution can be expressed through the parameters of our model based on the close relation between the lognormal and normal distributions (e.g., Crow and Shimizu 1988). If random variable X is lognormally distributed, then $\ln X$ is normally distributed and there exist the following links between their means and variances, which will be used in our derivations:

$$\mu_{\ln X} = \ln \mu_X - (1/2) \ln(\mathrm{CV}_X^2 + 1), \quad (9a)$$

$$\sigma^2_{\ln X} = \ln(\mathrm{CV}_X^2 + 1), \quad \text{and} \quad (9b)$$

$$\mu_X = \exp[\mu_{\ln X} + (1/2)\sigma^2_{\ln X}]. \quad (9c)$$

Using (9a) and (9b), and after some rearrangements, one can obtain the expressions for the five parameters in (8):

$$\mu_{\ln R} = \ln \mu_R - (1/2) \ln(\mathrm{CV}_R^2 + 1), \quad (10a)$$

$$\sigma^2_{\ln R} = \ln(\mathrm{CV}_R^2 + 1), \quad (10b)$$

$$\mu_{\ln Z_m} = \ln \mu_Z - (1/2) \ln(\mathrm{CV}_Z^2 + 1) - (1/2) \ln(\mathrm{CV}_{\mathrm{E_Z}}^2 + 1), \quad (10c)$$

$$\sigma^2_{\ln Z_m} = \ln(\mathrm{CV}_Z^2 + 1) + \ln(\mathrm{CV}_{\mathrm{E_Z}}^2 + 1), \quad \text{and} \quad (10d)$$

$$r_{\ln R;\ln Z_m} = \sqrt{\frac{\ln(\mathrm{CV}_Z^2 + 1)}{\ln(\mathrm{CV}_Z^2 + 1) + \ln(\mathrm{CV}_{E_Z}^2 + 1)}}, \quad (10e)$$

where CV_Z is related to CV_R through

$$\ln(\mathrm{CV}_Z^2 + 1) = b^2 \ln(\mathrm{CV}_R^2 + 1). \quad (11)$$

For two dependent normal random variables, one variable conditioned on the other also is normally distributed. This relation implies that R_a conditioned on a fixed $Z_m = z_m$ (or equivalently on $\ln Z_m = \ln z_m$) has to be lognormally distributed. Applying standard textbook formulas (e.g., Hogg and Craig 1995), one obtains the following expressions for the conditional mean and variance of the $\ln R_a$:

$$E\{\ln R_a | z_m\} = \mu_{\ln R} + r_{\ln R;\ln Z_m} \frac{\sigma_{\ln R}}{\sigma_{\ln Z_m}} (\ln z_m - \mu_{\ln Z_m}), \quad (12a)$$

and

$$\mathrm{Var}\{\ln R_a | z_m\} = \sigma^2_{\ln R}(1 - r^2_{\ln R;\ln Z_m}). \quad (12b)$$

From the mean and variance of the $\ln R_a$ conditioned on z_m, we now can obtain easily the sought expression for the conditional mean of R_a itself by applying (9c):

$$\begin{aligned} R_{p1}(z_m) &= E\{R_a | Z_m = z_m\} \\ &= \exp[E\{\ln R_a | z_m\} + (1/2)\,\mathrm{Var}\{\ln R_a | z_m\}]. \end{aligned} \quad (13)$$

Substituting from (10a)–(10e) and (12a)–(12b) the terms that pertain to z_m (the other terms affect the Z_m–R_{p1} multiplier only), one finally obtains

$$R_{p1}(z_m) = c_1 \exp\left(\ln z_m \frac{1}{b(1 + D_{\mathrm{E_Z}})}\right). \quad (14)$$

Thus, the asymptotic large sample estimate of the exponent d_1 (and consequently of its inverse β_1) is

$$d_1 = \frac{1}{b(1 + D_{E_Z})}, \quad (15a)$$

and

$$\beta_1 = 1/d_1 = b(1 + D_{E_Z}), \quad (15b)$$

where

$$D_{E_Z} = \frac{\ln(\mathrm{CV}_{E_Z}^2 + 1)}{\ln(\mathrm{CV}_Z^2 + 1)}. \quad (16)$$

Quantity D_{E_Z} describes the impact of the reflectivity measurement error relative to the natural variability of the true reflectivity itself. Hereinafter D_{E_Z} will be referred to as a coefficient of radar measurement error.

b. Reverse nonlinear regression

By the ''reverse regression'' we mean adjusting the Z_m–R_p parameters in (4) using a regression procedure in which the reflectivity is a response variable and the measured rainfall R_m is a predictor. This approach is quite common and probably stems from the traditional form of Z–R relationship [(1)]. In this arrangement, reflectivity is a parametric power-law function of rainfall and it seems natural to take the reflectivity as a response variable in the regression. The reverse regression can be defined generally as the conditional mean of radar reflectivities conditioned on a fixed value of rain gauge–measured rainfall:

$$Z_p(r_m) = E\{Z_m | R_m = r_m\} = \alpha_2 r_m^{\beta_2}, \quad (17)$$

where Z_p is the reflectivity predicted for measured rainfall equal to r_m. The estimated parameters α_2 and β_2 are then applied to predict rainfall from measured reflectivity:

$$R_{p2}(z_m) = \alpha_2^{-1/\beta_2} z_m^{1/\beta_2}. \quad (18)$$

A quick solution for the estimate of β_2 can be achieved from the previous result for d_1 [see (17a)] by utilizing a symmetry between (6) and (17). Note that the physics of the model defined by (1) and (2) is invariant to the following transformation of the variable and parameter space:

$$R \rightarrow Z, \quad (19a)$$

$$Z \rightarrow R, \quad (19b)$$

$$b \rightarrow 1/b, \quad \text{and} \quad (19c)$$

$$A \rightarrow A^{-1/b}, \quad (19d)$$

which naturally has to be followed by appropriate exchange of all the other model components (E_R with E_Z, etc.). Specifically, the parameter d_1 in (6) is replaced by β_2 in (17), and the straightforward consequence of this symmetry is the solution

$$\beta_2 = \frac{b}{1 + D_{E_R}}, \quad (20)$$

where

$$D_{E_R} = \frac{\ln(\mathrm{CV}_{E_R}^2 + 1)}{\ln(\mathrm{CV}_R^2 + 1)}. \quad (21)$$

Quantity D_{E_R} is an analog of D_{E_Z} in (16) and characterizes the rain gauge measurement error relative to the variability of the true grid-averaged rainfall intensity. Hereinafter, it will be called a coefficient of rainfall measurement error.

c. Probability matching method

The PMM (Atlas et al. 1990; Rosenfeld et al. 1994, and references therein) constructs a predictive Z_m–R_p conversion function,

$$R_{p3}(z_m) = \alpha_3^{-1/\beta_3} z_m^{1/\beta_3}, \quad (22)$$

so that the resulting radar rainfall products R_{p3} and the rain gauge–measured rainfalls R_m have the same (matched to each other) marginal probability distributions:

$$R_{p3}(Z_m) \sim R_m. \quad (23)$$

This particular Z–R tuning criterion perhaps is stimulated by the desire to make the radar rainfall products mimic statistical behavior of the rain gauge data with which the hydrometeorology community has been familiar for decades. In our model, estimates of the two parameters α_3 and β_3 that fulfill (23) exist and are unique because of the fact that the lognormal distribution also is determined uniquely by its two parameters. If two variables have the same distribution, then their functions also are distributed identically. Thus, we can rearrange (22) and (23) as follows:

$$Z_m = \alpha_3 R_{p3}^{\beta_3} \sim \alpha_3 R_m^{\beta_3}, \quad (24a)$$

and

$$\ln Z_m \sim \ln(\alpha_3 R_m^{\beta_3}) = \ln\alpha_3 + \beta_3 \ln R_m. \quad (24b)$$

This rearrangement implies the equality for the variances of the two sides in (24b) and the immediate solution for the estimate of the exponent β_3:

$$\mathrm{Var}\{\ln Z_m\} = \beta_3^2 \, \mathrm{Var}\{\ln R_m\}, \quad (25a)$$

and

$$\beta_3 = \sqrt{\frac{\mathrm{Var}\{\ln Z_m\}}{\mathrm{Var}\{\ln R_m\}}}. \quad (25b)$$

After using (1), (2), (11), and simple rearrangements, we obtain the explicit expression

$$\beta_3 = \sqrt{\frac{\ln(\mathrm{CV}_Z^2 + 1) + \ln(\mathrm{CV}_{E_Z}^2 + 1)}{\ln(\mathrm{CV}_R^2 + 1) + \ln(\mathrm{CV}_{E_R}^2 + 1)}} = b\sqrt{\frac{1 + D_{E_Z}}{1 + D_{E_R}}}, \quad (26)$$

which contains coefficients of both reflectivity and rainfall measurement errors as defined earlier.

4. Discussion of the results

Three different methods were applied to estimate the parameters of a Z_m–R_p relationship, which could be used to convert radar-measured reflectivities into radar-predicted rainfall. The three Z_m–R_p exponents originating from these procedures are

$$\beta_1 = b(1 + D_{E_Z}), \quad (27a)$$

$$\beta_2 = \frac{b}{1 + D_{E_R}}, \quad \text{and} \quad (27b)$$

$$\beta_3 = b\sqrt{\frac{1 + D_{E_Z}}{1 + D_{E_R}}}, \quad (27c)$$

for direct regression, reverse regression, and the probability matching method, respectively.

The first noticeable feature of these results is that none of them is equal to the intrinsic parameter b, which in our model plays the role of an exponent of physical Z_a–R_a dependency. One can see that, based on direct nonlinear regression, the estimated exponent β_1 always exceeds the assumed b. This result is due to the errors in measurements of the surface reflectivity, which are numerous (Zawadzki 1984) and undoubtedly large enough to make the coefficient of the radar measurement error D_{E_Z} significant. Applying the reverse regression procedure, the estimated β_2 is always smaller than b because of the extreme variability of rainfall intensity fields and the resulting area–point differences. In both cases the departure of the estimated exponent from the original b depends on the error factor of the regressor variable only. This dependence reflects a general fact that regression can filter out the random measurement errors of the response variable (Seber 1989). In contrast to the regression techniques, the β_3 that results from the PMM depends on a combined effect of both radar and rain gauge error factors. From the point of view of correct retrieval of the intrinsic Z_a–R_a relationship, these factors work in opposite direction in (27c) and can compensate for each other. This compensation can happen only in favorable conditions when the coefficients D_{E_Z} and D_{E_R} have similar values. Unfortunately, at present we have neither control nor information on these errors to know the actual outcome of this combination in practice. One interesting property of the three solutions is that they are related closely to each other through the following identity:

$$\beta_3 = \sqrt{\beta_1 \beta_2}, \tag{28}$$

which says that the PMM solution for the Z_m–R_p exponent is a geometric mean of the two solutions given by the direct and reverse nonlinear regressions. At present, we cannot tell whether this feature is characteristic for our model only or perhaps has more general meaning.

The dependence of the estimated *Z–R* exponent on the estimation method and on the uncertainties can be explained based on statistical theory of the ''error-in-variable'' problem (Fuller 1987). In a nutshell, errors in both the predictor and the response variables make the estimation of an underlying dependency a mathematically ill-posed problem. In practice, this fact leads to biased parameter estimates of a functional dependency between the true variables for any standard estimation method. The theory also shows that the problem cannot be solved without additional information that can make the mathematical system complete. This requirement poses a practical question about the conditions under which one could solve fully the system described by our radar–rain gauge model in terms of unbiased estimation of all the model parameters. An advantage of this complete solution would be identification of the underlying physical system and its separation from the distorting effects caused by the observation and data analysis tools (Krajewski et al. 1996). The problem is not trivial, especially for a nonlinear system like that in our model (Carroll et al. 1995; Seber 1989). Additional information on statistical properties of small-scale rainfall, which could be collected from specially designed rain gauge networks (Krajewski et al. 1998), probably could help in answering such questions. However, further pursuit of this direction is beyond the scope of this study.

5. Summary and conclusions

In this study, an idealized and analytically tractable model of joint radar–rain gauge rainfall observations was developed. It was applied to analyze three common *Z–R* estimation schemes. We showed that under large observational uncertainties the three techniques result in quite different estimates of the *Z–R* exponent. All three also can be significantly different from the value of the intrinsic *Z–R* exponent assumed in the model. The analysis indicates a strong impact of the magnitude of radar and rain gauge measurement errors on the estimation results. This impact suggests that, even for the same estimation method and the same rainfall regime, significant differences of the estimated *Z–R* relationships still might exist among different radar sites. These differences are caused by the fact that the sites often have their specific technical characteristics that can impact the D_{E_Z} factor. On the other hand, different rainfall regimes often have different spatial characteristics that can directly impact the area–point differences synthetically described by the D_{E_R} factor. This fact can result in different *Z–R* estimates that might not be associated with any physical *Z–R* differences among the regimes, if the estimates are obtained using reverse regression or the PMM. We believe that all these effects add to the abundance of the reflectivity–rainfall relationships encountered in the literature (Battan 1973). We also suspect that, apart from the problems investigated in this study, several other commonly applied procedures such as arbitrary manipulations with data samples or nonlinear variable transformations can affect the estimated *Z–R*s in a way that often is not understood fully. Perhaps more specific information on the applied data analysis methods, together with conceptual tools that we started to develop here, could facilitate systematic interpretation of the literature results.

Different *Z–R* relationships used in hydrometeorological practice imply different properties of the resulting radar–rainfall products. To assess the utility of these products from the user point of view, appropriate criteria have to be applied. We do not address this subject in this study, and we refrain from discussing the relative value of the three estimation methods for radar-based rainfall prediction. Instead, we advocate an explanatory potential of the simple model developed here to provide insight into the impact of observational uncertainties on the radar estimates.

The possibly large differences between the estimated and the actual *Z–R* relationships discussed here can be important from both the physical and practical points of view. This study shows that distinguishing the natural variability of the *Z–R* exponent from the observational effects is not a trivial task. Statistical literature on the error-in-variable problem suggests that, apart from joint radar–rain gauge data, additional information on the measurement errors is required to solve it. We hope that the new experimental efforts under way, such as the Tropical Rainfall Measuring Mission (Simpson et al. 1988) field campaigns, geared toward collecting small-scale rainfall data, will provide the needed information.

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