Point-to-Area Rescaling of Probabilistic Quantitative Precipitation Forecasts

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ABSTRACT

A probabilistic quantitative precipitation forecast (PQPF) is prepared judgmentally by a meteorologist based on a guidance PQPF. The predictand of a judgmental PQPF is the spatially averaged precipitation amount. The predictand of a guidance PQPF produced by a statistical model is the point precipitation amount. Therefore, a procedure is needed for point-to-area rescaling of the PQPF. Theoretically based equations for rescaling are presented. The equations incorporate two predictive parameters, which characterize the precipitation field being forecast: the quotient of the area covered by a precipitation cell to the area of averaging (cell/area quotient), and the degree of certainty about the precipitation pattern (pattern certainty factor). Both parameters can be judgmentally quantified by the meteorologist during PQPF preparation. The same parameters can be entered into an inverse procedure for area-to-point rescaling of the judgmental PQPF.

1. Introduction

Probabilistic hydrometeorological forecasting is rapidly advancing from theory to application (National Weather Service 1999; Krzysztofowicz 1998). One of the issues in developing an operational procedure for probabilistic quantitative precipitation forecasting (PQPF) is the spatial scale of the predictand. In some forecast systems, the predictand is defined as the *point precipitation amount*. For example, the guidance PQPF produced via the model output statistics technique is for point amounts (Bermowitz and Zurndorfer 1979; Carter et al. 1989). In other forecast systems, the predictand is defined as the *spatially averaged precipitation amount*. For example, the guidance QPF prepared by the Hydrometeorological Prediction Center based on numerical model outputs is for spatially averaged amounts (Olson et al. 1995).

The task of a field forecaster is to combine information from various guidance forecasts, numerical models, observations, and local analyses with his knowledge of local hydrometeorological influences into the final forecast (Krzysztofowicz et al. 1993). To aid the forecaster in performing this complex task, every guidance forecast should pertain to the same predictand. Hence there is a need for a rescaling procedure.

This article coalesces several theoretical results into an operational procedure for rescaling a PQPF of the point amount to a PQPF of the spatially averaged amount. (Rescaling in the opposite direction can be done by rearranging scaling equations, each of which is one-to-one.) In contrast to existing methods for rescaling climatic intensity-duration-frequency distributions of rainfall, whose primary purpose is hydrologic simulation studies, a procedure for rescaling the PQPF should satisfy two operational requirements: (i) it should incorporate essential predictive parameters of the *current meteorologic situation*, as judged by the forecaster, and (ii) it should be implementable in a field office. Meeting these requirements is the aim of the proposed procedure.

The next section defines the predictand and the PQPF. The subsequent three sections present procedures for rescaling elements of the PQPF: probability of precipitation occurrence, conditional distribution of the amount, and expected fractions of the temporal disaggregation. The sixth section formulates a model for judgmental quantification of the variance reduction factor. The last section summarizes the main properties of the overall procedure.

2. Probabilistic forecast

a. Predictand

Consider a fixed period beginning at a designated hour of the day and divided into n subperiods of equal or unequal length. Precipitation amounts to be defined are either point amounts or spatially averaged amounts.

Let W denote the total precipitation amount accumulated during the period. Let W_i denote the precipitation amount accumulated during the ith subperiod, $i \in \{1, \ldots, n\}$. Thus

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$$W \geq 0; \qquad W_i \geq 0, \qquad i = 1, \ldots, n;$$
$$W_1 + \cdots + W_n = W.$$

Conditional on precipitation occurrence during the period, $W > 0$, define variate

$$\Theta_i = \frac{W_i}{W}, \qquad i = 1, \ldots, n,$$

which represents a fraction of the total amount accumulated during subperiod *i*. Thus

$$0 \leq \Theta_i \leq 1, \qquad i = 1, \ldots, n;$$
$$\Theta_1 + \cdots + \Theta_n = 1.$$

The vector of fractions $(\Theta_1, \ldots, \Theta_n)$ defines the *temporal disaggregation* of the total amount into *n* subperiods. Because one of the fractions can always be expressed in terms of the remaining fractions through the unit sum constraint, only $n - 1$ fractions must be forecast. The predictand is the vector $(W; \Theta_1, \ldots, \Theta_n)$.

b. Forecast

The forecast uncertainty is completely characterized in terms of an *n*-variate generalized distribution of the vector $(W; \Theta_1, \ldots, \Theta_n)$. The term generalized distribution stems from the fact that each variate is discrete-continuous. Specifically, $W = 0$ may have probability p $(0 < p < 1)$, while the remaining probability $(1 - p)$ is spread over the unbounded interval $(0, \infty)$ in accordance with some probability density function. Likewise, $\Theta_i = 0$ and $\Theta_i = 1$ may have probabilities p_0 and p_1, respectively $(0 \leq p_0 < 1, 0 \leq p_1 < 1, 0 < p_0 + p_1 < 1)$, while the remaining probability $1 - p_0 - p_1$ is spread over the open interval (0, 1) in accordance with some probability density function. Because direct assessment of an *n*-variate generalized distribution by a field forecaster is infeasible, an operational forecast specifies only key elements of the distribution. This forecast consists of two parts (Krzysztofowicz and Sigrest 1997).

The first part is a probabilistic forecast of total amount *W*. It specifies a probability of precipitation occurrence π during the period,

$$\pi = P(W > 0),$$

and a cumulative distribution function *G* of amount *W*, conditional on the hypothesis $W > 0$; with *P* standing for probability and ω for a fixed amount,

$$G(\omega) = P(W \leq \omega | W > 0), \qquad \omega > 0.$$

Practically, a continuous distribution is obtained by fitting a parametric model to three conditional exceedance fractiles (x_{75}, x_{50}, x_{25}) of *W* that are assessed by the forecaster. With *p* denoting a probability number, $0 < p < 1$, the $100p$% conditional exceedance fractile of *W* is an estimate x_{100p} such that $P(W > x_{100p} | W > 0) = p$.

The second part is a deterministic forecast of temporal disaggregation $(\Theta_1, \ldots, \Theta_n)$. It specifies a vector of expected fractions $\mathbf{z} = (z_1, \ldots, z_n)$, conditional on the hypothesis $W > 0$; with *E* standing for expectation,

$$z_i = E(\Theta_i | W > 0), \qquad i = 1, \ldots, n,$$

where $0 \leq z_i \leq 1$ for every subperiod *i*, and $z_1 + \cdots + z_n = 1$.

In summary, the PQPF specifies π, *G*, and $\mathbf{z}$, and these are the elements for which a rescaling procedure is desired.

c. Examples

In numerical examples presented throughout the paper, the spatially averaged amount is for the Lower Monongahela River basin above Connellsville, which covers 3429 km² (1325 square miles) in Pennsylvania and Maryland. The point amounts are for three stations within the basin with the following abbreviations and elevations: Connellsville (Cn, 900 ft), Confluence (Cf, 1490 ft), and Sines Deep Creek (Sd, 2040 ft). The spatially averaged amounts were calculated from these and three other stations. The source data were hourly amounts recorded by rain gauges from 1943 to 1993. All examples are for months of March and July, 24-h period beginning at 1200 UTC, and disaggregation into four 6-h subperiods. The sample sizes (equal to the number of complete data records) are 725 for March and 937 for July. The area of spatial averaging is roughly comparable to 5000 km² initially specified by the National Weather Service as the nominal scale for the PQPF. This is the scale at which the PQPF should be prepared and verified.

Henceforth, variables pertaining to point precipitation will acquire subscript *O*, and variables pertaining to spatially averaged precipitation will acquire subscript *A*.

3. Rescaling probability of precipitation

a. Area probability

Let $\pi_O = P(W_O > 0)$ denote the point probability, that is, the probability of observing measurable precipitation at a fixed point *O* within the area, and let $\pi_A = P(W_A > 0)$ denote the area probability, that is, the probability of observing measurable precipitation at some point within the area. A theoretical relationship between π_O and π_A was derived by Epstein (1966) under two assumptions: (i) both the area and the precipitation cells are circular, and (ii) the precipitation cells have identical diameter and are distributed at random over space large compared to the area of averaging (see Fig. 1). Then π_O is identical at every point and

$$\pi_A = 1 - (1 - \pi_O)^{[1+(1/Q)^{1/2}]^2}, \qquad (1)$$

where $Q = C/A$ is the quotient of the area covered by a precipitation cell, *C*, to the area of averaging, *A*; the *cell/area quotient* for short. One can see that $\pi_A > \pi_O$

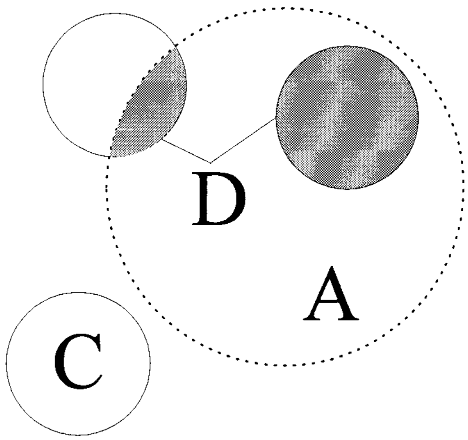

FIG. 1. Epstein's model for point-to-area rescaling of precipitation probabilities consists of a circular area of averaging, A, and circular precipitation cells, each covering area C. The forecaster's task is to judge the quotient $Q = C/A$. The uncertain wetted area is D.

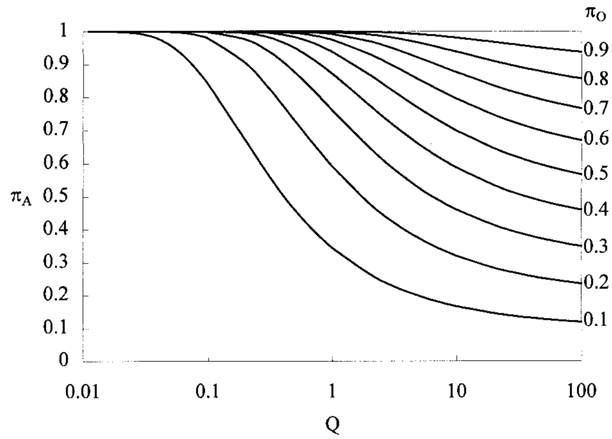

FIG. 2. Relation between point probability π_O and area probability π_A of precipitation occurrence as a function of the cell/area quotient Q, based on Epstein's model.

whenever $\pi_O < 1$, and that π_A converges to π_O as Q increases. These relations are displayed in Fig. 2.

For example, when a small convective storm is forecast for which $Q = 0.5$ and $\pi_O = 0.3$, then $\pi_A = 0.87$. When a large cyclonic system is forecast for which $Q = 5$ and $\pi_O = 0.3$, then $\pi_A = 0.53$. The forecaster's thought process can thus progress from the type of storm (convective vs stratiform) to the spatial character of precipitation (spotty vs widespread), to the judgment of the cell/area quotient Q. Judging Q is a simple cognitive task, relative to other tasks performed by forecasters. Hence (1) offers an operational method for rescaling point probabilities to area probabilities.

Given π_O and π_A, the cell/area quotient can be found from (1) as

$$Q = [(\gamma^{1/2} + 1)/(\gamma - 1)]^2,$$

where $\gamma = [\ln(1 - \pi_A)/\ln(1 - \pi_O)]$. Climatic values of π_O, π_A, and Q are shown in Table 1. At each point, Q is higher in March than it is in July. This implies that, on average, the size of precipitation fields is larger in March than in July, as one would expect. In each month, π_O and Q increase with the elevation of a point. This suggests that a spatial nonhomogeneity of precipitation occurrence can be attributed to local orographic effects.

b. Coverage fraction

The uncertainty about the occurrence of precipitation induces uncertainty about the spatial coverage, which is modeled as follows (see Fig. 1). Let D denote the area covered by precipitation cells within the area of averaging, so that $0 \le D \le A$. Then $\Phi = D/A$ denotes the fraction of area that is covered by the precipitation cells, and hence is wetted. Because D is uncertain, Φ is a random variable, constrained by $0 \le \Phi \le 1$. What one needs for further development is the conditional moments of Φ.

Let I_O denote a Bernoulli variate indicating the occurrence of precipitation at a point and taking on value $I_O = 0$ if $W_O = 0$ or $I_O = 1$ if $W_O > 0$. By employing the total probability law, one can find the probability of precipitation occurrence at a point, conditional on the hypothesis that precipitation occurs within the area: $P(W_O > 0 \mid W_A > 0) = \pi_O/\pi_A$. Consequently, the conditional mean and the conditional variance of the indicator are, respectively, $E(I_O \mid W_A > 0) = \pi_O/\pi_A$ and $\text{var}(I_O \mid W_A > 0) = (\pi_O/\pi_A)(1 - \pi_O/\pi_A)$.

When the probability of precipitation occurrence π_O is identical at every point within the area, the conditional mean and the conditional variance of the coverage fraction Φ can be expressed in terms of the conditional moments of the indicator I_O as follows:

$$E(\Phi \mid W_A > 0) = \frac{\pi_O}{\pi_A}, \tag{2}$$

$$\text{var}(\Phi \mid W_A > 0) = \frac{\pi_O}{\pi_A}\left(1 - \frac{\pi_O}{\pi_A}\right)\tau^2, \tag{3}$$

TABLE 1. Point probability and area probability of precipitation occurrence, the inferred cell/area quotient and variance reduction factor, and the calculated conditional mean and variance of the coverage fraction; Monongahela basin, climatic data, 1943–93.

		Point*			Area
Month	Variable	Cn	Cf	Sd	π_A
Mar	π_O	0.36	0.42	0.50	0.60
	Q	5.09	10.39	39.38	
	τ^2	0.497	0.580	0.721	
	$E(\Phi \mid W_A > 0)$	0.594	0.690	0.825	
	$\text{var}(\Phi \mid W_A > 0)$	0.120	0.124	0.104	
Jul	π_O	0.34	0.36	0.39	0.62
	Q	3.45	4.22	6.09	
	τ^2	0.449	0.472	0.514	
	$E(\Phi \mid W_A > 0)$	0.542	0.572	0.625	
	$\text{var}(\Phi \mid W_A > 0)$	0.112	0.116	0.121	

* Cn—Connellsville, Cf—Confluence, Sd—Sines Deep Creek.

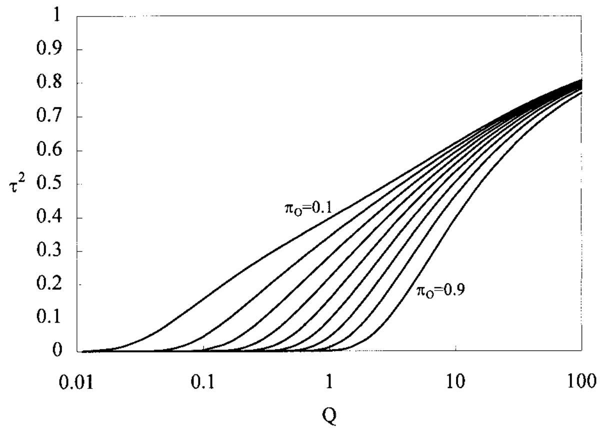

FIG. 3. Relation between the cell/area quotient Q, to be judged by the forecaster, and the variance reduction factor τ^2, for several values of the point probability π_O of precipitation occurrence.

where τ^2 is the variance reduction factor such that $0 \leq \tau^2 \leq 1$. Equation (2) follows from the fact that $E(\Phi | W_A > 0) = E(I_O | W_A > 0)$. Equation (3), which is exact, has been derived in the appendix together with an approximate expression for τ^2, which takes the form

$$\tau^2 = \left(\frac{\pi_O}{\pi_B} - \frac{\pi_O}{\pi_A}\right) \Bigg/ \left(1 - \frac{\pi_O}{\pi_A}\right) \quad \text{if } \pi_B < \pi_A, \tag{4}$$

and $\tau^2 = 0$ if $\pi_B \geq \pi_A$. Probability π_B lies within the interval $\pi_O < \pi_B < 1$ and is specified by the formula

$$\pi_B = 1 - (1 - \pi_O)^{\{1+[(\pi_O/\pi_A)/Q]^{c/2}\}^2}, \tag{5}$$

where $c > 1$ is a parameter. Its estimate $\hat{c} = 1.7$, to be used herein, has been determined empirically from plots of $\mathrm{var}(\Phi | W_A > 0)$ versus A reported by Seo and Smith (1996, Figs. 6 and 7) for precipitation fields observed by two radars over areas ranging from 140 km^2 to 100 000 km^2. Figure 3 shows the relation between π_O, Q, and τ^2; the order of calculations is (1), (5), and (4).

Table 1 shows climatic values of τ^2 calculated from the climatic values of π_O and Q. Also shown are $E(\Phi | W_A > 0)$ and $\mathrm{var}(\Phi | W_A > 0)$, which are calculated according to (2) and (3). The conditional mean of the coverage fraction Φ is higher in March than in July at each point, as one would expect, and it increases with the elevation of a point in each month, which is the result of increasing π_O. Interestingly, the conditional variance of Φ varies little; in fact, it appears to be one of the steadier spatial statistics of precipitation within the Monongahela basin.

In summary, the conditional moments of the coverage fraction Φ are determined solely by the point probability π_O and the cell/area quotient Q. This model of coverage differs from the models of Eagleson and Wang (1985) and Seo and Smith (1996). Their models are suited to analyzing observed precipitation fields—situations in which only the conditional probability of point precipitation occurrence is meaningful; this probability is given by the ratio $p = \pi_O/\pi_A$. Epstein's (1966) theory could be used to express this conditional probability p as a function of Q. Only then could the behavior of $E(\Phi | W_A > 0)$ and $\mathrm{var}(\Phi | W_A > 0)$ be compared with the behavior of their counterparts in climatic models. One distinction is conspicuous as A varies from 0 to ∞. When π_O and C are held fixed in the present model, the conditional mean of Φ decreases monotonically from 1 to an asymptote at π_O. When C is held fixed in the model of Eagleson and Wang (1985, Fig. 5), the conditional mean of Φ decreases from 1 to 0.

4. Rescaling conditional distribution of amount

a. Conditional moments

Define the mean and variance of each amount, W_O and W_A, conditional on the hypothesis that the amount is positive:

$$\mu_O = E(W_O | W_O > 0), \qquad \sigma_O^2 = \mathrm{var}(W_O | W_O > 0),$$

$$\mu_A = E(W_A | W_A > 0), \qquad \sigma_A^2 = \mathrm{var}(W_A | W_A > 0).$$

Next assume that within the area of averaging (i) probability π_O is identical at every point, and (ii) the field of point precipitation amounts, conditional on precipitation occurrence at every point, is covariance stationary; this implies, in particular, that each μ_O and σ_O^2 exists and is identical at every point at which $\pi_O > 0$.

When $\pi_O = 1$ at every point within the area of averaging, assumption (ii) is equivalent to a homogeneous precipitation field used in hydrologic rescaling models (e.g., Rodriguez-Iturbe and Mejia 1974; Sivapalan and Blöschl 1998). Mathematical formulations of these models usually suppress the conditioning of amount, W_O or W_A, on precipitation occurrence, $W_O > 0$ or $W_A > 0$, respectively, and this blurs the distinction between the conditional and unconditional moments of W_O or W_A. This distinction is essential in forecasting because $\pi_O < 1$ more often than not, which has a profound impact on scaling of conditional moments (Seo and Smith 1996).

When $\pi_O < 1$, the wetted area may be less than the averaging area. To account for this possibility, the conditional moments of point amount W_O are first rescaled to conditional moments of W_D, the spatially averaged amount for the wetted area D. This is accomplished by adapting results from hydrologic rescaling models (e.g., Rodriguez-Iturbe and Mejia 1974; Wood and Hebson 1986; Gupta and Waymire 1990) that are based on assumption (ii) and the hypothesis $W_O > 0$ at every point within area A. In the present case, the hypothesis $W_A > 0$ is equivalent to the hypothesis $W_D > 0$, which in turn is equivalent to the hypothesis $W_O > 0$ at every point within area D. Hence,

$$E(W_D | W_A > 0) = E(W_O | W_O > 0) = \mu_O, \tag{6}$$

$$\mathrm{var}(W_D | W_A > 0) = \mathrm{var}(W_O | W_O > 0)\kappa^2 = \sigma_O^2\kappa^2, \tag{7}$$

where κ^2 is the variance reduction factor such that $0 < \kappa^2 \leq 1$. In general, κ^2 depends upon the spatial correlation structure of the precipitation field, as well as the size and shape of the wetted area. In the present case, however, the wetted area is random and this makes κ^2 different from the variance reduction factor found in the hydrologic models. The effect of this randomness will be modeled later.

Mass conservation implies $AW_A = DW_D$. Hence $W_A = \Phi W_D$. Assuming that conditional on event $W_A > 0$, variates Φ and W_D are mutually stochastically independent, the conditional moments of W_A may be derived via three equations:

$$E(W_A | W_A > 0) = E(\Phi | W_A > 0)E(W_D | W_A > 0),$$

$$\begin{aligned} E(W_A^2 | W_A > 0) &= E(\Phi^2 | W_A > 0)E(W_D^2 | W_A > 0) \\ &= [\mathrm{var}(\Phi | W_A > 0) + E^2(\Phi | W_A > 0)] \\ &\quad \times [\mathrm{var}(W_D | W_A > 0) \\ &\quad + E^2(W_D | W_A > 0)], \end{aligned}$$

$$\mathrm{var}(W_A | W_A > 0) = E(W_A^2 | W_A > 0) - E^2(W_A | W_A > 0).$$

With all inputs specified by (2)–(3) and (6)–(7), one finds

$$\mu_A = \frac{\pi_O}{\pi_A}\mu_O, \qquad (8)$$

$$\sigma_A^2 = \frac{\pi_O}{\pi_A}\left\{\sigma_O^2\kappa^2\left[\tau^2\left(1 - \frac{\pi_O}{\pi_A}\right) + \frac{\pi_O}{\pi_A}\right] + \mu_O^2\tau^2\left(1 - \frac{\pi_O}{\pi_A}\right)\right\}. \qquad (9)$$

Equation (8) establishes a *ratio transformation* between the conditional mean of the point amount, μ_O, and the conditional mean of the spatially averaged amount, μ_A. Equation (9) establishes a *linear transformation* between the conditional variance of the point amount, σ_O^2, and the conditional variance of the spatially averaged amount, σ_A^2.

When precipitation is certain to occur at the point, $\pi_O = \pi_A = 1$ and $\sigma_A^2 = \sigma_O^2\kappa^2$; thus (9) is consistent with the hydrologic rescaling models that assume $\pi_O = \pi_A = 1$. However, when $\pi_O < \pi_A < 1$, Eqs. (8) and (9) reveal that the ratio of the point probability to the area probability, π_O/π_A, becomes the scaling factor for the conditional mean and a primary scaling factor for the conditional variance, along the variance reduction factors τ^2 and κ^2. In addition, μ_O enters the intercept of (9).

Unlike the case with certain occurrence of point precipitation, which leads to rescaling of the conditional variance via a ratio transformation, the case with uncertain occurrence of point precipitation leads to rescaling of the conditional variance via a linear transformation. This may be rationalized by considering a situation wherein, conditional on the hypothesis that precipitation will occur at the point, $W_O > 0$, the forecaster predicts the point amount W_O with certainty, so that $P(W_O = \mu_O | W_O > 0) = 1$ and consequently $\sigma_O^2 = 0$. Now (9) yields $\sigma_A^2 > 0$. In other words, conditional on the hypothesis that precipitation will occur at one point or more within the area, the spatially averaged amount W_A is not predicted with certainty. The reason is that the spatial coverage of precipitation remains uncertain, as indicated by $\pi_O < 1$. Only when this uncertainty vanishes as well, so that $\pi_O = 1$ and $\pi_O/\pi_A = 1$, does $\sigma_O^2 = 0$ imply $\sigma_A^2 = 0$.

b. Conditional distribution

Because forecasters judge conditional exceedance fractiles, not conditional moments, expressions for rescaling fractiles are desired. Toward this end, (point or spatially averaged) amount W, conditional on precipitation occurrence, $W > 0$, is modeled in terms of the Weibull family of distributions:

$$G(\omega) = 1 - \exp[-(\omega/\alpha)^\beta], \qquad \omega > 0, \qquad (10)$$

where $\alpha > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. The moments of W are

$$\mu = \alpha\Gamma(1 + 1/\beta), \qquad (11)$$

$$\sigma^2 = \alpha^2[\Gamma(1 + 2/\beta) - \Gamma^2(1 + 1/\beta)]. \qquad (12)$$

For any probability p, $0 < p < 1$, the $100p$% conditional exceedance fractile of W can be found from equation $p = 1 - G(\omega)$ and takes the form

$$\omega = \alpha(-\ln p)^{1/\beta}. \qquad (13)$$

It is the closed form of (10) and (13) that makes the Weibull model advantageous for operational forecasting. Most importantly, the model usually fits well to empirical distributions of daily amounts (e.g., Hershenhorn and Woolhiser 1987; Wilks 1989; Selker and Haith 1990). However, I do not know if a proof exists that a Weibull distribution of point amount is sufficient for a Weibull distribution of spatially averaged amount. Good fits of the Weibull model to both amounts suggest that at least some approximate sufficiency exists. This observation parallels the assumption of Wood and Hebson (1986) that a gamma distribution of point amount is approximately sufficient for a gamma distribution of the spatially averaged amount.

c. Conditional distribution parameters

Under the assumption that the conditional distribution of each amount, W_O and W_A, belongs to the Weibull family, point parameters (α_O, β_O) are uniquely rescaled to area parameters (α_A, β_A) via equations derived from (8)–(9) and (11)–(12). These equations are analytic, but not closed form:

$$\alpha_A = \frac{\pi_O}{\pi_A}\frac{\Gamma(1 + 1/\beta_O)}{\Gamma(1 + 1/\beta_A)}\alpha_O, \quad (14)$$

$$\left[\frac{\Gamma(1+2/\beta_A)}{\Gamma^2(1+1/\beta_A)} - 1\right]\frac{\pi_O}{\pi_A} = \left[\frac{\Gamma(1 + 2/\beta_O)}{\Gamma^2(1 + 1/\beta_O)} - 1\right] \times \left[\frac{\pi_O}{\pi_A} + \left(1 - \frac{\pi_O}{\pi_A}\right)\tau^2\right]\kappa^2 + \left(1 - \frac{\pi_O}{\pi_A}\right)\tau^2. \quad (15)$$

When expressed as functions of all inputs, the equations take the following general form: $\alpha_A = \alpha(\alpha_O, \beta_O, \pi_O/\pi_A, \tau^2, \kappa^2)$ and $\beta_A = \beta(\beta_O, \pi_O/\pi_A, \tau^2, \kappa^2)$. It is now transparent that three factors determine the scaling: the ratio of the point probability to the area probability, π_O/π_A; the factor reducing the conditional variance of the point precipitation occurrence, τ^2; and the factor reducing the conditional variance of the point amount, κ^2. Interestingly, the area scale parameter α_A depends upon both point parameters, scale α_O and shape β_O, whereas the area shape parameter β_A depends only upon the point shape parameter β_O.

The equations imply that (i) $\beta_A = \beta_O$ if and only if $\alpha_A = \alpha_O(\pi_O/\pi_A)$, and (ii) $\beta_A = \beta_O = \beta$ if and only if

$$\kappa^2 = \frac{(\pi_O/\pi_A)\gamma - (1 - \pi_O/\pi_A)\tau^2}{[\pi_O/\pi_A + (1 - \pi_O/\pi_A)\tau^2]\gamma},$$

where $\gamma = \Gamma(1 + 2/\beta)/\Gamma^2(1 + 1/\beta) - 1$. If, in addition, $\pi_A = \pi_O$, then $\kappa^2 = 1$. Because in reality $\kappa^2 < 1$, the equalities $\beta_A = \beta_O$ and $\pi_A = \pi_O$ cannot occur simultaneously. Hence, the shape parameters for areal amount and point amount can be identical only when $\pi_A > \pi_O$, which, according to (1), occurs whenever $\pi_O < 1$. Last, the equations imply that (iii) $\alpha_A = \alpha_O$ if and only if $\Gamma(1 + 1/\beta_A)\pi_A = \Gamma(1 + 1/\beta_O)\pi_O$. If, in addition, $\pi_A = \pi_O$, then $\beta_A = \beta_O$ and $\kappa^2 = 1$, which is unrealistic. Hence in practice, the scale parameters can be identical only if the shape parameters are not, and vice versa.

Climatic estimates of the distribution parameters are shown in Table 2. In both months and for all points, $\alpha_A < \alpha_O$ and $\beta_A < 1 < \beta_O$. The latter two inequalities imply that W_A has the mode at zero, whereas W_O has the mode at a positive amount (Johnson and Kotz 1970, p. 251). Thus the difference between β_A and β_O, which may appear small, has a significant impact on the shape of the distribution. In summary, spatial averaging of precipitation has two effects on the Weibull conditional distribution of the amount: (i) both parameters decrease, especially the scale parameter, and (ii) the mode of the amount shifts toward zero.

Table 2 also shows estimates of the variance reduction factor κ^2 inferred via (9) from climatic values of $(\pi_O, \mu_O, \sigma_O^2)$ and $(\pi_A, \mu_A, \sigma_A^2)$. At each point, κ^2 is larger in March than in July. This suggests that κ^2 decreases as precipitation becomes more scattered (Q decreases) and point amount becomes more uncertain (σ_O^2 increases).

TABLE 2. Parameters of the Weibull distributions of point and areal amounts, calculated moments, and the inferred variance reduction factor; Monongahela basin, climatic data, 1943–93. Note: μ_O and σ_O are in in./24 h.

Month	Parameter	Point*			Area
		Cn	Cf	Sd	
Mar	π_O	0.36	0.42	0.50	0.60 $= \pi_A$
	α_O	0.270	0.280	0.248	0.151 $= \alpha_A$
	β_O	1.079	1.139	1.039	0.930 $= \beta_A$
	μ_O	0.262	0.267	0.244	0.157 $= \mu_A$
	σ_O^2	0.059	0.055	0.055	0.028 $= \sigma_A^2$
	$\mu_O\pi_O$	0.094	0.111	0.121	0.095 $= \mu_A\pi_A$
	κ^2	0.724	0.589	0.514	
Jul	π_O	0.34	0.36	0.39	0.62 $= \pi_A$
	α_O	0.359	0.383	0.337	0.185 $= \alpha_A$
	β_O	1.088	1.132	1.003	0.973 $= \beta_A$
	μ_O	0.347	0.366	0.337	0.187 $= \mu_A$
	σ_O^2	0.102	0.105	0.113	0.037 $= \sigma_A^2$
	$\mu_O\pi_O$	0.117	0.130	0.131	0.117 $= \mu_A\pi_A$
	κ^2	0.573	0.509	0.506	

* Cn—Connellsville, Cf—Confluence, Sd—Sines Deep Creek.

Finally, Table 2 contains evidence for an assertion that station Cn is representative of the basin in the sense of satisfying the scaling condition $\mu_A\pi_A = \mu_O\pi_O$. On the contrary, stations Cf and Sd illustrate cases wherein the scaling condition is violated, albeit not excessively. Inasmuch as $\mu_O\pi_O$ increases with the elevation of a point, as does π_O, a spatial nonhomogeneity of the mean amount can be attributed to effects of local orography on precipitation occurrence. Implications are discussed at the end of the paper.

d. Conditional exceedance fractiles

An equation for rescaling the 100p% conditional exceedance fractile ω_O of the point amount W_O into the corresponding 100p% conditional exceedance fractile ω_A of the areal amount W_A can be obtained for any p, $0 < p < 1$, as follows. Letting $c = (\pi_O/\pi_A)[\Gamma(1 + 1/\beta_O)/\Gamma(1 + 1/\beta_A)]$, Eq. (14) takes the form $\alpha_A = c\alpha_O$. Starting from (13) for ω_A yields

$$\omega_A = \alpha_A(-\ln p)^{1/\beta_A} = c\alpha_O(-\ln p)^{(1/\beta_O)(\beta_O/\beta_A)}$$
$$= c\alpha_O^{1-\beta_O/\beta_A}[\alpha_O(-\ln p)^{1/\beta_O}]^{\beta_O/\beta_A} = c\alpha_O^{1-\beta_O/\beta_A}\omega_O^{\beta_O/\beta_A},$$

which, after substituting the expression for c, takes the form

$$\omega_A = m\frac{\pi_O}{\pi_A}\omega_O^n, \quad (16)$$

where

$$m = \frac{\Gamma(1 + 1/\beta_O)}{\Gamma(1 + 1/\beta_A)}\alpha_O^{1-\beta_O/\beta_A},$$

$$n = \beta_O/\beta_A.$$

TABLE 3. Approximate rescaling of conditional exceedance fractiles from the point amounts to the spatially averaged amount; Monongahela basin, climatic data, 1943–93. Note: Amounts are in in./24 h.

Month	p	Variable	Point* Cn	Cf	Sd	Area ω_A
Mar		π_O/π_A	0.594	0.690	0.825	
		ν	1.057	1.195	1.202	
	0.75	ω_O	0.085	0.094	0.075	
	0.75	$\hat{\omega}_A$	0.044	0.041	0.037	0.040
	0.50	ω_O	0.192	0.203	0.174	
	0.50	$\hat{\omega}_A$	0.104	0.103	0.101	0.102
	0.25	ω_O	0.365	0.373	0.339	
	0.25	$\hat{\omega}_A$	0.205	0.212	0.225	0.215
Jul		π_O/π_A	0.542	0.572	0.625	
		ν	1.057	1.175	1.109	
	0.75	ω_O	0.114	0.127	0.097	
	0.75	$\hat{\omega}_A$	0.055	0.051	0.047	0.051
	0.50	ω_O	0.256	0.277	0.234	
	0.50	$\hat{\omega}_A$	0.128	0.127	0.125	0.127
	0.25	ω_O	0.484	0.511	0.467	
	0.25	$\hat{\omega}_A$	0.252	0.260	0.269	0.259

* Cn—Connellsville, Cf—Confluence, Sd—Sines Deep Creek.

The rescaling equation takes the form of a *power transformation.* When $\beta_A = \beta_O$, the rescaling equation becomes linear: $\omega_A = (\pi_O/\pi_A)\omega_O$. The earlier analysis implies that the necessary condition for this to occur is $\pi_A > \pi_O$, assuming $\kappa^2 < 1$.

The scaling factors are functions of all inputs; specifically, $m = m(\alpha_O, \beta_O, \pi_O/\pi_A, \tau^2, \kappa^2)$ and $n = n(\beta_O, \pi_O/\pi_A, \tau^2, \kappa^2)$. Thus a formal implementation of (16) requires operational estimates of all inputs on each forecasting occasion. An estimate of π_O is given; an estimate of π_A can readily be calculated from Epstein's formula; an estimate of τ^2 can likewise be calculated. Estimates of (α_O, β_O) can be obtained when a guidance forecast is probabilistic but not when a guidance is deterministic (i.e., specifies only one estimate of W_O). An estimate of κ^2 poses a roadblock because it is not provided by any guidance currently available.

It is worthwhile, therefore, to consider an approximation wherein m and n are held fixed across all forecasting occasions and only π_O/π_A varies. To test the potential of such an approximation, $m = 1$ has been fixed, whereas n has been estimated for each station and month. Such a stratification is consistent with the dependence of n on τ^2 and κ^2, each of which varies with storm type and hence season. Next, the estimate ν of n, together with the climatic value of π_O/π_A for each station and month, have been inserted into (16) to rescale climatic exceedance fractiles ω_O for $p = 0.75$, 0.50, 0.25. The resultant estimates

$$\hat{\omega}_A = (\pi_O/\pi_A)\omega_O^{\nu} \qquad (17)$$

are compared with the climatic exceedance fractiles ω_A in Table 3. The maximum error of 10% and the average error of just 3.3% suggest that formula (17) alone may offer an approximation suitable for operational application. (This conclusion notwithstanding, the problem of quantifying κ^2 operationally is solved in section 6.)

TABLE 4. Expected fractions defining the expected temporal disaggregation of the point amounts and the spatially averaged amount; Monongahela basin, climatic data, 1943–93.

Month	Expected fraction	Point* z_{Oi} Cn	Cf	Sd	Area z_{Ai}
Mar	z_1	0.32	0.29	0.29	0.30
	z_2	0.18	0.21	0.20	0.22
	z_3	0.21	0.20	0.19	0.20
	z_4	0.29	0.30	0.32	0.28
Jul	z_1	0.23	0.24	0.24	0.22
	z_2	0.33	0.32	0.30	0.36
	z_3	0.23	0.21	0.22	0.21
	z_4	0.21	0.23	0.24	0.21

* Cn—Connellsville, Cf—Confluence, Sd—Sines Deep Creek.

5. Rescaling expected fractions

Daily precipitation exhibits the conditional disaggregative invariance—a property demonstrated empirically for point amounts and spatially averaged amounts (Krzysztofowicz and Pomroy 1997). Operational forecasting takes advantage of this property by assuming that on any forecasting occasion, the vector of fractions $(\Theta_1, \ldots, \Theta_n)$ is stochastically independent of the total amount W. This implies, inter alia, that for every $i \in \{1, \ldots, n\}$,

$$E(\Theta_i | W > 0) = \frac{E(W_i | W > 0)}{E(W | W > 0)}.$$

Therefore, the expected fraction can be expressed as follows:

$$z_i = \frac{E(W_i | W > 0)}{E(W | W > 0)} \frac{P(W > 0)}{P(W > 0)} = \frac{E(W_i)}{E(W)}.$$

Next assume that within the area of averaging the field of point precipitation amounts in subperiod i is covariance stationary; thus $E(W_{Oi})$ is identical at every point. The immediate implication is that $E(W_{Ai}) = E(W_{Oi})$ for each subperiod $i = 1, \ldots, n$. This parallels the equality $E(W_A) = E(W_O)$ for the period assumed earlier. Consequently,

$$z_{Ai} = \frac{E(W_{Ai})}{E(W_A)} = \frac{E(W_{Oi})}{E(W_O)} = z_{Oi}, \qquad i = 1, \ldots, n. \qquad (18)$$

The vector of expected fractions $(z_{A1}, \ldots, z_{An})$ defines the expected temporal disaggregation of spatially averaged amount W_A, whereas the vector of expected fractions $(z_{O1}, \ldots, z_{On})$ defines the expected temporal disaggregation of point amount W_O. Equation (18) states that the expected temporal disaggregation is scale invariant.

To illustrate the property, Table 4 compares climatic estimates of expected fractions $z_{Oi} = E(\Theta_{Oi} | W_O > 0)$

of point amounts with climatic estimates of expected fractions $z_{Ai} = E(\Theta_{Ai}|W_A > 0)$ of spatially averaged amounts. These estimates have been purposely calculated as sample means of fractions, not as ratios of sample means of amounts, in order to examine the more general hypothesis $E(\Theta_{Ai}|W_A > 0) = E(\Theta_{Oi}|W_O > 0)$ for $i = 1, 2, 3, 4$. Data in Table 4 suggest that the hypothesis is plausible and offers an approximation sufficient for the purpose of operational forecasting. Moreover, no elevation effect can be discerned. It appears that, at least within the Monongahela basin, a field of the expected fraction is more homogeneous than either a field of the precipitation probability or a field of the mean amount.

6. Quantifying variance reduction factor

a. Theoretical relations

It is difficult to imagine how a forecaster could judge a parameter as abstract as the variance reduction factor κ^2. Therefore, a measurement model is needed whereby κ^2 is related to another parameter that admits an intuitive interpretation.

A simple model of the precipitation field (e.g., Rodriguez-Iturbe and Mejia 1974; Wood and Hebson 1986) assumes that, conditional on precipitation occurrence at every point within the area of averaging, the correlation coefficient $\rho(d)$ between two point amounts separated by distance d is isotropic (within the field) and exponential (with distance):

$$\rho(d) = \exp(-d/\lambda). \tag{19}$$

Function ρ is the *spatial correlogram,* and parameter $\lambda > 0$ is the spatial correlation length that characterizes the random field.

A relation between λ and κ^2 was estimated by Sivapalan and Blöschl (1998) under the assumption that the averaging area A is square. The relation was reported as a plot of κ^2 versus dimensionless argument A/λ^2. We have fitted a parametric function to this plot, which takes the form

$$\kappa^2 = [1 + a(A/\lambda^2)^b]^{-4}, \tag{20}$$

where $a > 0$ and $b > 0$ are parameters with estimates $\hat{a} = 0.134$ and $\hat{b} = 0.484$. A simple transformation of (20) can reveal that $\sqrt{\kappa}$ is a logistic function of $\ln(\lambda/\sqrt{A})$.

(One may note that a square area is assumed to rescale the conditional variance of the amount, and a circular area is assumed to rescale the probability of precipitation occurrence. While this introduces an aesthetically displeasing inconsistency, I do not suppose it will affect the calibration of operational PQPFs in any significant measure.)

b. Coverage uncertainty

In a forecasting situation with $\pi_O < 1$, the wetted area D is uncertain, as discussed in sections 3b and 4a. Consequently, the conditioning of κ^2 in (20) on the occurrence of precipitation at every point of A is not satisfied. An approximate way of modeling this situation is to replace A with an estimate of D. Inasmuch as $D = \Phi A$, one possible estimate is the mean of D, conditional on the hypothesis that precipitation occurs within the area: $E(D|W_A > 0) = E(\Phi|W_A > 0)A$. With the conditional mean of coverage fraction Φ specified by (2), one finds $E(D|W_A > 0) = (\pi_O/\pi_A)A$. Using this estimate in lieu of A in (20) yields

$$\kappa^2 = \left[1 + a\left(\frac{\pi_O}{\pi_A}\frac{A}{\lambda^2}\right)^b\right]^{-4}. \tag{21}$$

In effect, κ^2 is now conditioned on event $W_A > 0$, which is consistent with conditioning the variance of amount W_D on event $W_A > 0$ in (7).

c. Pattern uncertainty

The statistical definition of the correlation coefficient, which serves well for estimating $\rho(d)$ from climatic data, is not meaningful for quantifying $\rho(d)$ by a forecaster. In the context of forecasting, for which subjective probability is the appropriate theory, the correlation coefficient should be interpreted as a *measure of uncertainty about the precipitation pattern.*

For a realization of the precipitation field, its pattern can be represented in terms of isopleths. If a clairvoyant supplied isopleths (drawn at fixed fractions of the maximum point amount), but without the amounts attached, then only uncertainty about the amounts would remain. The forecaster could quantify this uncertainty by assessing a distribution of the amount W_O at a point on any isopleth. A realization of the field could next be obtained by generating a realization of W_O from the assessed distribution and scaling the remaining isopleths. Across all probable realizations, the ratio of amounts at two fixed points would be constant. Hence, the amounts would be perfectly positively correlated, $\rho(d) = 1$. At the other extreme, if the forecaster were totally ignorant about the pattern (to the point of not knowing how a realization of the isopleths can look), then in his judgment any realization of point amounts would be probable. To generate all such realizations, the amounts at any two points would have to be uncorrelated, $\rho(d) = 0$. In reality, the forecaster is neither clairvoyant nor ignorant. At the minimum, he is familiar with precipitation patterns that are climatically probable. Hence, a climatic estimate of $\rho(d)$ constitutes a reference point (which may vary with storm type, season, and possibly other climatic predictors of pattern uncertainty).

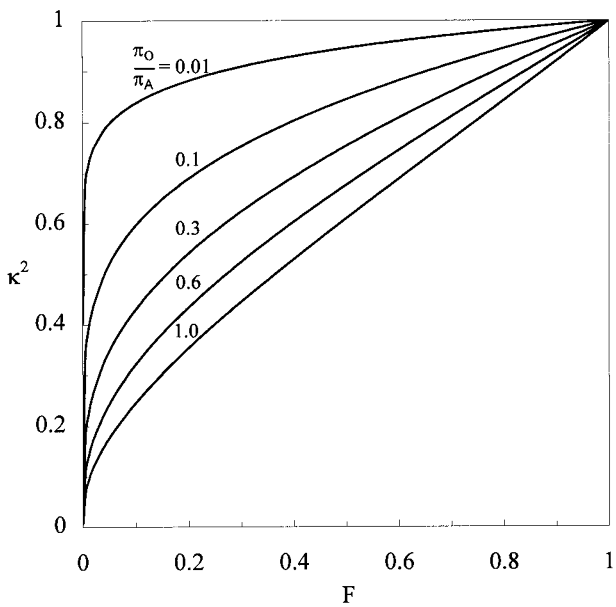

FIG. 4. Relation between the pattern certainty factor F, to be judged by the forecaster, and the variance reduction factor κ^2, for several values of the ratio of the point probability to the area probability π_O/π_A.

d. Measurement model

With the exponential correlogram ρ, the limiting cases may be characterized in terms of parameter λ. When the precipitation pattern is certain, we let $\lambda \to \infty$; hence $\rho(d) \to 1$ and $\kappa^2 \to 1$. When the precipitation pattern is totally uncertain, we let $\lambda \to 0$; hence $\rho(d) \to 0$ and $\kappa^2 \to 0$. However, λ is not suitable for quantifying and communicating the forecaster's uncertainty. First, it is not an intuitive measure of uncertainty because it has a unit of distance (e.g., kilometers or miles). Second, it is unbounded from above, which makes it difficult to judge the degree of uncertainty relative to the limiting cases.

A judgmental measure of uncertainty is constructed as follows. Define a *pattern certainty factor, F*, with range $0 < F < 1$, lower bound $F = 0$ if the pattern is totally uncertain, and upper bound $F = 1$ if the pattern is certain. The pattern in question is the pattern of point precipitation over the averaging area A. (It is, in fact, the uncertainty about this pattern that is the primary reason for forecasting the spatially averaged amount rather than the point amount.) To connect F with λ, two scaling conditions are imposed: (i) the area of averaging is square (consistent with an earlier assumption), and (ii) the pattern certainty factor equals the correlation coefficient between the amount at the center of the area and the amount at a most distant point of the area. In a square, such maximum distance is half of the diagonal, $\sqrt{A}/\sqrt{2}$. The second scaling condition states that $F = \rho(\sqrt{A}/\sqrt{2})$. When (19) is substituted for ρ, the following relation results:

TABLE 5. Variance reduction factor, inferred spatial correlation length, and inferred pattern certainty factor; Monongahela basin, climatic data, 1943–93. Note: $A = 3429$ km²; λ is in km.

		Point*		
Month	Parameter	Cn	Cf	Sd
Mar	κ^2	0.724	0.589	0.514
	λ	72.88	46.00	39.00
	F	0.57	0.41	0.35
Jul	κ^2	0.573	0.509	0.506
	λ	38.47	31.95	33.06
	F	0.34	0.27	0.29

* Cn—Connellsville, Cf—Confluence, Sd—Sines Deep Creek.

$$\lambda = -\frac{\sqrt{A}}{\sqrt{2}\,\ln F}. \tag{22}$$

After inserting (22) into (19), one obtains

$$\rho(d) = F^{d\sqrt{2}/\sqrt{A}}. \tag{23}$$

In effect, the correlogram has been reparameterized by supplanting λ with F and A. Finally, (21) and (22) yield

$$\kappa^2 = \left\{1 + a\left[2\frac{\pi_O}{\pi_A}(\ln F)^2\right]^b\right\}^{-4}. \tag{24}$$

This relation is plotted in Fig. 4. The plot shows that the variance reduction factor κ^2 increases with the pattern certainty factor F, and decreases with the ratio of the point probability to the area probability π_O/π_A. For instance, the same value $\kappa^2 = 0.69$ results in two distinct situations: (i) when the forecaster is relatively uncertain about the pattern, $F = 0.2$, and precipitation is spotty, $\pi_O/\pi_A = 0.1$, and (ii) when the forecaster is considerably more certain about the pattern, $F = 0.6$, and precipitation is widespread, $\pi_O/\pi_A = 1$. Inasmuch as π_O/π_A depends upon the cell/area quotient Q, the variance reduction factor κ^2 is affected by both parameters, Q and F.

Given the climatic estimates of κ^2 in Table 2, climatic estimates of λ and F may be inferred via (21) and (24), respectively. Their values are listed in Table 5. The pattern certainty factor F is higher in March than in July at each point, and decreases with the elevation of a point, decisively in March but only somewhat in July.

e. Judgmental quantification

To quantify the pattern certainty factor F on a particular occasion and for a particular area A, the forecaster should ponder the question, ''How certain am I about the pattern of point precipitation over the area?'' The forecaster's task is to express his degree of certainty on a scale between $F = 0$ (totally uncertain) and $F = 1$ (certain). A reference point F_c, which is the climatic value of F (say, for a day within a season), should be located on the scale to facilitate calibration of judgment. When the forecaster judges the predictability of the precipitation pattern to be higher on this particular day than on an average rainy day of the season, his F should be higher than F_c. When the forecaster judges the predictability of the precipitation pattern to be lower on this particular day than on an average rainy day of the season, his F should be lower than F_c. Otherwise, he should set F equal to F_c. As a surrogate for predictability, the forecaster may judge the difficulty of predicting the pattern on a particular occasion *relative* to the average difficulty, which he knows from experience.

Judgment of F is also *relative* to area A. This can be inferred from (22). For a fixed λ, factor F decreases with area A. In other words, the degree of certainty about the pattern of point precipitation increases as the area for which the pattern must be predicted decreases. Assuredly, the easiest isopleth to predict is that for a point. The relativity of F with respect to A is desirable, from a cognitive point of view, because the forecaster's mind naturally focuses on the area of averaging.

In summary, quantification of F requires a judgment on predictability of the precipitation pattern. Such judgment is a part of the methodology used by forecasters who prepare PQPFs operationally (Krzysztofowicz et al. 1993). Thus the only novelty will be the quantification task.

7. Conclusions

A procedure has been formulated for rescaling a PQPF of the point amount to a PQPF of the spatially averaged amount, or vice versa. The procedure is intended to meet operational needs for (i) rescaling a guidance PQPF for points to a guidance PQPF for a nominal area, and (ii) rescaling a PQPF prepared by the forecaster for a nominal area to a PQPF for points.

To prevent a degradation of informativeness of the PQPF due to rescaling, theoretically based scaling equations must incorporate predictive parameters, which characterize the precipitation field being forecast, as opposed to climatic parameters, which characterize a sample of precipitation fields observed in the past. Two such predictive parameters are the cell/area quotient, Q, and the pattern certainty factor, F. When both Q and F are quantified by the forecaster, they are sufficient for rescaling all elements of the PQPF. When only Q is quantified, one other parameter must be estimated from climatic data (it may be the variance reduction factor κ^2, or the exponent ν in the approximate formula that rescales the conditional exceedance fractiles). The quantification of each predictive parameter appears to lie well within the scope of judgmental tasks routinely performed by forecasters.

The assumption that the uncertainty about the precipitation field being forecast is spatially homogeneous leads to compact scaling equations. The validity of this assumption is obviously limited, as demonstrated via examples. To stay within the limits of validity, at least approximately, the nominal area of averaging should be chosen so that any *local hydrometeorological influences,* such as elevation effect or lake effect, are uniform throughout the area. To establish the robustness of the rescaling procedure, further tests should be conducted on data from other river basins.

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APPENDIX

Variance of Coverage Fraction

a. Conditional variance

Recall that $E(\Phi\,|\,W_A > 0) = \pi_O/\pi_A$ and denote $s = E(\Phi^2\,|\,W_A > 0)$. Next rearrange the expression defining the conditional variance as follows:

$$\begin{aligned}\mathrm{var}(\Phi\,|\,W_A > 0) &= E(\Phi^2\,|\,W_A > 0) - E^2(\Phi\,|\,W_A > 0)\\ &= s - (\pi_O/\pi_A)^2 = \frac{\pi_O}{\pi_A}\left(\frac{\pi_A}{\pi_O}s - \frac{\pi_O}{\pi_A}\right)\\ &= \frac{\pi_O}{\pi_A}\left(1 - \frac{\pi_O}{\pi_A}\right)\tau^2, \qquad \text{(A1)}\end{aligned}$$

where

$$\tau^2 = \left(\frac{\pi_A}{\pi_O}s - \frac{\pi_O}{\pi_A}\right)\bigg/\left(1 - \frac{\pi_O}{\pi_A}\right). \qquad \text{(A2)}$$

b. Bounds

The unknown quantity is s. Its bounds are

$$\left(\frac{\pi_O}{\pi_A}\right)^2 \le E(\Phi^2\,|\,W_A > 0) \le \frac{\pi_O}{\pi_A}. \qquad \text{(A3)}$$

The lower bound is implied by the nonnegativity of variance, $\mathrm{var}(\Phi\,|\,W_A > 0) \ge 0$. To find the upper bound, suppose that at all points within area A, indicators I_O are equal either to zero or to one; in other words, the correlation between indicators I_O at any two points is one. In such a case, either $\Phi = 1$ or $\Phi = 0$. Consequently $\Phi = I_O$ and $\mathrm{var}(\Phi\,|\,W_A > 0) = \mathrm{var}(I_O\,|\,W_A > 0) = \pi_O/\pi_A - (\pi_O/\pi_A)^2$, which implies $E(\Phi^2\,|\,W_A > 0) = \pi_O/\pi_A$. The bounds of s imply that $0 \le \tau^2 \le 1$.

c. Approximation

Toward finding s, recall that $Q = C/A$ and observe that $\Phi = D/A = Q(D/C)$. Hence,

$$E(\Phi \mid W_A > 0) = QE\left(\frac{D}{C}\middle| W_A > 0\right) = \frac{\pi_O}{\pi_A}.$$

Analogously, $\Phi^2 = \Phi Q(D/C)$ and

$$E(\Phi^2 \mid W_A > 0) = QE\left(\Phi\frac{D}{C}\middle| W_A > 0\right),$$

which forms a basis for a two-step approximation to s. In the first step, we let

$$\begin{aligned} s &\simeq QE(\Phi \mid W_A > 0)E\left(\frac{D}{C}\middle| W_A > 0\right) \\ &= Q\frac{\pi_O}{\pi_A}E\left(\frac{D}{C}\middle| W_A > 0\right) = QE\left(\frac{\pi_O}{\pi_A}\frac{D}{C}\middle| W_A > 0\right) \\ &= \frac{\pi_O}{\pi_A}Q_B E\left(\frac{D}{C_B}\middle| W_A > 0\right) = \frac{\pi_O}{\pi_A}\frac{\pi_O}{\pi_B}, \end{aligned} \qquad \text{(A4)}$$

where $C_B = C/(\pi_O/\pi_A)$, $Q_B = C_B/A = Q/(\pi_O/\pi_A)$, and π_B is the area probability calculated from the point probability π_O and the cell/area quotient Q_B. When (A4) is inserted into (A2), one obtains (4).

The approximation in the first line of (A4) consists of replacing the expectation of a product of variates with a product of the expectations of individual variates. This leads to a certain linearization of the expectation of Φ^2 with respect to Q. In effect, $E(\Phi^2 \mid W_A > 0)$ is overvalued for small Q and undervalued for large Q. This effect is compensated for in the second step, whereby Q_B is redefined as

$$Q_B = [Q/(\pi_O/\pi_A)]^c, \qquad \text{(A5)}$$

with parameter $c > 1$ to be estimated empirically. When (A5) is inserted into (1), one obtains (5).

Finally, one must verify that the approximation to s given by (A4)–(A5) lies within the bounds specified by (A3). This is the case provided $\pi_O < \pi_B < \pi_A$. The left inequality is assured by (5). The right inequality is satisfied if $Q < [Q/(\pi_O/\pi_A)]^c$. When $c = 1$, this holds always because $\pi_O/\pi_A < 1$. When $c > 1$ and $Q \geq 1$, this holds always. When $c > 1$ and $Q < 1$, the inequality is violated for Q sufficiently close to 0 (and π_A sufficiently close to 1). But the convergence of Q to zero also implies that precipitation becomes spotty and hence the indicators I_O at any two points become uncorrelated. In effect, $\mathrm{var}(\Phi \mid W_A > 0)$ converges to 0, while s converges to its lower bound. This suggests that whenever $\pi_B \geq \pi_A$, approximately $\tau^2 = 0$.

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