NOTES AND CORRESPONDENCE

A Note on the Idealized Radar Range Equation¹

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1. Introduction

An idealized radar range equation is considered here. The complete solution for a simple case is derived and compared to the narrow-beam approximation given by Probert-Jones (1962).

Earth surface waves and reflections are not considered. The decrease in signal strength by attenuation is the only propagation effect included. Multiple scattering is neglected. The distribution of scatterers is to be random and the signal is presumed to be suitably averaged. The radiation is assumed to be emitted by a paraboloidal antenna of circular cross section into the half-space containing the target in a single lobe whose normalized polar diagram may be represented by a Gaussian distribution of elliptical cross section.

FIG. 1. Spherical coordinate system.

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2. Derivation of the range equation

Let a spherical coordinate system with origin at the antenna be introduced as indicated in Fig. 1. The point P has coordinates (R, θ, ϕ) where R is radial distance, θ is polar angle and ϕ is the azimuthal angle in the plane normal to the direction of propagation. The increment of power received by a transmitting-receiving antenna from a collection of meteorological targets at (R, θ, ϕ) is

$$dP_r = \frac{P_t G_t(\theta, \phi) A_e(\theta, \phi) \kappa(\theta, \phi)}{(4\pi R^2)^2} \eta(\theta, \phi, R) dV. \quad (1)$$

Here P_t is the transmitted power, G_t the transmitter gain function, A_e the effective aperture of the antenna, κ the attenuation factor and η the radar cross section per unit volume. The increment of volume dV is $R^2 \sin\theta d\theta d\phi dR$.

Presuming a lossless antenna and a proper impedance match in the receiving system, the effective area is related to the gain function by

$$A_e(\theta, \phi) = \frac{G_t(\theta, \phi) \lambda^2}{4\pi}, \quad (2)$$

where λ is wavelength. Then, for a rectangular pulse shape the total average power is

$$P_r = \frac{P_t \lambda^2 \bar{h} \eta \kappa}{2R(R+h/2)(4\pi)^3} \int_0^{2\pi} \int_0^{\pi/2} G_t^2(\theta, \phi) \sin\theta d\theta d\phi, \quad (3)$$

where h is the pulse length and the bar denotes a mean-value point evaluation.

The gain function considered here is

$$G_t(\theta, \phi) = G_0 \exp \left\{ -4 \ln 2 \theta^2 \left(\frac{\cos^2 \phi}{\Phi_0^2} + \frac{\sin^2 \phi}{\Phi_1^2} \right) \right\}, \quad (4)$$

where Φ_0 and Φ_1 are the (total) half-power beam widths,

$G_0 = G_t(0, \phi)$ and $\ln$ denotes the natural logarithm. The integral in Eq. (3) can be transformed and written as

$$2G_0^2 \int_0^{\pi/2} \exp\{-b\theta^2\} \sin\theta \int_0^\pi \exp\{-a\theta^2 \cos x\} dx d\theta,$$

where x is a dummy variable of integration,

$$b = 4\ln 2 \left(\frac{1}{\Phi_0^2} + \frac{1}{\Phi_1^2} \right), \quad (5)$$

and

$$a = 4\ln 2 \left(\frac{1}{\Phi_0^2} - \frac{1}{\Phi_1^2} \right). \quad (6)$$

Since

$$\int_0^\pi \exp\{-a\theta^2 \cos x\} dx = \pi I_0(a\theta^2), \quad (7)$$

where I_0 is the modified (hyperbolic) Bessel function of the first kind and order zero, the power received is

$$P_r = \frac{P_t \lambda^2 h \eta \kappa G_0^2}{64\pi^2 R(R+h/2)} \int_0^{\pi/2} I_0(a\theta^2) \sin\theta \exp(-b\theta^2) d\theta. \quad (8)$$

It will be shown that the value of the integral over an unbounded interval is of the order of $1/(2b)$. The asymptotic behavior of I_0 for positive a may be introduced to observe the inequality

$$\left| \int_{\pi/2}^\infty I_0(a\theta^2) \sin\theta \exp(-b\theta^2) d\theta \right| < \frac{1}{2\sqrt{2a(b-a)}} \operatorname{erfc}\left(\frac{\pi}{2}\sqrt{b-a}\right). \quad (9)$$

When $a=0$, $I_0(0)=1$ and the pertinent inequality becomes

$$\left| \int_{\pi/2}^\infty \sin\theta \exp(-b\theta^2) d\theta \right| < \frac{1}{2b} \exp\left\{-b\left(\frac{\pi}{2}\right)^2\right\}. \quad (10)$$

Comparison with $1/(2b)$ for typical values of a and b indicates that the relative contribution of the extended range of integration is substantially below the error due to neglecting side lobes, as indicated by the result of Probert-Jones (1962).

The integral in Eq. (8) is evaluated by expanding $I_0(a\theta^2)$ and using the integral formula given by Gröbner

and Hofreiter (1958), i.e.,

$$\int_0^\infty I_0(a\theta^2) \sin\theta \exp(-b\theta^2) d\theta = \frac{1}{2b} \sum_{j=0}^\infty \left(\frac{a}{2b}\right)^{2j} \frac{\Gamma(2j+1)}{(j!)^2} F\left(2j+1, \frac{3}{2}; -\frac{1}{4b}\right). \quad (11)$$

Here F is the confluent hypergeometric function

$$F\left(2j+1, \frac{3}{2}; -\frac{1}{4b}\right) = \sum_{\nu=0}^\infty \frac{\Gamma(3/2)\Gamma(2j+1+\nu)}{\nu!\Gamma(3/2+\nu)\Gamma(2j+1)} \left(-\frac{1}{4b}\right)^\nu. \quad (12)$$

The power received becomes

$$P_r = \frac{P_t \lambda^2 h \eta \kappa G_0^2}{128\pi^2 R(R+h/2)b} \sum_{j=0}^\infty \left(\frac{a}{2b}\right)^{2j} \frac{1}{j!j!} \times \sum_{\nu=0}^\infty \frac{\Gamma(3/2)\Gamma(2j+1+\nu)}{\nu!\Gamma(\nu+3/2)} \left(-\frac{1}{4b}\right)^\nu. \quad (13)$$

3. Comparison with previous results

When the beam cross section is circular, $a=0$ and the range equation becomes

$$P_r = \frac{P_t \lambda^2 h \eta \kappa G_0^2 \Phi_0^2}{1024 \ln 2 \pi^2 R(R+h/2)} \times \left[1 - \frac{\Phi_0^2}{48 \ln 2} + \frac{\Phi_0^2}{3840 (\ln 2)^2} - \dots \right]. \quad (14)$$

Since $\Phi_0 \ll 1$ for all applications of interest, only the leading term of unity need be retained. This agrees with the result obtained by Probert-Jones.

When the cross section is elliptical, say $\Phi_1^2 = k\Phi_0^2$, the range equation can be written as

$$P_r = \left\{ \frac{P_t \lambda^2 h \eta \kappa G_0^2 \Phi_0 \Phi_1}{1024 \ln 2 \pi^2 R(R+h/2)} \right\} K, \quad (15)$$

where

$$K = \frac{2\sqrt{k}}{k+1} \sum_{j=0}^\infty \frac{1}{j!j!} \left[\frac{k-1}{2(k+1)} \right]^{2j} \times \sum_{\nu=0}^\infty \frac{\Gamma(3/2)\Gamma(2j+\nu+1)}{\nu!\Gamma(\nu+3/2)} \left[-\frac{k\Phi_0^2}{16(k+1) \ln 2} \right]^\nu. \quad (16)$$

Here K is the factor by which the present result differs from that given by Probert-Jones. It is insensitive to variations in Φ_0 up to at least 10° . It is unity to four significant figures for $k \leq 8$; for beam width ratios ($k^{\frac{1}{2}}$) of 3, 4 and 5 the factor K is 0.9998, 0.9951 and 0.9754, respectively. The correction is -0.5 and -1.0 db for beam width ratios of 7 and 8.88, respectively.

The small discrepancy can be explained by defining new angular coordinates γ and ψ through the transformation

$$\left. \begin{aligned} \gamma &= \theta \cos \phi \\ \psi &= \theta \sin \phi \end{aligned} \right\} \quad (17)$$

When these are introduced in the gain function, the form used by Probert-Jones is obtained. However, in transforming the integral of the gain function, the product $\sin(\sqrt{\gamma^2 + \psi^2})J(\theta, \phi/\gamma, \psi)$ has been taken as unity. Since the Jacobian is $(\gamma^2 + \psi^2)^{-\frac{1}{2}}$, his approximation is good, in general, only for small angle, ellipticity and an

appropriate region of integration. However, the nature of the gain function here permits the approximation.

4. Conclusions

For narrow, nearly circular, beams the Probert-Jones (1962) correction to the radar equation is quite adequate. The approximation is good to within -0.1 db for a beam width ratio as large as five. An additional correction as large as one decibel is necessary only when beam width ratio equals or exceeds 8.88.

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