# Numerical Analysis of Atmospheric Soundings

JOHN D. STACKPOLE

*National Meteorological Center, Weather Bureau, ESSA, Washington, D. C.*

(Manuscript received 30 December 1966, in revised form 8 February 1967)

ABSTRACT

Numerical methods of calculating the pseudo-adiabatic characteristics of saturated air parcels are presented, based on the Rossby definition of the pseudo-equivalent potential temperature. With these methods it is then possible to perform routine automatic analysis of soundings. As examples, techniques for determining the lifted condensation level, the level of free convection, and the convective condensation level are presented.

## 1. Introduction

Many atmospheric phenomena are of interest to the synoptic meteorologist concerned with the weather elements of greatest direct impact on the public, i.e., cloudiness, precipitation, thunderstorms, tornadoes and the like. One of the more important of the phenomena is the stability characteristics of soundings as measured by rising moist parcels. A number of measures of the stability have been devised, perhaps the best known of which is the justly honored Showalter Index (Showalter, 1953). This index, involving as it does only the parcel characteristics at 850 mb and the temperature at 500 mb, has been criticized as being occasionally unrepresentative when the atmospheric structure is complex between 850 mb and 500 mb or the 850-mb moisture is unrepresentative. The difficulty, however, with devising a stability measure of more general application for routine use is that such a measure, if current practices were followed, would have to be computed either from tables or graphically, which could become an onerous task. The graphs or charts from which these indices are usually computed are the familiar Stüve pseudo-adiabatic charts and variations thereon (Skew $T-\log p$, etc.), the pseudo-adiabatic portions of which are drawn from Table 78 of the Smithsonian Meteorological Tables: "Temperature and Pressure Along Saturation Pseudo-Adiabats" (List, 1958).

In essence what the charts or tables allow the analyst to do is:

1) Given a particular pseudo-adiabat identified by either its equivalent potential temperature or pseudo-wet bulb potential temperature and given a particular pressure to find the Celsius temperature of the parcel, or
2) Given a pseudo-adiabat and temperature, to find the associated pressure, or
3) Given temperature and pressure and assuming saturated conditions to arrive at the associated pseudo-adiabat.

This enables the analyst to move up and down the pseudo-adiabat with the parcel and to "step off," so to speak, wherever he wishes to make comparisons between the parcel and the environment. From these comparisons, the stability indices and energetical characteristics of a given sounding are derived. These steps are straightforward but sufficiently involved, particularly when detailed study of a sounding is contemplated, to preclude routine calculations by hand.

With the availability of high-speed computers, it would seem that these difficulties can be for the most part overcome once satisfactory mathematical formulae for "getting on" and "getting off" the pseudo-adiabats can be found. To such formulae we shall now turn our attention.

## 2. Calculation of pseudo-adiabatic lapse rates

For the identification of pseudo-adiabats we shall employ the Rossby definition (Rossby, 1932) of the pseudo-equivalent potential temperature, i.e.,

$$\theta_{SE}=\theta_d \exp(Lm_s/C_pT), \tag{1}$$

where the symbols, and others to follow, have these meanings:

$t$ = temperature (°C),
$T$ = absolute temperature (°A) $= t+273.16$,
$p$ = pressure (mb),
$C_p$ = specific heat of air at constant pressure = 0.24 cal gm$^{-1}$ (°C)$^{-1}$,
$e_s$ = saturation vapor pressure (mb),
$L$ = latent heat of vaporization (cal gm$^{-1}$),
$m_s$ = saturation mixing ratio $=0.622\,[e_s/(p-e_s)]$,
$\theta_d$ = partial potential temperature (°A) $=T[1000/(p-e_s)]^{2/7}$.

This definition of pseudo-equivalent temperature differs from the implicit definition of the Smithsonian Tables only in the neglect of the effect of the specific heat of water times the mixing ratio. For the present purposes (i.e., the numerical emulation of pseudo-adiabatic chart calculations) the saturation vapor pressure and latent heat of vaporization are very well approximated by the formula

$$e_s = 6.11 \times 10^{at/(t+b)}, \quad (2)$$

due to Tetens (1930), where $a=7.5$ and $b=237.3$, and

$$L = 596.73 - 0.601t, \quad (3)$$

from the Smithsonian Tables.

There would have been no difficulty in using the Goff–Gratch expression for $e_s$ from the Smithsonian Tables (Table No. 94) or Bosen's (1960) empirical expression for the same (both are perfectly straightforward computations), but the results for tests using these indicate that the computational sophistication and additional time involved is not necessary.

Thus, given values of temperature and pressure and assuming saturated conditions, there is no difficulty using Eqs. (1), (2) and (3) to calculate the associated $\theta_{SE}$.

However, given $\theta_{SE}$ and a value of either $t$ or $p$, any attempt to calculate $p$ or $t$, respectively, involves somewhat more effort and must be accomplished by approximate numerical methods (which are still accurate to any desired degree). In either case, the method is as follows: (for concreteness let us say we are seeking $p$ given $\theta_{SE}$ and $t$):

Consider the quantity

$$A \equiv \theta_d \exp(Lm_s/C_pT) - \theta_{SE}. \quad (4)$$

Clearly, if we can find a value for $p$ such that $A=0$, this will be the value which we are seeking, provided, of course, that $A$ does not cross zero more than once. This restriction seems to be satisfied well beyond the meteorological range of the variables. From a practical point of view, we will be satisfied with a value for $p$ such that $|A|$ is less than some small number $\epsilon$ where the magnitude of $\epsilon$ is specified in terms of the degree of accuracy required for $p$.

The computational procedure or algorithm is then:

1) Guess at values for $p$ and $\Delta p$.
2) Calculate $A$ from $p$ and test for $|A|<\epsilon$. If $A$ passes this test then $p$ is taken as the desired pressure. If not,
3) Form $p'=p+\Delta p$ and an associated $A'$. Again, test for $|A'|<\epsilon$. If $A'$ passes, then $p'$ is the desired pressure. If not,
4) Compare the signs of $A$ and $A'$. If they are different, i.e., the desired zero crossing of $A$ is between $p$ and $p'$.
5) Divide $\Delta p$ by 2 and return to Step 3.
6) If at Step 4 the signs of $A$ and $A'$ were the same, we compare the absolute values of $A'$ and $A$.
7) If $|A'|<|A|$ we set $p$ equal to the previous value of $p'$, $A$ equal to the previous value of $A'$, and return to Step 3.
8) If $|A'|>|A|$ we change the sign of $\Delta p$ and return to Step 3.
9) If $|A'|=|A|$ the computation cannot continue. This will occur only if the slope of the $A(p)$ vs. $p$ curve never changes sign or becomes zero. The requirement that this does not occur is more restrictive than the one above relating to the zero crossings of $A$ (the latter indeed is a consequence of the former), but does not seem to be violated under even extreme meteorological conditions.

As mentioned above, essentially the same procedure is followed when we are seeking $t$ given $\theta_{SE}$ and $p$ initially. Experience has indicated that initial values of $p=700$ mb, $\Delta p=100$ mb, or $t=0$C, $\Delta t=10$C have given quite satisfactory results with $\epsilon=0.05$.

As a test of the utility and accuracy of this algorithm, a computation was performed which was essentially a re-calculation of Table 78 of the Smithsonian Meteorological Tables using the method described above. The numbers are not identical (one hesitates to say that one table is more correct than the other in view of the physical simplifications that went into both sets of calculations), but they are sufficiently close to one another, differing by at most 6% (and this only for the warmest pseudo-adiabats) that one feels confident that the use of the algorithm will not introduce errors of such magnitude as to warrant concern. The utility is particularly enhanced by noting the computation required only about 30 sec of IBM-7094 computer time to generate.

A companion calculation, with which there is no published table to compare, was also made, which gave the temperature along saturation pseudo-adiabats as a function of pressure. It was possible to make some comparison between the calculated values and the plots of Smithsonian Table 78 found on pseudo-adiabatic charts. Such comparisons indicate the largest readable departures from the plotted values to be no more than 2C and this is only for the warmest pseudo-adiabat at the lowest plotted pressure (100 mb). This is less than 1% of the absolute temperature and may again serve as an indication of the utility of the formulae and algorithm. The calculation required only 20 sec of 7094 computer time.

## 3. Calculation of the lifted condensation level

As is well known, the lifted condensation level is defined as the pressure-temperature point at which a dry-adiabatically rising air parcel of initially specified pressure, temperature, and moisture content (usually specified in terms of the dew point) reaches saturation, i.e., the point at which the mixing ratio of the parcel becomes equal to the saturation mixing ratio given by the initial pressure and dew point of the parcel. On a pseudo-adiabatic chart, this point is the intersection of the line of constant potential temperature $\theta$ drawn through the initial pressure temperature point $(p,t)$ and the line of constant saturation mixing ratio $m_s$ drawn through the parcel's initial pressure-dew point $(p,t_d)$.

In mathematical terms, this intersection point can be thought of as the solution of two equations, one describing the constant $m_s$ line the other describing the constant $\theta$ line, for two unknowns: the pressure $(p_{LCL})$ and temperature $(t_{LCL})$ of the lifted condensation level. As before, this solution can be found by straight-forward numerical procedures.

The known value of the saturation mixing ratio for given $p$ and $t_d$ is given by

$$m_s=0.622\left(\frac{e_s}{p-e_s}\right),$$

where Eq. (2) above is used to determine the saturation vapor pressure for the dew-point temperature and initial pressure. The known value of the potential temperature $\theta$ is given directly by its definition.

The desired quantities $p_{LCL}$ and $t_{LCL}$ are then the two unknowns in the pair of equations

$$m_s=0.622\left(\frac{6.11\times10^D}{p_{LCL}-6.11\times10^D}\right), \tag{5}$$

$$\theta=(t_{LCL}+273.16)(1000/p_{LCL})^{2/7}, \tag{6}$$

where for convenience we have introduced a variable

$$D=\frac{7.5t_{LCL}}{237.3+t_{LCL}}. \tag{7}$$

Some manipulation of (5) and (6) gives

$$p_{LCL}=6.11\times10^D\left(\frac{0.622+m_s}{m_s}\right)\equiv f(D), \tag{8}$$

$$p_{LCL}=1000\left[\frac{D(237.3-273.16)+(273.16)(7.5)}{\theta(7.5-D)}\right]^{3.5}\equiv g(D), \tag{9}$$

where $p_{LCL}$ has been written as two functions of $D$. The solution for $p_{LCL}$ and $D$ [and $t_{LCL}$ directly from (7)] will give the lifted condensation level.

Since as can be easily shown (over a meteorological range of the variables)

$$\left|\frac{\partial f}{\partial D}\right|>\left|\frac{\partial g}{\partial D}\right|,$$

and $f(D)$ can be solved for $D$ as a function of $p$, i.e.,

$$D=\log_{10}\left[\frac{p_{LCL}m_s}{6.11(0.622+m_s)}\right]\equiv F(p_{LCL}), \tag{10}$$

the following numerical method of iteration (Scarborough, 1930) may be employed to find $p_{LCL}$ and $D$ (and $t_{LCL}$):

1) Guess a value of $D=D_1$.
2) Compute a corresponding $p_1$ from (9), i.e., $p_1=g(D_1)$.
3) Compute a $D_2$ from (10), i.e., $D_2=F(p_1)$.
4) Return to Step 2 using $D_2$ in (9) to find a $p_2$.
5) Continue the iterative procedure until the change in $p$ is sufficiently small from one step to the next such step that satisfactory convergence may be said to have been achieved, i.e., $|p_n-p_{n-1}|<\epsilon$, and use the corresponding value of $D_n$ to obtain $t_n$.

The resulting $p_n$ and $t_n$ are then to be identified with the desired $p_{LCL}$ and $t_{LCL}$. Experience indicates that $D_1=0.5$ ($t=18$C) and $\epsilon=1$ mb give satisfactory results.

A series of tests were run using this computational method with results that were indistinguishable from the corresponding graphical calculations.

Since, by definition, $p_{LCL}$ and $t_{LCL}$ are the values of the pressure and temperature that the parcel has at saturation, they serve to define the pseudo-equivalent potential temperature of the then saturated parcel, and, by going down the pseudo-adiabat to 1000 mb and finding the temperature there, the pseudo wet-bulb (potential) temperature (°C).

## 4. Analysis of atmospheric soundings—LFC and CCL

With the mathematical-computational tools described above, namely, the ability to find the lifted condensation level of a parcel and the ability to move freely up and down the associated pseudo-adiabatic lapse rate, analysis of the convective, stability, and energetic characteristics of a given atmospheric sounding can proceed apace, completely automatically.

As an example, we here describe a method of finding two familiar items in the analysis of soundings, the level of free convection (LFC) and the convective condensation level (CCL). For both, the method is essentially the same and to fix attention we will describe the determination of the LFC while indicating the slight differences required for the CCL computation.

Given then a sounding and a parcel of air for which the $\theta_{SE}$ has been determined, we work our way up the

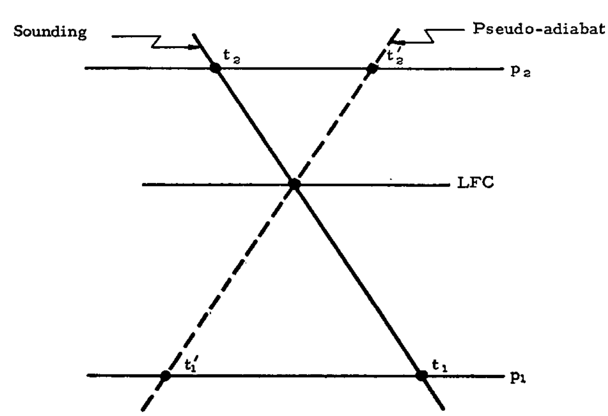


FIG. 1. Determination of LFC (or CCL).

sounding, comparing, at each $(p,t)$ point of the sounding, the sounding temperature and the temperature of the pseudo-adiabat at the same pressure. The latter is found by the methods described in Section 2. This process continues until the temperature difference changes sign from one sounding point to the next. This situation is illustrated in Fig. 1.

In this figure, the solid line represents the sounding, the dashed line the pseudo-adiabat, and $p_1$ and $p_2$ are two neighboring pressure levels for which sounding $(p,t)$ points are measured and at which the differences $t-t'$ have opposite signs. If we define $\Delta = t-t'$ and assume a linear variation of the temperatures between $p_1$ and $p_2$, an exercise in geometry serves to show the pseudo-adiabatic and sounding lines to cross at

$$t_{LFC}=\frac{t_1\times\Delta_2-t_2\times\Delta_1}{\Delta_2-\Delta_1}.$$

The known values of $\theta_{SE}$ and $t_{LFC}$ then are sufficient to determine $p_{LFC}$ with the algorithm of Section 2 and the LFC is thus known. It is perfectly possible (indeed frequent) that a given parcel-sounding pair may have no LFC; such a circumstance presents no particular problem in the machine analysis, the search merely terminates at the end of the sounding and the non-existence of an LFC is appropriately indicated. Further, it should be noted that this procedure will find the first LFC; if, due to an inversion, there are more than one, there is nothing to prevent the process continuing further if the existence and location of higher level LFC's are desired.

If the CCL is sought, the procedure is identical except that the line of constant saturation mixing ratio (defined by the parcel's original pressure and dew point) is substituted for the line of constant $\theta_{SE}$. This calculation proceeds somewhat more rapidly as the equation defining $m_s$ [where $e_s$ is given by (2)] can be solved analytically for $t$ as a function of $p$ and $m_s$ (or $p$ as a function of $t$ and $m_s$) and thus no iterative numerical procedures are needed.

## 5. Other applications

One need not, of course, cease the analysis of a given sounding with the determination of the various parameters discussed above. One is limited only by willingness to write the necessary computer programs.

Another ready application of the methods described is the determination of the "positive" and "negative" areas between the sounding line and the rising parcel line on a pseudo-adiabatic chart. This simply involves the stepwise integration of the hydrostatic equation to obtain the geopotential thickness of a given layer for both temperature lines. The thickness difference between the two is the energy either required or released by the rising parcel which (on a tephigram) is just the area between the lines. Various stability indices can be formulated from the energy information thus obtained. One such index, in use at the National Meteorological Center and elsewhere, involves a comparison of the negative energy required to lift a parcel to its LFC with the positive energy released by the parcel between the LFC and 500 mb. These again are familiar quantities in sounding analysis but their routine calculation by hand has been precluded by the amount of time required.

Another energetic quantity of obvious utility is the energy area measured between the sounding and the dry adiabat drawn from the CCL to the ground. This, as is well known, respresents the amount of energy which must be supplied to the lower atmosphere in order to induce convective overturning from diurnal heating. This too can be readily calculated.

One question that has been left hanging is that of the determination of representative initial pressure and dew-point values for the parcel under consideration. Use of simply the surface values has obvious deficiencies, an average of some sort through a layer of the atmosphere being in order. At the National Meteorological Center, averages are calculated through two such layers, one 100 mb thick, the other 160 mb thick. These layers do not necessarily rest on the ground: the 100-mb layer with the largest pseudo wet-bulb potential temperature within the first 160 mb of the atmosphere is first found and then the average temperature and dew point with the pressure of the center of the layer are taken as the initial conditions of the parcel. The 160 mb thick layer having the highest moisture content within the first 240 mb is taken as defining the characteristics of a second parcel. Condensation level data and various energy based indices are then calculated for each of these "parcels." The utilization of these data in routine forecasting operations will be reported upon elsewhere.

Finally, it should be emphasized that all of the calculations described here are based purely on the parcel method with no account taken of compensatory down drafts, entrainment, or the destabilization of whole layers as is found in frontal overrunning situations. A brief investigation of these effects seems to indicate that there would be no insurmountable difficulties in taking account of them (at least in an approximate form) if such was desirable.

REFERENCES

Bosen, J., 1960: A formula for approximation of the saturation vapor pressure over water. *Mon. Wea. Rev.*, **88**, 275–276.

List, R. J., 1958: *Smithsonian Meteorological Tables.* Sixth Revised Edition, Smithsonian Institution, Washington D. C., 318–322.

Rossby, C. G., 1932: Thermodynamics applied to air mass analysis. *MIT and Woods Hole Oceanographic Institution Papers in Physical Oceanography and Meteorology (Massachusetts Institute of Technology Meteorological Papers)*, **1**, No. 3, 41 pp.

Scarborough, J. B., 1930: *Numerical Mathematical Analysis.* Fourth Edition, 1958, Baltimore, Johns Hopkins Press, 199–203.

Showalter, A. K., 1953: A stability index for thunderstorm forecasting. *Bull. Amer. Meteor. Soc.*, **34**, 250–252.

Tetens, O., 1930: Ôber einige meteorologische Vegriffe. *Z. Geophys.*, **6**, 297–309.