A Sharp Cutoff Spectral Differentiator

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ABSTRACT

Design data and experimental results are presented for an eight-pole optimally flat filter which makes available the first seven derivatives of the filtered input signal.

1. Introduction

The differentiator described in this paper was developed for some current research projects at The Pennsylvania State University. These projects include studies of the small scale structure of turbulence in the atmosphere and in the laboratory. Atmospheric data are obtained from airborne hot-wire anemometers, the turbulent velocity signals being recorded in flight on magnetic tape.

The information about the smaller scales of turbulence occurs at the higher frequencies in the signal, and typically at such low amplitude that it is in danger of being lost in the noise of the recording equipment. High-pass filtering, such as differentiation, is necessary between anemometer and tape recorder to solve this problem. The anemometer signal, however, has its own noise, as shown in a typical power spectral density plot in Fig. 1. The power spectral density of the first derivative signal is also shown in this figure. It can be seen that the noise becomes important in the derivative; the noise is differentiated, too, and will dominate the signal in derivatives of sufficiently high order. This is a problem always associated with differentiation and is discussed in analog computer texts such as Jackson (1960). It is customary to use differentiation circuits which provide low-pass filtered derivatives, therefore, in order to improve the signal-to-noise ratio.

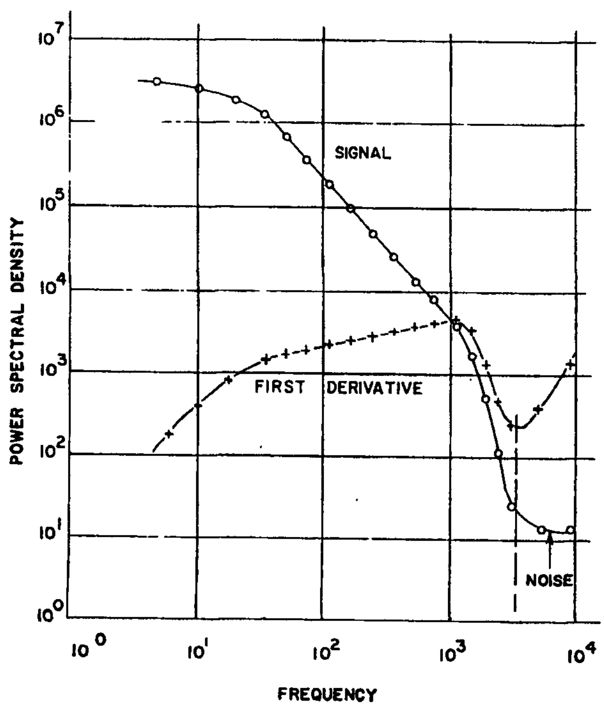

FIG. 1. Typical power spectral densities.

Additionally, one property of the small-scale turbulent motion is often sought—the dissipation, which is considered to play a key role in the universality of turbulent structure. It is common practice to use the isotropic relation, $15\nu\overline{(\partial u/\partial x)^2}$ (where ν is the kinematic viscosity, u the velocity fluctuation in the x direction), to determine the dissipation. Then, Taylor's hypothesis, $\bar{U}^2\overline{(\partial u/\partial x)^2}=\overline{(\partial u/\partial t)^2}$, shows that the mean-square first time derivative of the velocity signal is proportional to dissipation. Thus, a differentiator is commonly used for simple, direct dissipation measurements, particularly in the case of analog data processing.

The device used here to perform these tasks is an eight-pole Butterworth filter. The Butterworth is a particularly useful filter for this application since it has maximally-flat amplitude vs. frequency characteristics. The principles of the Butterworth filter have been discussed in White Electromagnetics, Inc. (1963); as ordinarily realized, however, it cannot be used for differentiation. The somewhat novel realization described here combines differentiation (first seven derivatives) and low-pass filtering (−48 db per octave cutoff). Design data, experimental results and information on the use of the filter are presented here.

2. The circuit

The circuit is shown in Fig. 2. The cutoff frequency is ω_0, and $e_i(t)$ and $e_0(t)$ are the input and output signals. The outputs of the integrators are e_0 and its derivatives, which are denoted by

$$\frac{d^n e_0(t)}{dt^n} = e_0^{(n)}.$$

A simpler circuit also having the Butterworth transfer function and also using operational amplifiers has been given by Hansen (1965). However, the advantage of the present circuit is that the filtered signal derivatives are available.

The maximally-flat amplitude vs. frequency characteristics of the Butterworth filter are due to the pole locations. They are located on the left half of a circle centered at the origin in the imaginary plane, and they are symmetrically located with respect to the real axis. This pole pattern gives the following relations between resistance, capacitance and cutoff frequency values for an eight pole filter:

$\omega_0 R_7 C_7 = 0.195$ $\omega_0 R_4 C_4 = 0.850$ $\omega_0 R_1 C_1 = 2.563$

$\omega_0 R_6 C_6 = 0.390$ $\omega_0 R_3 C_3 = 1.176$ $\omega_0 R_0 C_0 = 5.126$

$\omega_0 R_5 C_5 = 0.601$ $\omega_0 R_2 C_2 = 1.663$

For a detailed discussion of the response characteristics of the Butterworth filter, the reader is referred to the article by Peless and Murakami (1957). They discuss the influence of pole location on amplitude characteristics, transient response and cutoff rate.

3. Use of the filter

A few remarks should be made about the choice of cutoff frequency. To lose the least information it should be chosen at the point, shown in Fig. 1, where the signal spectrum breaks upward as a result of noise. As shown in Fig. 2 the signal level of the mth derivative decreases as ω_0^{-m}, so that, in general, amplification between Butterworth and tape recorder is required; as ω_0 increases, the noise level of the filter itself will become significant in the highest derivatives.

The outputs are proportional to the derivatives of the output signal and the output signal itself. It would be valuable to be able to consider them as derivatives of the input signal; a brief discussion of the validity of this is relevant.

It is not difficult to show that the power spectral densities of the mth derivatives of the input and output signals are equal within a constant factor, below the cutoff frequency. For spectral purposes, therefore, the derivative signals may be considered to be input signal derivatives. The situation for other statistics is not as simple. A Fourier component is attenuated and phase-shifted between input and output; at frequencies well below the cutoff it can be shown that the effect of the phase shift can be neglected, and input and output statistics are the same. Closer to the cutoff, the phase shift becomes important. However, this has no influence on spectra, and, therefore, all even-order moments of any Fourier component are unaffected. If the incoming signal may be regarded as a Gaussian process [not a bad assumption for even order statistics of turbulence signals, according to Frenkiel and Klebanoff (1967)], then all even-order statistics will be unaffected, since Fourier components at different wave numbers are statistically independent for such a signal. Finally, since the system is linear, if odd-order moments of the input vanish, those of the output will vanish also, i.e., the system will not introduce any skewness.

4. An existing unit

It was found that it was necessary to use operational amplifiers having a unity-gain frequency much higher than the filter cutoff frequency. This was thought to be due to the errors in integration at high frequencies due to the finite open loop gain of the amplifiers.

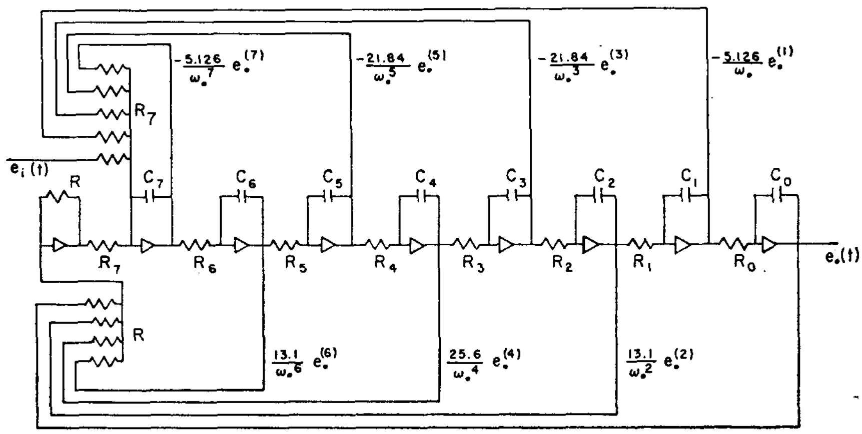

FIG. 2. Circuit diagram for complete filter.

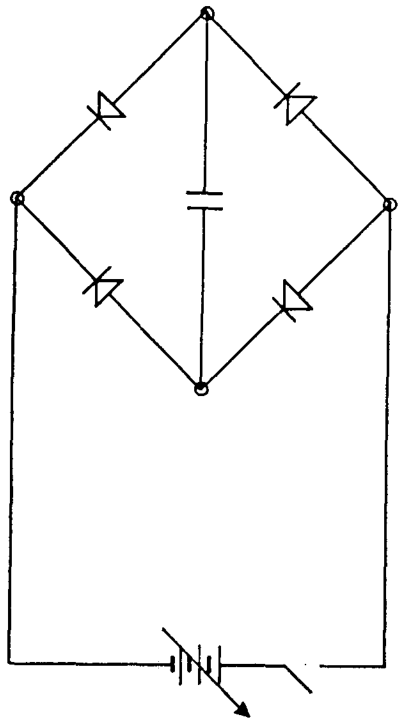

FIG. 3. Diode quad shorting switch.

Philbrick P65AHU operational amplifiers, having a unity gain frequency of 25 MHz, were used in our 20-kHz cutoff frequency unit.

It was also found that a shorting switch across each feed-back capacitor was needed in order to make the filter operate. For low cutoff frequency filters (below say 3kHz), mechanical switches or relays were found to suffice. At the 20-kHz cutoff frequency, however,

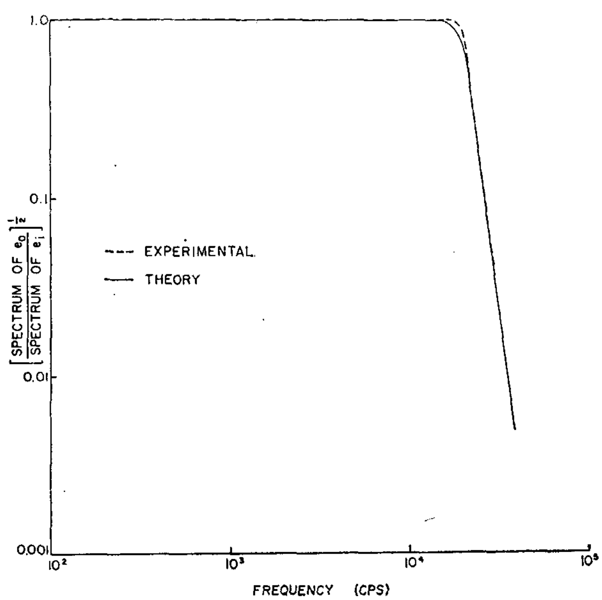

FIG. 4. Root mean square relative filter output vs. frequency.

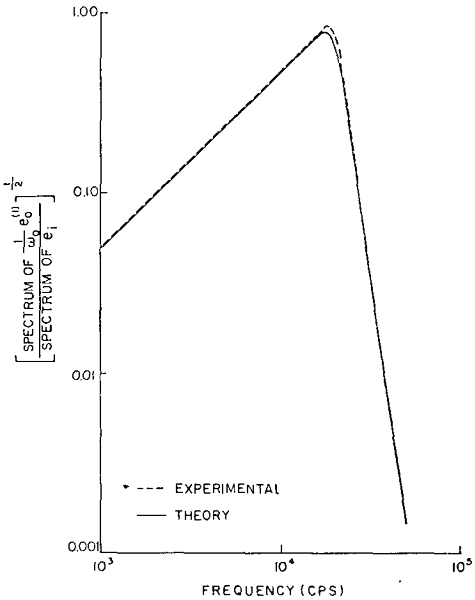

FIG. 5. Root mean square relative output of filtered first derivative vs. frequency.

these were found inadequate and each capacitor was fitted with a diode quad shorting switch, as shown in Fig. 3. These were installed in parallel and the bias varied until the value was found that would allow the filter to start. The filter was started by momentarily closing the single mechanical switch in the bias circuit, which shorts all the feedback capacitors and drives the amplifier output voltages to nearly zero. When the

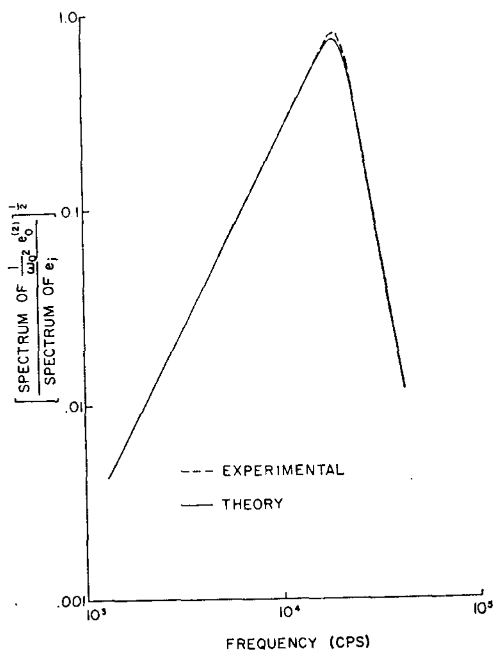

FIG. 6. Root mean square relative output of filtered second derivative vs. frequency.

switch is opened, the diodes effectively simultaneously stop conducting and the filter starts operating.

In Figs. 4–6 are shown amplitude-frequency curves for the output and its first two derivatives, obtained from our unit. The curves predicted by the Butterworth theory are shown for comparison. Response curves for the third and fourth derivatives are not shown here but agree equally well.

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