A High-Precision and Fast Solution Method of Gamma Raindrop Size Distribution Based on 0-Moment and 3-Moment in South China

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ABSTRACT: According to the high-accuracy linear shape–slope ($\mu - \Lambda$) relationship observed by several two-dimensional video disdrometers (2DVD) in South China, a high-precision and fast solution method of the gamma ($\Gamma$) raindrop size distribution (RSD) function based on the zeroth-order moment ($M_0$) and the third-order moment ($M_3$) of RSD has been proposed. The 0-moment $M_0$ and 3-moment $M_3$ of RSD can be easily calculated from rain mass mixing ratio $Q_r$ and total number concentration $N_m$ simulated by the two-moment (2M) microphysical scheme, respectively. Three typical heavy-rainfall processes and all RSD samples observed during 2019 in South China were selected to verify the accuracy of the method. Relative to the current widely used exponential RSD with a fixed shape parameter of zero in the 2M microphysical scheme, the $\Gamma$ RSD function using the linear constrained gamma (C-G) method agreed better with the $\Gamma$-fit RSD from 2DVD observations. The characteristic precipitation parameters (e.g., rain rate, $M_2$, $M_3$, and $M_4$) obtained by the proposed method are generally consistent with the parameters calculated by $\Gamma$-fit RSD from 2DVD observations. The proposed method has effectively solved the problem that the shape parameter in the 2M microphysical scheme is set to a constant, and therefore the $\Gamma$ RSD functions are closer to the observations and have obviously smaller errors. This method has a good potential to be applied to 2M microphysical schemes to improve the simulation of heavy precipitation in South China, but it also paves the way for in-depth applications of radar data in numerical weather prediction models.

KEYWORDS: Precipitation; Drop size distribution; Optimization; Parameterization; Field experiments

1. Introduction

Heavy rainfall occurs frequently in South China and always leads to flooding, urban waterlogging, and mountain torrents. These heavy-precipitation processes seriously threaten the safety of people’s lives and property and often cause significantly economic losses. Accurate forecast of heavy rainfall in South China is essential for government disaster prevention and mitigation.

Numerical weather prediction (NWP) model is the main technical means and tool for modern weather forecast. As computing power continues to increase, the grid spacing of operational NWP models have reached the convective-scale resolution ($\leq 4$ km). At this resolution, the traditional cumulus parameterization scheme is no longer applicable, and the explicit processes of cloud and precipitation must use the microphysics schemes in NWP models. The microphysics processes have complex non-linear interactions with dynamics and radiation process, and significantly affect the evolution of precipitation system (Johnson et al. 2014; Zheng and Chen 2014; Qian et al. 2018). Most of the microphysical parameterization schemes in NWP models pre-assume the PSDs as using the gamma, or $\Gamma$ function:

$$N(D) = N_0 D^\mu \exp(-AD),$$

where $N_0$ (mm$^{-1}$ m$^{-3}$) is the shape parameter, $\mu$ is the slope parameter, and $D$ (mm) is the particle diameter. When $\mu > 0$, the curve is sunken. When $\mu < 0$, the curve is raised. It degenerates into an exponential distribution if $\mu = 0$.

Bulk cloud microphysical parameterization schemes mainly include the one-moment (1M) scheme and the two-moment (2M) scheme. The 1M scheme only predicts the mass content $Q_r$ of the hydrometeor. The values of $N_0$ and $\mu$ are fixed, and $\Lambda$ is obtained by solving the $Q_r$. The evolution of PSD is determined only by $\Lambda$, and it imposed great restrictions in comparison with observation results. Commonly used 1M schemes are the Lin (Lin et al. 1983), WSM6 (Hong and Lim 2006), Goddard (Tao et al. 1989), and State University of New York at Stony Brook bulk microphysical parameterization (BMP) scheme (SBU-YLIN: Lin and Colle 2011). Moreover, 2M schemes simultaneously predict the $Q_r$ and total number density $N_{\infty}$ of rainwater. The value of $\mu$ is fixed in most schemes (generally at zero), and the $N_0$ and $\Lambda$ can be solved by $Q_r$ and $N_{\infty}$. For the 2M scheme, the description of particle spectrum is improved to some extent. The lack of physical constraints between the slope and intercept parameters often leads to the mismatch between them (Xu and Duan 1999).
Commonly used 2M schemes are the Morrison (Morrison et al. 2005), WDM6 (Lim and Hong 2010), NSSL (Mansell et al. 2010), and Milbrandt (Milbrandt and Yau 2005a) schemes. Thompson (Thompson et al. 2008) is also a 2M scheme when simulating rain and ice.

All 1M and most of 2M schemes assume that \( \mu \) is a constant (usually 0 for raindrops). However, extensive observational studies (e.g., Zhang et al. 2001; Bringi et al. 2002; Chen et al. 2013; Tang et al. 2014; Wen et al. 2016, 2017; Liu et al. 2018) have shown that the \( \mu \) is not constant, but often varies from negative to positive values. This key technical bottleneck needs to be solved urgently.

Morrison et al. (2019) has proposed a general N-moment raindrop size distribution (RSD) normalization method. However, the uncertainty of estimating RSD is very large by using zeroth-moment \( (M_0) \) and third-moment \( (M_3) \) two-moment normalization (as shown in Fig. 5, described in more detail later) (Morrison et al. 2019). For the 2M microphysical scheme, after considering the air and rainwater density, \( M_0 \) is equivalent to \( N_r \), and \( M_3 \) is proportional to \( Q_r \). The accuracy of general RSD normalization method is not enough and needs to be further improved.

If we need to predict \( \mu \), then it is necessary to introduce a third equation to close the system. Milbrandt and Yau (2005b) introduced a prediction equation for the radar reflectivity factor \( Z \) to predict \( \mu \). However, the \( Z \) itself is not an independent forecast quantity but rather is a derived variable that depends on \( Q_r \) and \( N_r \). Therefore, it cannot determine the value of \( \mu \) (D. H. Wang et al. 2014). Zhang et al. (2016) used actual observations in East Asia to infer the value of \( \mu \) (assuming that \( \mu \) is between 0 and 6) from the model-predicted \( A \) value. This method simply inferred the value of \( \mu \) through the \( \mu \)–\( A \) relationship; however, there was no closure between the derived \( \Gamma \) RSD and the 2M model forecasts \( (Q_r \text{ and } N_r) \), and no actual observation data were used to check whether the method is reasonable.

Extensive observation of RSD shows that the three parameters of the \( \Gamma \) function are not all completely independent (Zhang et al. 2001; Gorgucci et al. 2002; Brandes et al. 2004a,b; Chen et al. 2013; Wen et al. 2017; Liu et al. 2018; Wen et al. 2019). In particular, there is a positive correlation between \( \mu \) and \( A \) of the RSD. All of the present studies describe the \( \mu \)–\( A \) relationship with a parabolic function; these equations provide a one-variable sixth-order equation for \( \mu \). The one-variable sixth-order equation is difficult to solve analytically, and iterative solutions tend to oscillate back and forward without convergence. Solving this complex equation will consume a lot of computing resources. Furthermore, the equation can have as many as six solutions; therefore, it is difficult to eliminate invalid and wrong solutions and finally obtain a scientific and reasonable solution.

If the \( \mu \)–\( A \) relationship can be reduced to a linear correlation, then one obtains a one-dimensional cubic equation that can be solved analytically. In comparison with the one-variable sixth-order equation, the solution speed of this method will be significantly faster. However, this raises two questions: First, does the fitting accuracy decrease significantly after the \( \mu \)–\( A \) relationship is simplified to a linear relationship? Second, when there are two or three real solutions, how does one eliminate the invalid solutions and arrive at the unique solution that is scientifically valid?

To improve the understanding of precipitation microphysics characteristics and heavy-rainfall forecast skills during monsoon season in Southern China, unique program of the Southern China Monsoon Rainfall Experiment (SCMREX) project were conducted in Southern China from 2013 to 2018 (Luo et al. 2017). Several 2D video disdrometers (2DVDs) were collocated to observe the precipitation microphysics in South China in recent years, and the representative linear \( \mu \)–\( A \) relationship of RSD was observed by 2DVD in this region. According to the linear \( \mu \)–\( A \) relationship, a high-precision and rapid solution for \( \Gamma \) function based on the 0-moment and 3-moment has been established. The purpose of this study is to resolve the problem that current 2M microphysical schemes set \( \mu \) to a constant and thereby improve the ability of the 2M schemes to simulate the RSD of heavy rainfall in South China.
2. Data and method

a. 2DVD and dataset

Aimed at the needs of rainstorms mechanism analyzing and development of NWP model in South China, the Guangzhou Institute of Tropical and Marine Meteorology of the China Meteorological Administration (ITMM/CMA) and the Chinese Academy of Meteorological Sciences (CAMS) have jointly set up the Longmen Cloud Physics Field Experiment Base, CMA, since 2014.

The 2DVDs are important instruments for measuring precipitation characteristic at the field experiment base. The locations and observation periods of these 2DVDs are shown in Fig. 1. The RSD samples observed by these 2DVDs from 2016 to 2019 were used in this study. The detailed information of the 2DVDs has been introduced by Liu et al. (2018). The data quality control and processing method for the observations of 2DVDs are similar to those of Tokay et al. (2013), Wen et al. (2016), and Liu et al. (2018). The 2DVD observations are processed at 1-min interval. For each 1-min 2DVD observation, if the total number is less than 10 or if rain rate is less than 0.1 mm h$^{-1}$ then these 1-min data are considered as noise and are disregarded.

b. Method of solving $\Gamma$ raindrop size distribution function

The three parameters of the $\Gamma$ function ($N_0$, $\mu$, and $\Lambda$) can be solved using the order moment method. The moment of the $n$th order of the RSD is defined as follows:

$$M_n = \int_0^{D_{\text{max}}} D^n N(D) dD. \quad (2)$$

The three parameters ($N_0$, $\mu$, and $\Lambda$) of the RSD $\Gamma$ function are calculated with $M_2$, $M_3$, and $M_4$ following the methods as employed in Cao and Zhang (2009) and Liu et al. (2018).

Based on the $\mu$--$\Lambda$ relationship Zhang et al. (2001) proposed the constrained-gamma (C-G) method to retrieval RSD from polarimetric radar observations. According to the $\mu$--$\Lambda$ relations,
the independent parameters of \( \Gamma \) distribution function will reduce from 3 to 2. Previous studies generally used parabolic functions to describe the \( \mu - \Lambda \) relationships. In this study, linear functions were used to describe these relations and is referred to as the linear C-G method.

The RSD samples collected by 2DVDs from 2016 to 2018 in South China were used for the statistics of the linear \( \mu - \Lambda \) relationship. Three typical heavy-rainfall processes occurred in 2019 and all RSD samples observed during 2019 in South China, as shown in Table 1, were mainly used to test the accuracy of linear (C-G) solution method.

If the RSD \( N(D); \text{mm}^{-1} \text{m}^{-2} \) is obtained, the corresponding radar reflectivity factor \( (Z; \text{mm}^6 \text{mm}^{-3}) \) and rain rate \( (R; \text{mm} \text{h}^{-1}) \), rainwater content \( (W; \text{g} \text{m}^{-3}) \), and raindrop total number concentration \( (N_t; \text{m}^{-3}) \) can be calculated following the methods as employed in Wen et al. (2016) and Liu et al. (2018). As can be seen from Eq. (2), \( Z \) and \( M_6 \) are equivalent, \( N_t \) is equal to \( M_0 \), and \( W \) can be easily calculated from \( M_3 \).

Because the moment values change greatly, spanning several orders of magnitude, we convert them to decibels (dB) as

\[
M_a(\text{dB}) = 10 \log_{10} \left[ \frac{M_a(\text{mm}^n \text{m}^{-m})}{1 \text{mm}^n \text{m}^{-m}} \right].
\] (3)

The joint normalized probability distribution functions (PDF) of 2DVD observed and \( \Gamma \)-fitted \( M_0, M_3, M_6, \) and \( M_9 \) are given in Fig. 2. No matter whether \( M_0, M_3, M_6, \) or \( M_9 \), \( \Gamma \) function–fitted results are very consistent with 2DVD observations. These suggest that the \( \Gamma \) RSD function solving method using in this study has very high fitting accuracy. Note that the accuracy of observed small raindrops (<1.0 mm) for 2DVD is relatively lower than that of medium (1–3 mm) and large (>3 mm) raindrops (Tokay et al. 2013; Feng et al. 2020). For most RSD samples, the number concentrations of small raindrops are very high. Meanwhile, medium and large raindrops contribute most of rainwater content. Therefore, the \( \Gamma \) function fitted \( M_3 \) more accurately than \( M_0 \), which resulted in some differences between the observation and the simulation in \( M_0 \) and \( M_3 \).

Based on the function features of \( \Gamma \) (Cao and Zhang 2009), we obtain the relationship of \( \mu \) and \( \Lambda \) from \( M_0 \) and \( M_3 \):

\[
\Lambda = \frac{M_0}{M_3} (\mu + 1) = \frac{M_1}{M_3} (\mu + 1) = \frac{M_2}{M_3} (\mu + 3) \quad \text{and} \quad (4)
\]

\[
\Lambda^3 = \frac{M_0}{M_3} (\mu + 3)(\mu + 2)(\mu + 1).
\] (5)

If the values of \( M_0 \) and \( M_3 \) are known, the ratio of \( M_0 \) to \( M_3 \) \( (M_0/M_3) \) can be calculated, and then the three parameters \( (N_0, \mu, \text{and } \Lambda) \) of the \( \Gamma \) function can be obtained theoretically through the \( \mu - \Lambda \) correlation. However, most existing research uses the parabolic function \( (\Lambda = a\mu^2 + b\mu + c) \) to fit the \( \mu - \Lambda \) relationship. As a
FIG. 4. Scatterplots of (a)–(c) $m$–$L$, (d)–(f) $D_m$–$d_m$, (g)–(i) $\mu^m$–$\Lambda$, and (j)–(l) $\Delta \Lambda$–$M_3/M_1$ in three typical bins. (m)–(o) Scatterplots of $G$-fitted $\Lambda$ and improved $\mu$–$\Lambda$-fitted $\Lambda$ in three typical bins. In addition, the $m$–$L$ relationships from Zhang et al. (2003) (dark-yellow dotted lines), Cao et al. (2008) (green dotted lines), Liu et al. (2018) (blue dotted lines), and all samples (red dotted lines) are provided. The heavy dark-red lines denote the calculations of $D_m$ and $d_m$ from gamma RSDs with the corresponding linear $m$–$L$ relations. The orange crosses represent the estimated $\mu^m$ and $\Lambda$ calculated with $M_2$, $M_3$, and $M_4$ with 5% independent relative errors. The orange lines are the linear fitting results of these estimated $\mu^m$ and $\Lambda$. The pink dashed lines are the approximate linear relations given by Eq. (11) in Zhang et al. (2003).
result, Eq. (5) leads to \( f(\mu) = 0 \) through the simultaneous equations. This is a one-variable sixth-order function that is difficult to solve analytically. When solutions are sought with the iterative method, the solution often oscillates back and forth without convergence. Also, this process can yield as many as six solutions. It is difficult to quickly obtain a unique, effective, and accurate solution by using the parabolic function to fit the \( \mu - \Lambda \) relationship.

We attempt to fit the \( \mu - \Lambda \) relationship with a high-precision linear function and simplify Eq. (5) into a one-variable cubic function. If achieved, the equation \( f(\mu) = 0 \) can be easily solved analytically.

If the \( M_3/M_2 \) and the linear \( \mu - \Lambda \) relationship \( (\Lambda = a\mu + b) \) are known, Eq. (5) can be changed to

\[
 f(\mu) = (a^3 - c)\mu^3 + (3a^2b - 6c)a\mu^2 \\
 + (3ab^2 - 11c)a + b^3 - 6c = 0, 
\]  

where \( a \) is the slope parameter of linear fitting equation, \( b \) is intercept parameter, and \( c = M_2/M_3 \). In the equation \( f(\mu) = 0 \), only \( c \) (i.e., \( M_2/M_3 \)) is a variable. At this time, the solution of Eq. (6) is related to the ratio of \( M_3 \) and \( M_2 \). Therefore, Eq. (6) can be solved analytically using Shengjin’s formula (Fan 1989), which can directly calculate the solutions. In section 3b, we elaborate on the method for quickly obtaining the only reasonable analytical solution of \( \mu \).

c. Assessing the accuracy of calculated results

As the RSD of two-moment bulk microphysical scheme is continuous, the accuracy of the linear C-G method was verified by using the \( \Gamma \) fit from 2DVD observations to reduce truncation error. Since the existing two-moment bulk microphysical schemes use exponential function to describe RSD, the accuracy of the exponential function method was also verified in the same way.

To assess the accuracy of the calculated results using the above method, we regard the \( R, M_2, M_3 \), and \( M_3 \) values calculated from \( \Gamma \)-fitted RSDs of 2DVD observations as the true values. The \( R, M_2, M_3 \), and \( M_3 \) values calculated by RSDs from linear C-G method solutions and exponential method solutions are compared with these true values.

To quantitatively evaluate the fitting precision of the linear C-G method and the exponential method, we calculated the correlation coefficient (CC), the root-mean-square error (RMSE), the normalized absolute error (NAE), and the normalized relative error (NRE). The expressions are as follows:

\[
 CC = \frac{\sum_{i=1}^{N} (R_{D,i} - \bar{R}_O)(R_{O,i} - \bar{R}_D)}{\left[ \sum_{i=1}^{N} (R_{D,i} - \bar{R}_D)^2 \right]^{1/2} \left[ \sum_{i=1}^{N} (R_{O,i} - \bar{R}_O)^2 \right]^{1/2}}, 
\]

\[
 RMSE = \left[ \frac{1}{N} \sum_{i=1}^{N} (R_{D,i} - R_{O,i})^2 \right]^{1/2}, 
\]

\[
 NAE = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{R_{D,i} - R_{O,i}}{R_{O,i}} \right|, \quad \text{and} 
\]

\[
 NRE = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{R_{D,i} - R_{O,i}}{R_{O,i}} \right|, \quad \text{and} 
\]

FIG. 5. PDF of CC values for the original and improved \( \mu - \Lambda \) linear relationships in all \( (M_3/M_2, M_3) \) 1 dB × 1 dB bins. In comparing with the original \( \mu - \Lambda \) linear relationships, it is seen that the improved \( \mu - \Lambda \) linear relationships are further corrected by the \( \Delta \Lambda - M_3/M_3 \) relations. The red dotted line shows the CC value for \( \mu - \Lambda \) linear relationship of all DSD samples from 2016 to 2018.

FIG. 6. Variation of \( f(\mu) \) as a function of \( \mu \) for different values of \( M_3/M_2 \) in three typical bins.
In these expressions, \( N \) is the total number of RSD samples, \( R_{D,i} \) is rain rate calculated from the RSD of linear C-G method solutions or exponential method solutions of \( i \)th RSD samples, \( R_{O,i} \) is rain rate calculated from 2DVD \( i \)th observed RSD samples, \( \bar{R}_O \) is the average rain rate calculated from the RSD of linear C-G method solutions or the exponential method solutions, and \( \bar{R}_O \) is the average rain rate observed by the 2DVDs. The \( M_2, M_6, \) and \( M_9 \) are also calculated and evaluated using Eqs. (7)–(10).

3. A high-precision rapid solution method for \( \Gamma \) RSD suitable for 2M microphysical schemes

a. Linear \( \mu-\Lambda \) relationship

During 2016–18, about \( 2 \times 10^5 \) RSD samples have been collected by all of the 2DVDs at the Longmen Cloud Physics...
Field Experiment Base, CMA, as shown in Fig. 1. These mass RSD samples include different rainfall conditions in different seasons, which can well represent the precipitation characteristics in South China.

The rainwater content histogram of these RSD samples is shown in Fig. 3a. The RSD sample number decreases rapidly with increasing rainwater content. For each RSD sample collected by 2DVDs from 2016 to 2018 in South China, the $M_3$ are calculated from the derived $G$ distribution, and the $M_0$ are calculated from $N_0 L (2^{-1}G (m-1))$ to mitigate the error effects. The values of $M_0/M_3$ and $M_3$ are reported to dB values. As shown in Eq. (5), the ratio of $M_0$ and $M_3$ as well as $\mu-L$ correlation relationship is the key to solve $G$ RSD function. The PDF of $M_0/M_3$ is given in Fig. 3b. More
than 98% of RSD samples have $M_0/M_3$ values greater than 1. More than two thirds of RSD samples have $M_0/M_3$ values between 1 and 4. Joint normalized PDFs of $M_0/M_3$ and $M_3$ with units of decibels are shown in Fig. 3c. The gray crosses in Fig. 3d show a scatterplot between $\mu$ and $\Lambda$, and there is a positive correlation between them. The linear $\mu$-$\Lambda$ relation fit is given by

$$\Lambda = 1.250\mu + 2.858.$$  \hspace{1cm} (11)

The CC value of Eq. (11) is just 0.869. Meanwhile, the distributions of $\mu$ and $\Lambda$ are very divergent as shown in Fig. 3d. Therefore, the fixed $\mu$-$\Lambda$ linear relation of Eq. (11) cannot well represent the characteristics of all RSD samples.

As the accuracy of the equation solution of $\mu$ is closely related to the $\mu$-$\Lambda$ relationship, it is very important to increase the fitting accuracy of $\mu$-$\Lambda$ relationship. If there are too few samples used in fitting the $\mu$-$\Lambda$ relationship, the error would propagate to the fitted ones. To obtain high precision and
Fig. 12. RMSE histograms of (a),(b) rain rate, (c),(d) $M_2$, (e),(f) $M_6$, and (g),(h) $M_9$ for (left) the linear C-G method and (right) the exponential fit method in different rain-rate ranges.
reliability for the $\mu$–$\Lambda$ linear correlation relationship, the pair of moments $M_0$/$M_3$ and $M_3$ are discretize into $1 \text{ dB} \times 1 \text{ dB}$ bins by following the method proposed by Kumjian et al. (2019). Within each bin that has more than 50 RSD samples, the $\mu$–$\Lambda$ linear relationships are obtained. The RSD sample numbers in most bins are more than 200. Figs. 4a–c show the statistical results and linear fit results in three typical bins. The bin shown in Fig. 4a has most RSD samples. The RSD samples in the bin

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**Fig. 13.** As in Fig. 12, but for the values of NRE.
shown in Fig. 4b are heavy-rainfall cases. The RSD samples in
the bin shown in Fig. 4c are weak rainfall cases. The scatterplot
of $D_m$–$\delta_m$ ($\delta_m$: standard deviation of the mass-weighted di-
ameter distribution) calculated from RSDs samples agrees well
with the corresponding $\mu$–$\Lambda$ linear relationships, which indi-
cate they are not artifacts. Meanwhile, following the procedure
documented in Zhang et al. (2003), the numerical simulation
results by adding independent random errors to the moments
are given in Figs. 4g–i. The results shown that high linear
relationships between $\mu$ and $\Lambda$ can be caused by adding inde-
pendent random errors in estimated RSD moments. However,
these linear relationships between $\mu$ and $\Lambda$ are very close to the
approximate linear relations given by Eq. (11) in Zhang et al.
(2003), and obviously different with the linear
relationships derived from 2DVD measurements. These derived linear
$\mu$–$\Lambda$ relations are confirmed to be useful information that
can represent the average characteristics of actual RSD param-
eters. As shown in Figs. 4j–l, the $\mu$–$\Lambda$ linear $\Delta$-fit errors ($\Delta\Lambda$:
difference between the $\Lambda$ value calculated from $\mu$–$\Lambda$ linear
relations and the real $\Lambda$ value) are related to $M_0/M_3$. There are
linear correlations between $\Delta\Lambda$ and $M_0/M_3$. Since the $M_0/M_3$
is known, we can estimate the $\Delta\Lambda$. The $\mu$–$\Lambda$ linear fitting
accuracy can be further improved based on the linear corre-
lation between $\Delta\Lambda$ and $M_0/M_3$.

As shown in Fig. 5, in comparison with the fixed linear re-
lation of all RSD samples given by Eq. (11), the CCs of the
original (direct linear correlation between $\mu$ and $\Lambda$ in each
1 dB × 1 dB bin) and improved $\mu$–$\Lambda$ linear relationships are
greatly increased. The CCs of original $\mu$–$\Lambda$ linear relationships
are higher than 0.99 in more than 91% of the bins, and the CCs
are about 0.999 in 61% of the bins. The CCs of improved $\mu$–$\Lambda$
linear relationships (corrected by the linear correlation be-
 tween $\Delta\Lambda$ and $M_0/M_3$) are higher than 0.95 in all of the bins.
The CC values are higher than 0.99 in 98% of the bins, and
more than 85% of them are higher than 0.9995. These indicate
that the linear $\mu$–$\Lambda$ correlation relationships obtained by the
improved method have higher accuracy. The improved method
effectively overcome the defect of linear correlation by divid-
ing all RSD samples into 1 dB × 1 dB bins and can well rep-
resent the internal connection between the $\Gamma$ RSD parameters
$\mu$ and $\Lambda$ in South China. Since the accuracy of the $\Gamma$ function
solution result is closely related to the accuracy of the linear
$\mu$–$\Lambda$ correlation relationships, the step of improvement is very
important.

b. Three-parameter solution method based on the linear
constrained gamma function

Three typical bins are chosen to show the variations of
Eq. (6) for different values of $M_0/M_3$, and the results are given
in Fig. 6. When $f(\mu) = 0$, there may be one, two, or three real
number solutions. If there is only a unique real solution, then
$\mu$ directly adopts the solution. If there are two or three real
solutions, one of the real solutions is about $-2$ or even smaller
in most cases. Since $\mu$ is about $-2$ or even smaller, the corre-
sponding $\Lambda$ value obtained from the linear $\mu$–$\Lambda$ relationship
is less than 0. For an actual RSD, the value of $\Lambda$ must be greater
than 0; that is, the larger the raindrop size is, the lower is the
number density. Otherwise, the rain intensity will be infinite.

Therefore, this solution $\mu$ may be rejected since its corre-
sponding $\Lambda$ does not meet the actual atmospheric observation
results. If there are only two real solutions, the remaining $\mu$ is
the final solution. If there are three real solutions, one of the
real solutions can be removed by the negative value of $\Lambda$. Then,
only two real solutions of $\mu_1$ and $\mu_2$ remain, and the two values
of $\Lambda_1$ and $\Lambda_2$ are obtained. Since the $Z$ accumulates according
to $D^3$, a high density of large sized raindrops can increase the
value of $Z$. If the $\mu$ is obviously different with observations, the
obtained $Z$ is also significant deviate from normal $Z$–$R$
relationship. At this point, the $Z$–$R$ relationship can be used to
eliminate the unreasonable solutions and arrive at the only
valid $\mu$ solution.

To illustrate the above problems, three typical heavy-
rainfall processes and all RSD samples observed by 2DVs
during 2019 in South China are selected to perform cases test.
Table 1 gives the detailed information of these three rainfall
processes and RSD samples in 2019. Meanwhile, Fig. 7 shows
the solution results based on the above method in different
precipitation intensities. It can be seen that there are only one
or two valid solutions of $\mu$. If there is only one valid solution,
the $\Gamma$ RSD obtained by the linear constrained-gamma method
is very close to the original fitted $\Gamma$ RSD. When there are two
valid solutions, one of the solutions ($\mu_1$) will be very close to
the original $\Gamma$ function–fitted $\mu_0$. Another one ($\mu_2$) will be an
obviously higher or lower value in comparison with the $\mu_0$. In
this situation, the $R$ calculated by the $\Gamma$ RSD of $\mu_2$ is close to
the 2DVD observations, but the $Z$ is significantly higher or
lower than observation results. This makes its $Z$–$R$ relation-
ship significantly deviate from the average $Z$–$R$ relationship
in this bin (not shown). By comparing the degree of deviation
from $Z$–$R$ relationship, one of the solutions (here, $\mu_2$) can be
excluded.

Three lookup tables are established in each ($M_0/M_3, M_3$)
1 dB × 1 dB bin based on the RSD samples observed by
2DVs from 2016 to 2018. For each ($M_0/M_3, M_3$) 1 dB × 1 dB
bin, we collect the $M_0/M_3$, $M_3$, and $Z$ from
–40 dB to 80 dB. Then, we establish the lookup tables.
This method has the following four steps:

1) Calculate the $M_3$ and $M_0$ values of the rainwater. Then,
convert $M_0/M_3$ and $M_3$ to dB values.

2) Based on the three lookup tables, the improved linear $\mu$–$\Lambda$
relationship and $Z$–$R$ relationship are obtained through
the dB values of $M_0/M_3$ and $M_3$. If the $M_0/M_3$ or $M_3$
decibel values are out of the lookup table, the fixed $\mu$–$\Lambda$
relationship of Eq. (11) is suggested for use in this step.
3) Equation (6) is solved by using Shengjin’s formula. Through the positive or negative values of , as well as the degree of deviation from the - relationship, the unreasonable solutions of and the are eliminated, and only the reasonable solution of is left.

4) Calculate according to the and the - relationship. Calculate according to , , and . Then, the high-precision RSD is obtained.

4. Verification using real data

To check the fitting accuracy of the function parameter solving by the linear C-G method developed in this study, we selected the three typical heavy-rainfall processes, as well as all RSD samples observed by 2DVD during 2019 in South China as shown in Table 1. The weather conditions of these three processes include cold air, monsoon, and typhoon, which are typical conditions of heavy rainfall in South China during the flood season. The RSD samples of these three processes are all more than 1000, and the heaviest rain rates are all larger than 100 mm h⁻¹.

The values of and were calculated using the 1-min interval RSD measured by the 2DVDs. Using the linear C-G method, the three parameters of the function were solved from the and and values. In addition, the two parameters of the exponential function were solved from the and values. The R, , , and values calculated from the RSD parameters of the exponential function were derived from 2DVD measurements when setting as the reference standard. We conducted a quantitative assessment of the fitting accuracy of the RSD function described above using Eqs. (7)–(10). Comparisons of the values calculated from the linear C-G function and exponential function with the RSD from 2DVD observations are shown in Fig. 8. The comparison results for the , , , and values are also shown in Figs. 9–11, respectively.

As shown in Fig. 8, the values calculated from linear C-G method are generally consistent with the RSD measured by the 2DVDs. The CCs of the linear C-G method are all up to 0.997 in these three processes and all samples in 2019. Meanwhile, all of the RMSE values are less than 1.3 mm h⁻¹, the NAE values are less than 0.071, and the NRE values approach 0. The values calculated from the exponential method are higher than the RSD from 2DVD observations when rain rates are lower than 20 mm h⁻¹ but lower when rain rates are heavier than 50 mm h⁻¹. The errors of the exponential method are obviously larger than those of linear C-G method. The CCs are all less than 0.99. Meanwhile, the RMSE and NAE values of exponential method are more than 2 times larger than that of linear C-G method.

The values calculated from the linear C-G method are also consistent with the - RSD from 2DVD observations (see Fig. 9). The values in these three heavy-rainfall processes and all samples in 2019 are all larger than 0.993. The RMSE values are all smaller than 0.59 dB. Meanwhile, the NAE values are less than 0.019, and the NRE values are nearly 0. The values calculated from the exponential method are relatively smaller than the - RSD from 2DVD observations when are less than 30 dB but are higher when is larger than 35 dB. The errors of the exponential method are obviously larger than that of the linear C-G method. The RMSE and NAE values of the exponential method are about 3 times that of the linear C-G method.

The values calculated from the linear C-G method are about 3 times that of the linear C-G method. The RMSE and NAE values of the exponential method are much larger than that of the linear C-G method. The RMSE and NAE values of the exponential method are about 3 times that of the linear C-G method.

To analyze in depth the performance of the linear C-G method and exponential method under different rainfall intensities, rainfall samples are divided into six categories based on the rain-rate PDF in South China. They are weak rain (0.1–1.0 mm h⁻¹), light rain (1.0–5.0 mm h⁻¹), moderate rain (5.0–10.0 mm h⁻¹), heavy rain (10.0–20.0 mm h⁻¹), rainstorm (20.0–50.0 mm h⁻¹), and downpour (heavier than 50.0 mm h⁻¹). This classification method is based on the local characteristics of precipitation and climate and is different from the daily precipitation classification method.

The RMSE values of both the linear C-G and exponential method in every range for three heavy-rainfall processes and all samples in 2019 are shown in Fig. 12. Meanwhile, the NRE values are shown in Fig. 13. The RMSE values of , , , and of the linear C-G method are all obviously smaller than the exponential method in every range. The NRE values of , , , and of the linear C-G method are all very close to 0. However, the NRE values of , , and for the exponential method are significantly higher than those for the linear C-G method. The results show that, no matter what the precipitation-class situation is, the errors of the linear C-G method are obviously smaller than those of the exponential method. Thus, the linear C-G method can significantly improve the accuracies of simulated RSD for 2M bulk microphysical schemes.

5. Conclusions and discussion

Based on the linear - correlation relationship acquired by several 2DVDs in South China, we constructed a high-precision
fast $\Gamma$ function solution using a linear C-G method. In this solution, the three parameters ($N_0$, $\mu$, and $\Lambda$) of $\Gamma$ RSD function are calculated from $M_0$ and $M_3$, which can be easily obtained from the mass content $Q$, and total number density $N_0$ of the rainwater that is simulated by the 2M microphysical scheme. The obtained RSDs are obviously closer to the observations in comparison with the exponential function solution and avoid the problem of setting the shape parameter to a constant (usually 0) in the existing 2M microphysical scheme.

Based on about $2 \times 10^3$ RSD samples observed by several 2DVs from 2016 to 2018 at the Longmen Cloud Physics Field Experiment Base, CMA, the statistical linear $\mu$-$\Lambda$ correlation relationships in each $(M_0/M_3, M_3) 1 \text{ dB } \times 1 \text{ dB}$ bin were obtained. The CC values in 98% of the bins are higher than 0.99, and the values are higher than 0.9995 in more than 85% of the bins. These $\mu$-$\Lambda$ linear relationships reflect characteristics of actual RSDs and have good regional representations in South China. Based on these linear $\mu$-$\Lambda$ correlation relationships, we obtained a one-dimensional cubic equation for solving $\mu$ that is dependent only on the value of $M_0/M_3$. The value of $M_0/M_3$ can be easily calculated from $Q_0$ and $N_{tr}$. Furthermore, analytical solutions are obtained using Shengjin’s formula. When multiple solutions existed, the invalid solutions are excluded using the positive–negative relationship of $\Lambda$ as well as the deviation from the $Z$–$R$ relationship. Only one reasonable analytical solution of $\mu$ is obtained. Then, the values of $N_0$ and $\Lambda$ can be calculated.

Three typical heavy-rainfall processes and all RSD samples observed by 2DVs during 2019 were selected to verify the accuracy of the linear C-G method. Analysis results show that, relative to the exponential method usually employed in the 2M microphysical scheme, the $R$, $M_2$, $M_0$, and $M_3$ values obtained by the linear C-G method are significantly in better agreement with the $\Gamma$-fit RSD from 2DVD observations. The CC values of the linear C-G method are higher than those of the exponential method. Meanwhile, the RMSE, NAE, and NRE values are obviously lower.

In summary, a high-precision and fast linear C-G solution method based on $M_0$ and $M_3$ has been established in this study. The proposed method has effectively solved the problem that the shape parameter in the 2M microphysical scheme is set to a constant, and the obtained $\Gamma$ RSD are much closer to the observations. The simulated radar reflectivities agree well with observations. This method has great potential to be applied to the 2M microphysical scheme to improve the simulation of heavy precipitation in South China, and we will further analyze and apply this method in future studies.

Nevertheless, this work only conducted systematic studies of RSD characteristics in South China. Because the $\mu$–$\Lambda$ relationship changes depending on climatology, geographical location, and rain type (Liu et al. 2018), the lookup table values cannot be simply applied to other areas. Future in-depth research is needed, particularly for different climate regions, to ensure that this method is broadly applicable and impactful.

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