Direct Computation of Geostrophic Winds from Observed Winds Using the Balance Equation

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ABSTRACT

In certain meteorological problems it is desirable to make a grid-point analysis of observed winds. In order to combine a wind analysis with other quantities, or to use the wind analysis in a numerical model, one may wish to know the geostrophic winds (or the heights) that conform to the analyzed winds. Many previous studies have shown that the balance equation accurately describes relationships between winds and the height field. In this paper the balance equation is used to equate geostrophic vorticity at each grid point to a sum of terms evaluated from the wind analysis. Then the geostrophic winds are found by altering an initial guess field of zonal and meridional wind components until the geostrophic vorticity and divergence required at each point are met within a close tolerance. The final wind vectors are a geostrophic field that conforms to the original wind analysis according to the balance equation. Typical examples are shown of geostrophic wind vectors computed by this method in a sloping planetary boundary layer, and in the upper troposphere. In addition, geostrophic departures are compared with irrotational winds calculated for the same cases.

1. Introduction

The conventional use of the balance equation has been that of computing values of stream function from height fields analyzed at grid points on constant pressure surfaces. Balanced winds may then be determined from stream function gradients. Normally observed winds are used only in helping to establish height fields through use of the geostrophic relationship (e.g., Cressman, 1959). This procedure has proved to be satisfactory for the treatment of synoptic- and planetary-scale weather patterns outside the tropics. Recent articles bearing on this subject are given in the *Proceedings of the International Symposium on Numerical Weather Prediction* (Syōno, 1962). In the present paper, heights are not used; instead, geostrophic winds are computed directly from grid-point analyses of observed winds. The technique uses the balance equation to determine geostrophic vorticity from the vorticity and certain other terms computed from the observed winds; winds that have this geostrophic vorticity are then found.

There are three principal reasons for choosing an approach that begins with wind data. The first depends on certain experience in studying subsynoptic phenomena such as jet streams, areas of clear air turbulence, and severe storms. It appeared to the writer that kinematic quantities associated with these phenomena could be determined from observed winds over the United States with better resolution, simpler computations, and more meaningful results than from height data. Secondly, in the tropics it is commonly accepted that height data are much less reliable than wind observations; thus, the

wind analysis technique used here is believed to be applicable to tropical phenomena. Thirdly, it is interesting to approach the problem of wind-height balance from a different mathematical viewpoint than the customary one. However, the approach taken here is exploratory and there is no intention of evaluating the practical value of the method at this time.

To obtain a geostrophic wind field that is consistent with observed winds, the following steps are taken. The original grid-point analysis is made of winds measured by rawinsondes with no recourse to height data. The analysis scheme, which uses an inverse distance weighting factor and upstream-downstream enhancement, was developed by R. L. Mancuso (see Endlich and Mancuso, 1968). After the winds are specified, their relationship to the height field is assumed to be that described by the balance equation. This equation is written as a relationship between geostrophic vorticity and pertinent features of the wind analysis, as shown in Section 2. The next step is to compute the geostrophic vorticity at each grid point. Finally, winds that have these values of geostrophic vorticity are found. The method of solution begins with a first guess field (which may be the original grid-point wind analysis), and alters it until the desired geostrophic vorticity, and also the proper geostrophic divergence, are met at each grid point. The resulting wind field is a geostrophic one that conforms to the original wind analysis according to the balance equation. The iterative method of solution is essentially the same as that used previously to obtain nondivergent and irrotational winds from a grid-point wind analysis (Endlich, 1967).

An earlier method of solving the same general problem uses the balance equation to specify heights that conform to a stream-function field determined from observed winds. The solution for heights is performed using a relaxation technique. This possibility was proposed by Phillips (1959) and has been followed by Rosenthal (1960), Alaka *et al.* (1966) and others with satisfactory results. A first-guess field of heights and some appropriate boundary conditions are required. The alternating-direction implicit (ADI) method of relaxation makes the computation efficiently and determines boundary values during the iteration process (Mancuso *et al.*, 1967). The present paper differs from these mainly in that iterations are performed directly on wind components.

Previously, geostrophic departures have been computed from wind observations determined from constant-level balloon trajectories (e.g., Neiburger and Angell, 1956). The difference between motion vectors of an air parcel at two different times is a measure of the particle acceleration; this may be used to compute a geostrophic departure. Such a departure is an average over several hours time and an equivalent portion of the balloon trajectory. In contrast, the present procedure gives instantaneous departures. However, there is no conflict between the two methods; they could be used concurrently if relevant input data were available.

2. Use of the balance equation

Using standard notation, the balance equation may be written as

$$-2J(u, v) - f\zeta + \beta u + \nabla^2\Phi = 0. \quad (1)$$

It is well-known that this formulation is obtained from the divergence equation by neglecting dD/dt , divergence squared, a term containing vertical motion, and viscous forces. To obtain a relation for the geostrophic vorticity ζ_g for use in (1), the geostrophic equation is written in the form

$$\nabla\Phi = -f\mathbf{k} \times \mathbf{V}_g, \quad (2)$$

where the geostrophic wind $\mathbf{V}_g$ has components U and V , denoted by capitals to distinguish them from the observed u and v components. On operating on Eq. (2) with $\nabla \cdot$, one obtains

$$\nabla^2\Phi = f\zeta_g - \beta u. \quad (3)$$

Using this expression in (1) gives

$$\zeta_g = \zeta + f^{-1}[-2J(u, v) + \beta(U - u)]. \quad (4)$$

We also make use of the equation for the divergence of the geostrophic wind, i.e.,

$$D_g = \nabla \cdot \mathbf{V}_g = -(\beta/f)V. \quad (5)$$

Alternatively, if a quasi-geostrophic nondivergent wind

is desired, one can simply replace the right-hand side of (5) with zero.

As the analysis of zonal and meridional wind components is made on a spherical grid network, it is necessary to write the expressions for vorticity and divergence in spherical coordinates; thus, Eq. (4) is

$$V_x - U_y + (U/r) \tan \varphi = [v_x - u_y + (u/r) \tan \varphi + 2f^{-1} \times (u_x v_y - u_y v_x) - (\beta/f)u] + (\beta/f)U. \quad (6)$$

It will be convenient later to retain only the term $(V_x - U_y)$, which is the major portion of the geostrophic vorticity, as the unknown left-hand side of the equation. Then using $\beta/f = \cot(\varphi/r)$, (6) becomes

$$V_x - U_y = [v_x - u_y + (u/r)(\tan \varphi - \cot \varphi) + 2f^{-1}(u_x v_y - u_y v_x)] - (U/r)[\tan \varphi - \cot \varphi]. \quad (7)$$

Since in mid-latitudes the tangent and cotangent of latitude are nearly equal, the terms containing them could be neglected with little error; however, they are retained for generality. The entire term within brackets is determined at each point once and for all from the original wind analysis, or from nondivergent winds. The last term on the right side of (7) is recomputed after each iteration cycle using the most recent values of U . The complete right side of (7) is denoted by M . The final solution for U and V must satisfy (7) at every grid point.

Similarly, Eq. (5) is written as

$$U_x + V_y = (V/r)(\tan \varphi - \cot \varphi) = N. \quad (8)$$

As in the previous equation, the right-hand side of (8) is recomputed at each grid point after each cycle using the most recent values of V . Since the trigonometric functions can go to zero or infinity at the equator or poles, special treatment would be required to include these locations in the grid.

In summary, the two partial differential equations to be solved simultaneously over the grid for the geostrophic components U and V are

$$\left. \begin{aligned} V_x - U_y &= M \\ U_x + V_y &= N \end{aligned} \right\}, \quad (9)$$

where the values of M and N at each point are slowly varying functions recomputed after each iteration cycle.

The basic concept of the method of solving Eq. (9) for U and V from an initial guess field is as follows. Consider a typical grid point and its nearest four neighbors, each having an estimated U and V . The temporary geostrophic vorticity and divergence at the point of interest may be computed from the neighboring estimates of U and V . Let these temporary values be denoted as M' and N' . If they differ from the desired values M and N , alterations may be made to the U and V components at the neighboring points to force agreement with M and N . For example, if N' indicates more

convergence than the desired value N at a point, the excess can be removed by adding a small positive value to the U component at the point to the east, subtracting an equal amount from the U component at the point to the west, adding the small value to the V component at the point to the north, and subtracting it from the V component to the south. Similarly, vorticity can be changed by altering V components at the points to the east and west, and U components at the points north and south. Formulas for calculating the necessary alterations are given by Endlich (1967). One proceeds through the grid in the manner of a Gauss-Seidel iteration process (Todd, 1962, Chap. 11). The mathematical convergence of the method was found to be satisfactory by experiment, and was verified in a theoretical analysis made by Stephens (1967).

As a matter of convenience the analyzed values of u and v may be used as the first guess for the geostrophic components, as mentioned previously. In the computer program, the computations continue until the desired values of M and N at each point are met within the limit $1 \times 10^{-6} \text{ sec}^{-1}$. Then the values of U are adjusted so that they have the same average over the grid as the initial values of u ; $\bar{V}$ similarly is made equal to $\bar{v}$. [This is possible since the solution of (9) is not affected by addition of a constant to each value of U (or V) in the mesh.] Next, geostrophic departures ($u-U$) and ($v-V$) are computed at each point. Because of the prior adjustment of U and V , the departures each average to zero over the grid. This assumption seems reasonable from a synoptic viewpoint provided that the grid covers an area considerably larger than a single trough or ridge. Also, the adjustment has the property that it mini-

mizes the average energy (squared values) of the departures. A minimization of this type for irrotational wind components was used previously by Sangster (1960).

Examples of observed winds and computed geostrophic winds are given in the next section. Also, since nondivergent and irrotational winds may be of interest in comparison with geostrophic winds and departures, these were computed by the method referred to earlier, and will also be shown. For ease of inspection, the winds are plotted in vector form. Length scales are indicated in each diagram.

3. Examples

Three examples will be shown that are typical of computations made so far. The first is for the boundary layer in a case having a strong low-level front that was accompanied by severe thunderstorms and tornadoes. [The boundary layer is defined as lying between $P=10$ and $P=8.5$ using Sangster's (1960) nondimensional vertical coordinate. The top of the layer slopes with terrain at a height approximately 1.5 km above the surface.] Although it is not pertinent to the present discussion, it may be of interest that eight tornadoes were reported in the cross-hatched area of Fig. 1 in the three hours following map time. This example was chosen to represent an active weather system within an area of good data coverage.

Analyzed values of u and v components are plotted as vectors in Fig. 1, and the divergence of these winds is shown in Fig. 2. Note that the divergence pattern is smooth and has acceptable magnitudes. (Corresponding

FIG. 1. Objective wind analysis in the planetary boundary layer (surface to approximately 1.5 km) at 0000 GMT 28 April 1966.

FIG. 2. Horizontal divergence of the wind field of Fig. 1 (units 10^{-6} sec^{-1}).

vertical motions at the top of the boundary layer are as large as 5 cm sec^{-1} at 35°N , 100°W .) Irrotational winds necessary to account for the divergence field are shown in Fig. 3, as computed by the method mentioned previously. These are shown for comparison later with the geostrophic departures. The main feature is the flow into the area of large convergence. Fig. 4 gives the relative vorticity of the wind analysis of Fig. 1. As expected, the frontal zones can be identified by relatively large

positive values. The corresponding geostrophic vorticity computed from Eq. (7) is shown in Fig. 5. The general characteristics are, of course, quite similar to those of Fig. 4 with the differences being primarily due to the Jacobian term. The geostrophic wind field computed to conform to Fig. 5 using Eq. (9) is given in Fig. 6. Comparison with Fig. 1 shows that the geostrophic winds have smoother patterns than the observed winds, as desired. Also, in this particular case, the geo-

FIG. 3. Irrotational wind vectors that correspond to Fig. 1, and that have the divergence of Fig. 2.

FIG. 4. Relative vorticity of the wind field of Fig. 1 (units 10^{-6} sec^{-1}).

strophic winds are very similar to the nondivergent winds (not shown). The geostrophic departures are the vector differences between Fig. 1 and Fig. 6, as given in Fig. 7. These have maximum values along the front near 32°N, 100°W. They also have a strong resemblance, but are not identical, to the irrotational components of Fig. 3. This similarity implies that a major portion of the low-level geostrophic departure field in cases of this sort is due to divergent motions as one might have anticipated. In summary, we see that this method permits

computation of geostrophic winds in a sloping layer near the earth's surface without encountering problems of reducing pressure to an arbitrary level (e.g., sea level).

The second example selected is the flow in an upper tropospheric layer (400–300 mb) at the same time as Fig. 1. The wind analysis (Fig. 8) has a typical trough pattern and normal wind speeds. In this case, we show the nondivergent winds (Fig. 9) that correspond to Fig. 8; the two figures differ only in minor details. Due to the uncertainties of wind measurement and analysis,

FIG. 5. Geostrophic vorticity determined from Fig. 1 according to the balance equation (units 10^{-6} sec^{-1}).

FIG. 6. Geostrophic wind field that corresponds to Fig. 1, computed to have the vorticity of Fig. 5.

it may be argued that Fig. 9 is more reliable than Fig. 8. Therefore, in this instance, Fig. 9 was used as the first-guess field in computing the geostrophic wind field, and the geostrophic departures are taken as the difference between the nondivergent and geostrophic vectors. Since the departures are considerably smaller than the original wind vectors, they are shown in Fig. 10 plotted on a scale twice as large as that of Figs. 8 and 9. Dots

are shown where speeds are close to zero. The computed departures give a smooth field since irrotational components have been excluded from them. They reach magnitudes up to 10 m sec^{-1} , and have an anticyclonic rotation in an area centered near 45°N , 102°W . In other words, in this region of the trough, geostrophic speeds exceed the observed speeds, and geostrophic vorticity exceeds observed vorticity. Since the observed winds in

FIG. 7. Geostrophic departures determined as the difference between Fig. 1 and Fig. 6.

FIG. 8. Objective wind analysis in an upper tropospheric layer (400-300 mb) at 0000 GMT 28 April 1966.

the trough are subgeostrophic, they are apparently tending to conform to gradient winds as found previously in such situations (e.g., Angell, 1959; Endlich and McLean, 1960). Similar patterns of departures have been found in other cases.

The final example, not related to the previous two, is chosen to represent a strong jet stream that has sharp anticyclonic curvature on both sides of a shear line

(Fig. 11). Wind speeds are as large as 90 m sec^{-1} near 37°N , 85°W . In the present case, we show the irrotational vectors rather than their counterpart, the nondivergent winds. Fig. 12 gives them on a scale twice as large as that of Fig. 11. They have speeds up to 13 m sec^{-1} ; however, most are less than 5 m sec^{-1} . A careful inspection of Fig. 12 will reveal that the irrotational vectors tend to converge and diverge in certain well defined

FIG. 9. Nondivergent winds that correspond to Fig. 8.

FIG. 10. Geostrophic departures (not including irrotational components) that correspond to Fig. 9. Note that the length scale is twice as large as in Fig. 9.

patterns. These patterns correspond of necessity to the divergence field implied by Fig. 11. Here we do not probe the controversial question of the reliability of the computed divergence fields and corresponding irrotational winds. However, there is a tendency for patterns of irrotational fields determined independently at 12-hr intervals to persist, and this indicates that the fields are not entirely noise.

The geostrophic winds computed to conform to Fig. 11 are similar to it in general appearance, and will not

be shown. Geostrophic departures were computed as the difference between the nondivergent and geostrophic winds in order to exclude the irrotational components. These departures are shown in Fig. 13 using the same length scale as Fig. 12. Magnitudes up to 19 m sec^{-1} are apparently present near 35°N , 110°W . Since the departures in this region are in the same direction as the winds of Fig. 11, it follows that the flow is supergeostrophic. One would expect this from gradient wind arguments, since the curvature of flow is anticyclonic. In contrast,

FIG. 11. Objective wind analysis in the 300-250 mb layer at 0000 GMT 11 March 1966.

FIG. 12. Irrotational vectors that correspond to Fig. 11. Note that the length scale is twice as large as in Fig. 11.

the departures in the trough along the east coast imply subgeostrophic motions. Computations made for other maps of this series show that these patterns tended to continue for two or three synoptic times in association with the troughs and ridges in the flow. Finally, it is apparent that in this situation of strong winds and sharp curvature the computed geostrophic departures (Fig. 13) are generally larger than the irrotational winds (Fig. 12).

4. Discussion

Although the mathematical procedure of Section 2 is intricate, it is straightforward to use. The method has been written in ALGOL for use on the Burroughs 5500 computer and in FORTRAN for the Control Data Corporation 6400 computer. Various options permitting variations in the techniques can be introduced rather easily. At present, the general applicability of this method cannot be judged; however, in problems of ob-

FIG. 13. Geostrophic departures (not including irrotational components) that correspond to Fig. 11. Note that the length scale is twice as large as in Fig. 11.

jective analysis and numerical forecasting where observed winds are used as initial data, the technique may be of value.

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REFERENCES

- Alaka, M. A., D. T. Rubsam and G. E. Fisher, 1966: An experiment in the use of the balance equation in the tropics. *Proc. 1966 Army Conf. Tropical Meteorology*, U. S. Army Electronics Command, Fort Monmouth, N. J., 109-127.
- Angell, J. K., 1959: Some statistics on the magnitude of the ageostrophic wind obtained from constant level balloon data. *Mon. Wea. Rev.*, **87**, 163-170.
- Cressman, G. P., 1959: An operational objective analysis system. *Mon. Wea. Rev.*, **87**, 367-374.
- Endlich, R. M., 1967: An iterative technique for altering the kinematic properties of wind fields. *J. Appl. Meteor.*, **6**, 837-844.
- , and G. S. McLean, 1960: Geostrophic and gradient departures in jet streams. *J. Meteor.*, **17**, 135-147.
- , and R. L. Mancuso, 1968: Objective analysis of environmental conditions associated with severe thunderstorms and tornadoes. *Mon. Wea. Rev.*, **96**, 342-350.
- Mancuso, R. L., R. M. Endlich and W. Viezee, 1967: Tropical analyses based on computer techniques. To be published in *Proc. Fifth Tech. Conf. Hurricanes and Tropical Meteorology*, Caracas, Venezuela.
- Neiburger, M., and J. K. Angell, 1956: Meteorological applications of constant-pressure balloon trajectories. *J. Meteor.*, **13**, 166-194.
- Phillips, N. A., 1959: On the problem of initial data for the primitive equations. *Tellus*, **12**, 121-126.
- Rosenthal, S. L., 1960: Concerning the feasibility of computing the field of pressure-geopotential from observed winds by use of the divergence equation. Tech. Rept. No. 15, Contract NONR-1600(00), Florida State University, Tallahassee, 46 pp.
- Sangster, W. E., 1960: A method of representing the horizontal pressure force without reduction of station pressures to sea level. *J. Meteor.*, **17**, 166-176.
- Stephens, J. J., 1967: On convergence of the Endlich iteration method. *J. Appl. Meteor.*, **6**, 845-847.
- Syōno, S., 1962: *Proc. Intern. Symp. Numerical Weather Prediction*. Tokyo, Meteor. Soc. Japan, 656 pp.
- Todd, J., 1962: *Survey of Numerical Analysis*. New York, McGraw-Hill, 589 pp.