On Global Initialization of the Primitive Equations: Part I

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ABSTRACT

The problem of obtaining a realistic balanced relationship between the pressure field and the nondivergent velocity is examined for a global primitive equation model. Both a theoretical discussion of the atmospheric adjustment process and actual numerical examples of initialization are used to show that pressure at low latitudes must be the derived quantity and not the input data. For the two-layer version of the NCAR general circulation model, the latitudes of 27½N and S are shown to be acceptable for separating the equatorial belt, where velocity is used as input data, from regions of higher latitudes, where pressure is used as input data.

1. Introduction

The purpose of this study is to investigate the basic problem of obtaining initial conditions for the global primitive equations. The discussion in this paper is limited to determining appropriate pressure and nondivergent velocity fields over the entire globe. In a subsequent paper, methods of computing the divergent component of velocity will be discussed.

A basic problem of initialization is to establish a velocity-pressure relationship which will suppress gravity-inertial oscillations caused by data errors and motions too small in scale to be handled properly by the model. A concurrent problem is to minimize any distortions in those scales of motion which the model can handle properly.

Numerous velocity-pressure diagnostic relationships have been proposed and used in initialization of primitive equation models. These range from the simple geostrophic equation to elaborate balance conditions involving complicated iteration schemes such as those proposed by Fjørtoft (1962), Hollman (1966), Miyakoda and Moyer (1968). All of these diagnostic relationships have certain difficulties in the equatorial regions, usually problems of convergence or accuracy. It seems that most of the proposed diagnostic relationships differ appreciably from observed synoptic patterns at the equator.

In this study, no new types of diagnostics are introduced. Only variations of the standard balance equation (Charney, 1955) are considered. The main problem is to develop a method that will be mathematically suitable in the equatorial region and that can be applied to a spherical domain with no physical lateral boundaries. In Section 2 the equations are presented. In Section 3 the standard balance solution is extended to a global domain. The balance equation is solved in the usual Monge-Ampère form, i.e., the stream function is derived from the pressure field, and its deficiencies in the equatorial region are noted. Section 4 contains a theoretical discussion of the adjustment process fundamental to the problem of initialization. Adjustment theory determines where pressure or wind data should be used. Section 5 presents the method of solution of the balance equation which eliminates the deficiencies noted in Section 3 and is most rigorous from the point of view of the theoretical discussion in Section 4. Several sample solutions are shown for real data.

2. Basic equations

The balance equation is derived from the divergence equation. For deep atmospheric phenomena on a synoptic scale, the terms containing horizontal divergence and vertical motion are negligible in the divergence equation. The remaining terms from the balance equation are shown below with the scale factor beneath:

$$\left.\begin{array}{l} f\nabla^2\Psi = \dfrac{1}{\rho}\nabla^2 p + 2J(v,u) \\ \quad 1 \qquad\quad 1 \qquad\qquad \mathrm{Ro} \\ \qquad\qquad + \beta u + \dfrac{1}{a^2}\left(1+(\tan\phi)\dfrac{\partial}{\partial\phi}\right)(u^2+v^2) \\ \qquad\qquad\quad \dfrac{L}{a\tan\phi} \qquad\qquad \mathrm{Ro}\dfrac{L}{a}\left(\dfrac{L}{a}+\tan\phi\right) \end{array}\right\} \qquad (2.1)$$

The symbols are defined as follows:

- φ latitude (positive northward)
- λ longitude (positive eastward)
- z height above mean sea level

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u, v eastward and northward components of the horizontal velocity
f Coriolis parameter $=2\omega \sin\varphi$
ω earth's angular velocity
p pressure
ρ density

$$\beta=\frac{1}{a}\frac{\partial f}{\partial\varphi}$$

a mean radius of earth

∇^2 Laplacian operator

$$=\frac{1}{a^2\cos^2\varphi}\left[\frac{\partial^2}{\partial\lambda^2}+(\cos\varphi)\frac{\partial}{\partial\varphi}\left((\cos\varphi)\frac{\partial}{\partial\varphi}\right)\right]$$

$J(v,u)$ Jacobian operator

$$=\frac{1}{a^2\cos\varphi}\left[\frac{\partial u}{\partial\varphi}\frac{\partial v}{\partial\lambda}-\frac{\partial u}{\partial\lambda}\frac{\partial v}{\partial\varphi}\right]$$

ζ vertical component of vorticity

$$=\frac{1}{a\cos\varphi}\left[\frac{\partial v}{\partial\lambda}-\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]$$

d/dt total horizontal derivative operator
Ri Richardson number
Ro Rossby number
L horizontal length scale
Ψ stream function

Eq. (2.1) is accurate to 1% even in equatorial regions when Ri$=$O(100). If $(L/a)=O(1/10)$, then we observe that the spherical term (the last one) is important only at high latitudes, and the beta term is important only at low latitudes. The Jacobian term is important in jet stream areas and near the equator, where Ro$=$O(1).

Let us now examine the accuracy of the standard form of the general balance equation (2.1). To obtain this form, we approximate u and v by their nondivergent components which are

$$u=-\frac{1}{a}\frac{\partial\Psi}{\partial\varphi},\quad v=\frac{1}{a\cos\varphi}\frac{\partial\Psi}{\partial\lambda}.$$

According to Charney (1963), this approximation introduces an error of O(1/Ri Ro) in the terms where the substitution is made. Thus, the error to Eq. (2.1) due to the Jacobian term is of O(1/Ri), that due to the beta term,

$$\frac{L}{a\tan\varphi}\frac{1}{\text{Ri Ro}},$$

and that due to the spherical terms,

$$\frac{L}{a\,\text{Ri}}\left(\frac{L}{a}+\tan\varphi\right),$$

if these three terms do not dominate the equation.

This implies that the error in the Jacobian term is only 1% to rather low latitudes; the error in the spherical term is <1% except very near the poles; and the beta term error <1% except very near the equator, where $\tan\varphi\to 0$. If either the Jacobian, beta or spherical term becomes dominant in Eq. (2.1), as is likely very near the equator or the poles, the error may reach a maximum value of O(1/Ri Ro). Therefore, when the entire globe is considered, the maximum relative error for Eq. (2.1) is 10%. Thus, the standard form of the balance equation, written in the form

$$f\nabla^2\Psi=\frac{1}{\rho}\nabla^2 p+2J\left(\frac{1}{a}\frac{\partial\Psi}{\partial\varphi},\frac{1}{a\cos\varphi}\frac{\partial\Psi}{\partial\lambda}\right)-\frac{\beta}{a}\frac{\partial\Psi}{\partial\varphi}$$
$$+\frac{1}{a^2}\left(1+(\tan\varphi)\frac{\partial}{\partial\varphi}\right)\left[\left(\frac{1}{a}\frac{\partial\Psi}{\partial\varphi}\right)^2+\left(\frac{1}{a\cos\varphi}\frac{\partial\Psi}{\partial\lambda}\right)^2\right],\quad (2.2)$$

is suitable from the point of view of accuracy over major portions of the globe. (Excluded are the polar points, where the equation becomes singular.)

3. Solution of the standard balance equation

In this section we will solve (2.2) for Ψ, assuming p and ρ to be known. This has been the traditional way to obtain a balanced nondivergent wind. Although this procedure will be shown to be theoretically unreasonable over the globe, it is instructive to see just how it fails in a practical sense.

The Monge-Ampère equation (2.2) having the stream function as its unknown is subject to a constraint in order to yield a valid solution for the boundary value problem (Rellich, 1932; Lewy, 1937). This constraint for (2.2) is

$$\frac{1}{\rho}\nabla^2 p+\frac{f^2}{2}-\frac{\beta}{a}\frac{\partial\Psi}{\partial\varphi}$$
$$+\frac{1}{a^2}\left[\left(\frac{1}{a}\frac{\partial\Psi}{\partial\varphi}\right)^2+\left(\frac{1}{a\cos\varphi}\frac{\partial\Psi}{\partial\lambda}\right)^2\right]\geq 0,\quad (3.1)$$

as shown by Houghton (1968). With the exception of the spherical terms, (3.1) corresponds to the ellipticity condition derived by Bolin (1955). Scale analysis shows that the spherical terms will be of the order Ro$(L/a)^2$. Even at the equator, a velocity in excess of the speed of sound is required if these terms are to approach the magnitude of the beta term. Thus, we see that the spherical terms will do little to alleviate the severity of the ellipticity condition at the equator where $f\to 0$.

To solve (2.2) where $f=0$ in a manner that offers no mathematical difficulty, we rewrite it in the form suggested by Petterssen (1953), i.e.,

$$\nabla^2\Psi=-f\pm\left[f^2+\frac{2}{\rho}\nabla^2 p-2\frac{\beta}{a}\frac{\partial\Psi}{\partial\varphi}+A^2+B^2+2K^2\right]^{\frac{1}{2}}, \quad (3.2)$$

where

$$A=-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{1}{a}\frac{\partial\Psi}{\partial\varphi}\right)$$
$$-\frac{1}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial\Psi}{\partial\lambda}\right)-\frac{\tan\varphi}{a^2\cos\varphi}\frac{\partial\Psi}{\partial\lambda},$$

$$B=\frac{1}{a^2\cos^2\varphi}\frac{\partial^2\Psi}{\partial\lambda^2}-\frac{1}{a^2}\frac{\partial^2\Psi}{\partial\varphi^2}-\frac{\tan\varphi}{a^2}\frac{\partial\Psi}{\partial\varphi},$$

$$K^2=\frac{1}{a^2}\left[\left(\frac{1}{a}\frac{\partial\Psi}{\partial\varphi}\right)^2+\left(\frac{1}{a\cos\varphi}\frac{\partial\Psi}{\partial\lambda}\right)^2\right].$$

If the ellipticity condition (3.1) is always satisfied, then the expression under the radical in (3.2) is positive.

Eq. (3.2) shows that there are actually two solutions to the balance equation. In order to obtain the solution that most closely fits the observed conditions (positive absolute vorticities in the Northern Hemisphere and negative absolute vorticities in the Southern Hemisphere), we use the plus sign in the Northern Hemisphere and the minus sign in the Southern. At the equator the situation remains ambiguous. In order to eliminate this ambiguity and to provide a smooth transition between hemispheres, we take the radical to be 0, i.e., $\nabla^2\Psi=0$. This condition, while physically restrictive because it implies zero wind vorticity at the equator, is necessary for mathematical reasons.

Eq. (3.2) may be written in the form

$$\nabla^2\Psi=F(f,p,\rho,\Psi), \quad (3.3)$$

where F represents the right-hand side of (3.2). This equation is solved in a domain with no boundaries. Therefore, the method of solution for the elliptic equation must be altered. It might be thought that the poles, which are singularities in the coordinate system, should be taken as boundaries for the mathematical development. However, physical considerations suggest that only the condition of smoothly varying variables must be satisfied across the poles. Vorticity at a pole can be defined in generalized form by

$$\nabla^2\Psi=\frac{\cos\varphi}{(1-\sin\varphi)a^2(\pi/2-\varphi)}\times\left[\frac{1}{2\pi}\int_0^{2\pi}\Psi(\lambda,\varphi)d\lambda-\Psi_{\text{pole}}\right], \quad (3.4)$$

as $\varphi\to\pi/2$, where Ψ_{pole} is the value of the stream function at the pole. Eq. (3.4) is found by integrating the spherical Laplacian operator in finite difference form after applying Stokes' theorem to convert it to a line integral.

If the variables are to be continuous, the vorticity $\nabla^2\Psi$ must also be continuous. From (3.3) we see that F must likewise be continuous. This gives the definition for F at the pole as

$$F_{\text{pole}}=\frac{1}{2\pi}\int_0^{2\pi}F(\lambda,\varphi)d\lambda, \quad (3.5)$$

as $\varphi\to\pi/2$. Substituting (3.5) and (3.4) into (3.3) gives the appropriate formula for Ψ_{pole}, a formula consistent with the considerations previously mentioned:

$$\Psi_{\text{pole}}=\frac{1}{2\pi}\int_0^{2\pi}\Psi(\lambda,\varphi)d\lambda-\left[\frac{(1-\sin\varphi)a^2(\pi/2-\varphi)}{2\pi\cos\varphi}\right]\int_0^{2\pi}F(\lambda,\varphi)d\lambda. \quad (3.6)$$

Eq. (3.6) gives the value of Ψ_{pole} in terms of variables, at a distance $(\pi/2-\varphi)$ from the pole, which can be used in the balance equation. Thus, the entire globe can be covered using the balance equation.

In place of boundary conditions, the global domain requires an integral constraint. If we integrate Eq. (3.3) over the entire globe, the left-hand side is equal to 0. This means that

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}Fa^2\cos\varphi d\lambda d\varphi=0 \quad (3.7)$$

is a requirement for F.

It is easy to see the implications if (3.7) is not satisfied. Let us solve (3.3) by a simultaneous relaxation method (Thompson, 1961), i.e.,

$$\Psi^{\nu+1}=\Psi^{\nu}+\tfrac{1}{4}(\nabla^2\Psi^{\nu}-F^{\nu})a^2(\cos\varphi)\Delta\varphi\Delta\lambda, \quad (3.8)$$

where ν denotes iteration stages. We sum up (3.8) over the entire sphere to find the average stream function $\bar{\Psi}$ as

$$\bar{\Psi}^{\nu+1}=\bar{\Psi}^{\nu}-\tfrac{1}{4}\sum_{\substack{\text{whole}\\\text{globe}}}F^{\nu}a^2(\cos\varphi)\Delta\varphi\Delta\lambda. \quad (3.9)$$

The summation corresponds to the integral (3.7). If it is not zero, then $\bar{\Psi}^{\nu+1}\neq\bar{\Psi}^{\nu}$, i.e., there is a drift in the average value of Ψ. This will not affect gradients, but it may prevent convergence if the difference, $\Delta\bar{\Psi}=\bar{\Psi}^{\nu+1}-\bar{\Psi}^{\nu}$, happens to exceed the convergence criterion. For the usual Liebmann or sequential iteration method, it is not easy to determine the magnitude of this drift, but it appears likely that drifting will occur if (3.7) or its finite difference analog is not satisfied.

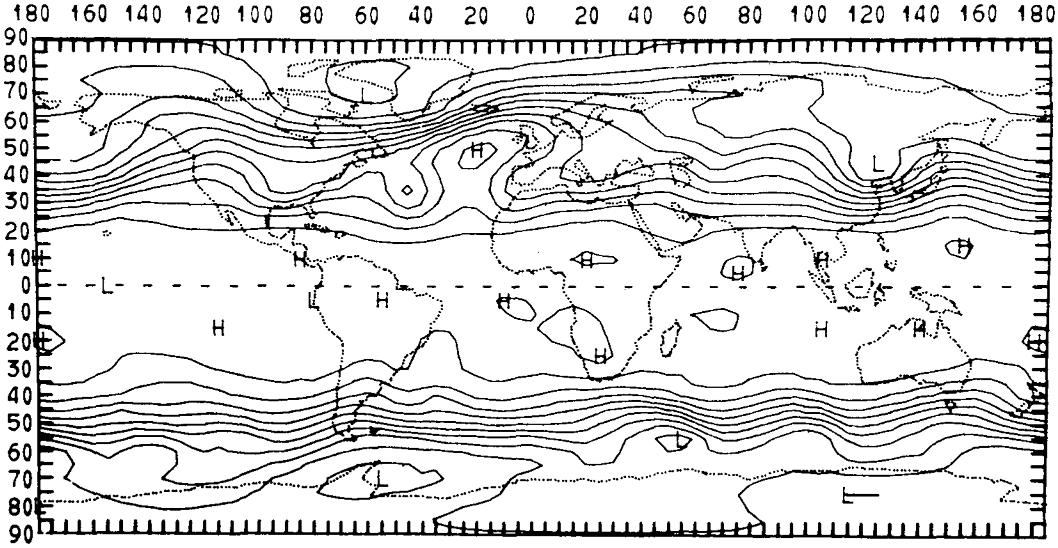

FIG. 1. The pressure field at 9 km on 15 January 1958, obtained by averaging the observed values at 6 and 12 km. Contours are drawn at 5-mb intervals.

Solution (3.3) was attempted using real global pressure data for 15 January 1958 (Fig. 1). The forcing function F was recomputed every five scans or cycles. At the same time, a constant was subtracted from all the values of F, except the equatorial value, in order to conform to the finite difference analog of (3.7). The value of this constant never exceeded 10^{-6} sec^{-1}, one order of magnitude smaller than the typical value of F. The Liebmann method of relaxation was used, and values at the poles were computed by (3.6).

Convergence was obtained after 295 cycles when $\Delta\Psi_{\max}/\Psi_{\max}<10^{-3}$. The final stream function field is shown in Fig. 2. Note that although in the middle and high latitudes the stream function field appears reasonable, the equatorial region stream function field implies a broad easterly flow, with velocities nearly as large as these of the jet streams in the middle latitudes. This easterly flow is a region of low vorticity magnitudes.

In the process of obtaining the solution, the ellipticity condition required lowering the pressure values in the equatorial region. Hence, the final pressure field (Fig. 3) has a general equatorial trough shown more clearly by the difference in pressure from the initial to the final state (Fig. 4).

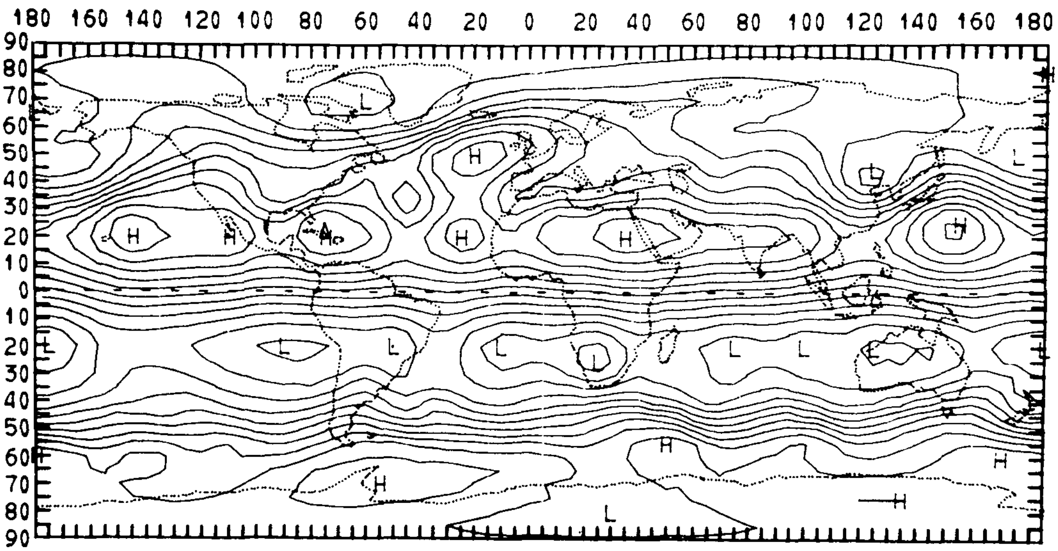

FIG. 2. The stream function field, obtained by the standard balance equation. Contours are drawn at intervals of 1×10^{11} cm^2 sec^{-1}.

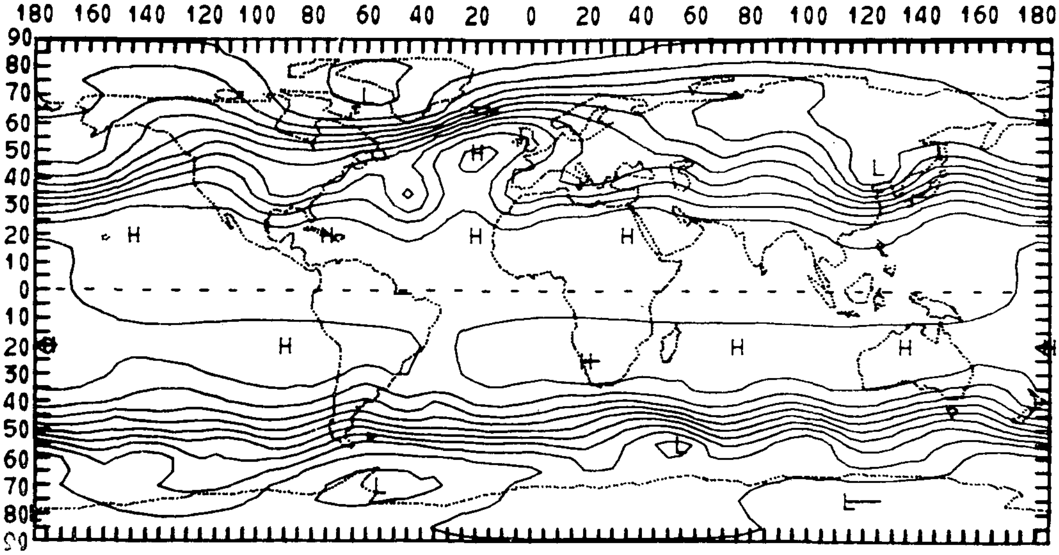

FIG. 3. The final pressure field, after solution of the standard balance equation. Contours are drawn at 5-mb intervals.

It is apparent that although a solution using real global pressure data is possible, the mathematical constraints render it physically unrealistic in the lower latitudes and, therefore, unsatisfactory for a global model. As will be shown in Section 4, the mathematical solution becomes physically unreasonable at the latitudes where the method itself cannot be justified theoretically.

4. Theoretical discussion

It is evident from Section 3 that solving the balance equation from pressure data is not satisfactory for the tropics. Because of the geostrophic relationship, small errors in pressure gradients near the equator may yield large geostrophic winds. An alternative in solving the balance equation for a global model is to use the velocity field to solve for the pressure field. While pressures may be more satisfactory as input data in the middle and high latitudes, winds (and, in particular, vorticity) seem the better source of data in the tropics. Thus, a mixed method seems the reasonable approach for a physically realistic global model. The important question, of course, is at what latitude it is best to change from one data source to the other. We shall examine this problem by using the atmospheric adjustment theory.

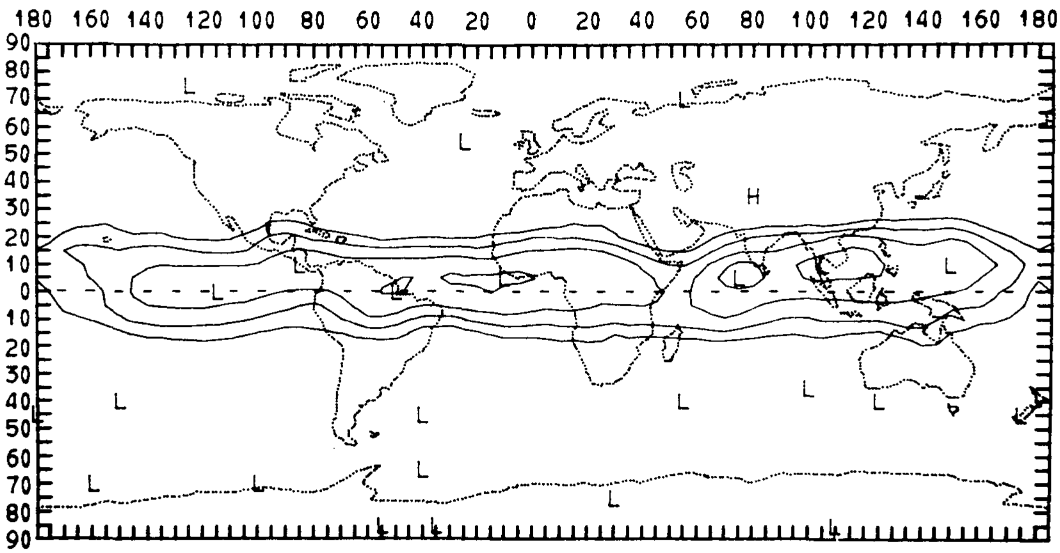

FIG. 4. The pressure change effected by the solution of the standard balance equation. Contours are drawn at 2-mb intervals, starting at ± 2mb.

For primitive equation models, the adjustment process is fundamental to the problem of initialization. In these types of models not only are the large-scale motions present, but also gravity-inertial motions. There is no clear-cut separation between the geostrophic and gravity-inertial modes, except possibly by period. The scale analyses by Charney (1947), Burger (1958) and Phillips (1963) have shown that atmospheric long waves have a period on the order of four days or longer, whereas gravity-inertial waves have a period of half a pendulum day or less. If either the unmodified pressure field or the wind field is used as initial data, unrealistic large-amplitude gravity-inertial waves can be created. These unwanted large-amplitude waves were first observed by Richardson (1922) in his ill-fated numerical integration of primitive equations, and later by Hinkelmann (1951). Platzman (1967), in his recent review of Richardson's work, reaffirmed Richardson's own conclusion that the reason for the poor forecast was that wind and pressure field observations were not sufficiently accurate to be introduced without modification into a primitive equation model. An example of modification of initial data is the use of certain wind approximations, for example, geostrophic winds, or balanced winds (Charney, 1955). An entire hierarchy of wind approximations has been outlined in articles by Hinkelmann (1962), Fjørtoft (1962), Hollman (1966) and Ellsaesser (1968).

The problem of suppressing the amplitude of unwanted gravity-inertial waves can be explored using the theory of atmospheric adjustment. The equations for the adjustment process in the homogeneous barotropic atmosphere have been discussed by Obukhov (1949) for an infinite plane. The infinite-plane approximation is most valid for motions with length scales shorter than the radius of the earth. For larger scales, spherical geometry should be taken into account. In this case the relevant equations are quite similar to those presented in Siebert's (1961) review of tidal and tsunamis theory. Another approach, not discussed here, is to include the beta-plane approximation (Jacobs, 1967).

The gravity waves manifest themselves almost entirely in the irrotational part of the fluid motion, whereas the quasi-geostrophic motions are almost entirely rotational. Therefore, we would expect that the potential vorticity would relate only to the large-scale motions. The potential vorticity for a linear barotropic atmosphere has the form (see Obukhov, 1949)

$$\zeta-\frac{fh}{H}, \tag{4.1}$$

where ζ is the relative vorticity, f the Coriolis parameter, h the height deviation of the free surface, and H the mean height. Phillips (1963), in his survey of geostrophic motion, shows that this is a Type 1 motion, characterized by small changes in potential vorticity.

In a one-dimensional model the relative vorticity can be expressed as a second derivative with respect to x of the stream function which uniquely determines the relationship between Ψ and h. Furthermore, in the steady state, a geostrophic relationship can be assumed, i.e.,

$$\Psi=\frac{gh}{f}. \tag{4.2}$$

Therefore, by combining (4.1) and (4.2), the equation relating the initial and final states of adjustment becomes

$$\frac{\partial^2\Psi_s}{\partial x^2}-\frac{f^2}{gH}\Psi_s=\frac{\partial^2\Psi_i}{\partial x^2}-\frac{fh_i}{H}. \tag{4.3}$$

In Eq. (4.3) the subscript s refers to the adjusted state and i to the initial state. The factor multiplying the second term of the left-hand side of (4.3) is related to the radius of deformation defined by Rossby (1938), $L_c=\sqrt{gH}/f$.

We now define L_c in a more general way as

$$L_c=\frac{c}{f}, \tag{4.4}$$

where c is a wave speed equal to $\sqrt{gH}$ for this one-layer model. If we substitute the harmonic solution of the form $\exp(ix/L)$ into (4.3), where L is a horizontal length scale, then, following Washington (1964), we find, for motions with length scales much larger than L_c (i.e., $L\gg L_c$), that

$$\Psi_s\simeq\frac{g}{f}h_i. \tag{4.5}$$

This implies that the wind field (at least the rotational component) adjusts to the initial height field. In the opposite case, where $L\ll L_c$,

$$\Psi_s\simeq\Psi_i \quad \text{and} \quad h_s\simeq\frac{\Psi_i f}{g}. \tag{4.6}$$

This shows that the atmosphere adjusts to the initial stream function field. From the above analysis we see that we should use the initial pressure if $L>L_c$ and the initial wind if $L<L_c$, where L is the length scale of the motions dominant in the adjustment process. Presumably this L is less than or equal to a typical synoptic scale.

A theoretical indication of the latitude which divides the region where pressure should be used from the region where wind should be used may be derived from (4.4). If we assume that $L=L_c$ and substitute $2\omega\sin\varphi$

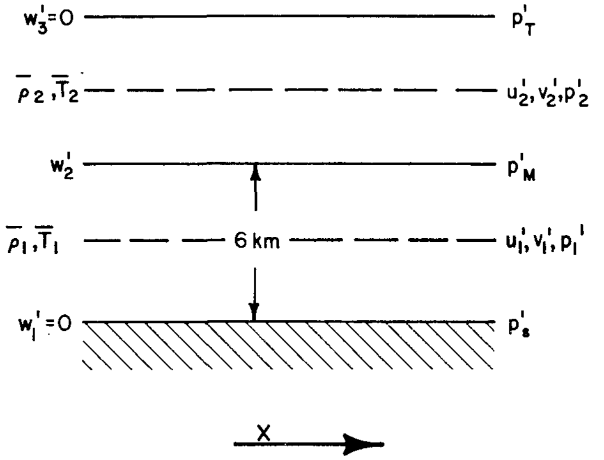

FIG. 5. The two-layer linearized model. Symbols are defined in the text.

for f, then

$$\varphi=\sin^{-1}\left[\frac{c}{2\omega L_c}\right], \tag{4.7}$$

gives the dividing latitude, where $\omega=7.29\times10^{-5}$ sec^{-1}.

In order to find a latitude appropriate for the NCAR general circulation model, we first establish the relevant wave speeds and wavelengths for the adjustment process. The wavelength is then equated to L_c in (4.7). We assume that the wave speeds needed in (4.7) correspond to the eigenmodes of the model without rotation. Since the two-layer model has a rigid upper boundary, we can expect free modes to exist.

We obtain representative values of c by the following linear analysis of the two-layer general circulation model. This analysis is similar in many respects to that given by Kasahara and Washington (1967), except that we assume no mean motion and we allow for different sound speeds in each of the two layers. Fig. 5 shows the vertical placement of the variables: pressure and vertical velocity are defined at 0, 6 and 12 km; while density, temperature, and horizontal wind components are defined at 3 and 9 km. We find the pressure at 3 and 9 km by linear interpolation of the pressure at 0, 6 and 12 km. The horizontal equations of motion are:

$$\frac{\partial u_1'}{\partial t}=-\frac{1}{\bar\rho_1}\frac{\partial p_1'}{\partial x}+fv_1', \tag{4.8a}$$

$$\frac{\partial v_1'}{\partial t}=-fu_1', \tag{4.8b}$$

$$\frac{\partial u_2'}{\partial t}=-\frac{1}{\bar\rho_2}\frac{\partial p_2'}{\partial x}+fv_2', \tag{4.8c}$$

$$\frac{\partial v_2'}{\partial t}=-fu_2'. \tag{4.8d}$$

The primes denote deviations from a basic state, where $\bar u=\bar v=0$, $p'=p-\bar p(z)$, and $\rho'=\rho-\bar\rho(z)$. In Kasahara and Washington (1967) a proof is given which shows that terms involving ρ' can be ignored in the equations of motion. The pressure tendency and Richardson's equations are:

$$\frac{\partial p_1'}{\partial t}=\frac{\partial p_T'}{\partial t}+g\left(\frac{\bar\rho_2+\bar\rho_1}{2}\right)\frac{w_2'}{2}-\frac{g\Delta z}{2}\bar\rho_1\frac{\partial u_1'}{\partial x}-g\Delta z\bar\rho_2\frac{\partial u_2'}{\partial x}, \tag{4.9a}$$

$$\frac{\partial p_2'}{\partial t}=\frac{\partial p_T'}{\partial t}+g\left(\frac{\bar\rho_2+\bar\rho_1}{2}\right)\frac{w_2'}{2}-\frac{g\Delta z}{2}\bar\rho_2\frac{\partial u_2'}{\partial x}, \tag{4.9b}$$

$$\bar\rho_1\frac{w_2'}{\Delta z}=-\bar\rho_1\frac{\partial u_1'}{\partial x}-\frac{1}{\sigma_1^2}\times\left[-g\Delta z\bar\rho_2\frac{\partial u_2'}{\partial x}-g\frac{\Delta z}{2}\bar\rho_1\frac{\partial u_1'}{\partial x}+\frac{\partial p_T'}{\partial t}\right], \tag{4.10a}$$

$$-\bar\rho_2\frac{w_2'}{\Delta z}=-\bar\rho_2\frac{\partial u_2'}{\partial x}-\frac{1}{\sigma_2^2}\left[-\frac{g\Delta z}{2}\bar\rho_2\frac{\partial u_2'}{\partial x}+\frac{\partial p_T'}{\partial t}\right], \tag{4.10b}$$

where $\sigma_i=(\gamma RT_i)^{\frac{1}{2}}$, the Laplacian sound speed for each layer, and γ is the ratio of specific heats, C_p/C_v. If we eliminate $\partial p_T'/\partial t$ and w_2' from (4.9) and (4.10), we find that

$$\frac{\partial p_1'}{\partial t}=A_1\bar\rho_1D_1+A_2\bar\rho_2D_2, \tag{4.11a}$$

$$\frac{\partial p_2'}{\partial t}=B_1\bar\rho_1D_1+B_2\bar\rho_2D_2, \tag{4.11b}$$

where

$$D_1=\frac{\partial u_1'}{\partial x},$$

$$D_2=\frac{\partial u_2'}{\partial x},$$

$$A_1=-g\frac{\Delta z}{2}-\frac{g\Delta z}{4}\left(1-\frac{g\Delta z}{2\sigma_1^2}\right)-\left[\frac{g\Delta z}{4}\left(\frac{1}{\sigma_2^2}-\frac{1}{\sigma_1^2}\right)+1\right]\frac{\bar\rho_2\left(1-\dfrac{g\Delta z}{2\sigma_1^2}\right)}{\left(\dfrac{\bar\rho_2}{\sigma_1^2}+\dfrac{\bar\rho_1}{\sigma_2^2}\right)},$$

$$A_2=-g\Delta z+\frac{g\Delta z}{4}\left(1-\frac{g\Delta z}{2\sigma_2^2}+\frac{g\Delta z}{\sigma_1^2}\right)$$

$$+\left[\frac{g\Delta z}{4}\left(\frac{1}{\sigma_2^2}-\frac{1}{\sigma_1^2}+1\right)\right]\frac{\bar{p}_1\left(\frac{g\Delta z}{2\sigma_2^2}-1\right)+\frac{g\Delta z}{\sigma_1^2}\bar{p}_2}{\left(\frac{\bar{p}_2}{\sigma_1^2}+\frac{\bar{p}_1}{\sigma_2^2}\right)},$$

$$B_1=A_1+\frac{g\Delta z}{2},$$

$$B_2=A_2+\frac{g\Delta z}{2}.$$

If we assume wave solutions of the form $\exp[i\alpha(x-c't)]$, where α is the horizontal wavenumber $2\pi/L$, then (4.8), (4.9) and (4.11) give the frequency equation

$$c'=\pm\left\{\frac{-(B_2+A_1)\pm[(B_2-A_1)^2+4A_2B_1]^{\frac{1}{2}}}{2}+\frac{f^2}{\alpha^2}\right\}^{\frac{1}{2}}. \tag{4.12}$$

We now ignore the inertial term in (4.12), f^2/α^2, to find c for Eq. (4.7). If the atmosphere is isothermal, with $\bar{T}_1=\bar{T}_2=255$K, then $c=\pm320$ m sec^{-1} and ±52.7 m sec^{-1}. If $\bar{T}_1=275$K and $\bar{T}_2=235$K, which is close to the stability of the standard atmosphere, $c=\pm322$ m sec^{-1} and ±27.0 m sec^{-1}. We see that there is a class of fast-moving Lamb waves (Lamb, 1945) and a class of slow-moving internal gravity waves. Both, of course, can take part in the adjustment process. Benwell and Bretherton (1968) discuss a corresponding family of waves for a 10-level model.

Fig. 6 gives the latitude at which either winds or pressures are equally valid as input data ($L=L_c$) for a given length scale L and wave speed c. Since the Lamb wave has such a large wave speed, L/L_c is always less than unity for synoptic-scale motions. This means that the model atmosphere adjusts to the initial non-divergent velocity field. The internal gravity wave, however, can give $L/L_c<1$ or $L/L_c>1$, depending on L and the latitude. Analysis of wave motion in the NCAR primitive equation model indicates that the Lamb mode is excited much less than the internal gravity mode. Therefore, we assume that the L_c relevant to this study is the one associated with the internal gravity wave.

In order to estimate a reasonable critical latitude, we determine bounds for the principal region of interest in Fig. 6. An internal gravity wave speed equal to 52.7 m sec^{-1} is a reasonable upper limit for c, and zero the lower limit (adiabatic atmosphere). Reasonable limits for L are between 1000 km, the synoptic scale, and 200 km, the smallest mesh size on the globe for the

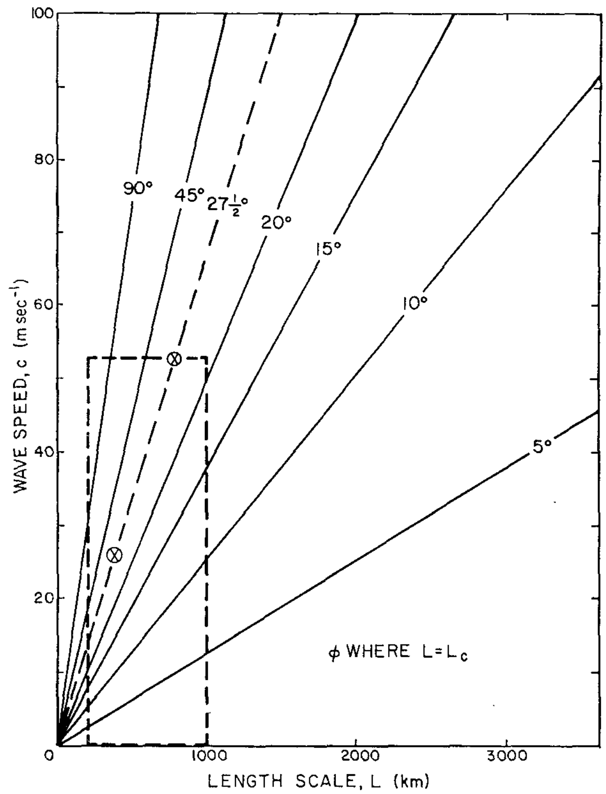

FIG. 6. The latitude where the given length scale equals the critical length scale. The abscissa is the length scale and the ordinate is the wave speed. The heavy dashed line encloses the region of interest discussed in the text. The latitude $\varphi=27\frac{1}{2}°$ is the one used in the mixed solution. The two crossed circles show the internal gravity wave speeds obtained for the isothermal and standard atmosphere lapse rates.

NCAR model. (The length scale is assumed to be one-fourth of the wavelength.)

We need to prove that the wave speeds c derived from linear analysis without the inertial term are identical to those found from potential vorticity arguments and used in Eq. (4.4), the definition for L_c. This can be done by first forming the vorticity equation from (4.8) and then eliminating the horizontal divergences, D_1 and D_2, from (4.11). This yields

$$\frac{\partial\Omega_{1,2}}{\partial t}=0, \tag{4.13}$$

where

$$\Omega_1=\frac{\partial^2\Psi_1}{\partial x^2}+\frac{f(B_2p_1'-A_2p_2')}{\bar{p}_1(A_1B_2-A_2B_1)}, \tag{4.14a}$$

$$\Omega_2=\frac{\partial^2\Psi_2}{\partial x^2}-\frac{f(B_1p_1'-A_1p_2')}{\bar{p}_2(A_1B_2-A_2B_1)}. \tag{4.14b}$$

Following the same procedure as with the one-layer analysis, we assume that in the final steady state a geostrophic relationship holds, i.e.,

$$\Psi_{1,2}=\frac{p'_{1,2}}{f\bar{\rho}_{1,2}}. \tag{4.15}$$

If (4.15) is substituted into (4.14), then

$$\Omega_1=\frac{\partial^2\Psi_{s1}}{\partial x^2}+f^2\left[\frac{B_2\Psi_{s1}-A_2\Psi_{s2}(\bar{\rho}_2/\bar{\rho}_1)}{A_1B_2-A_2B_1}\right], \tag{4.16a}$$

$$\Omega_2=\frac{\partial^2\Psi_{s2}}{\partial x^2}-f^2\left[\frac{B_1\Psi_{s1}(\bar{\rho}_1/\bar{\rho}_2)-A_1\Psi_{s2}}{A_1B_2-A_2B_1}\right]. \tag{4.16b}$$

The initial potential vorticities are Ω_1 and Ω_2, and the subscripts $s1$ and $s2$ refer to the final state values of layers 1 and 2, respectively. The same general properties hold here as in the one-layer model. The coefficient of the second terms in the right-hand side of (4.16) is inversely proportional to a length scale. We can find the critical length scale in the same manner as for the one-layer case by equating the coefficients of the unknowns Ψ_{s1} and Ψ_{s2} in nondifferentiated terms of (4.16) to $-1/L_c^2$. This gives a matrix equation for L_c^2 of the form

$$\frac{f^2}{(A_1B_2-A_2B_1)}\begin{pmatrix} B_2 & -A_2(\bar{\rho}_2/\bar{\rho}_1) \\ -B_1(\bar{\rho}_1/\bar{\rho}_2) & A_1 \end{pmatrix} = \begin{pmatrix} -1/L_c^2 & 0 \\ 0 & -1/L_c^2 \end{pmatrix}. \tag{4.17}$$

Therefore,

$$L_c=\left|\frac{[2(A_1B_2-A_2B_1)]^{\frac{1}{2}}}{f\{-(B_2+A_1)\pm[(B_2-A_1)^2+4A_2B_1]^{\frac{1}{2}}\}^{\frac{1}{2}}}\right|. \tag{4.18}$$

By comparing (4.18) with (4.4), we see that the wave speeds are now defined by

$$c=\left|\left\{\frac{2(A_1B_2-A_2B_1)}{-(B_2+A_1)\pm[(B_2-A_1)^2+4A_2B_1]^{\frac{1}{2}}}\right\}^{\frac{1}{2}}\right|. \tag{4.19}$$

This gives the same wave speeds as (4.12), but without the rotation term.

5. A mixed method of solution for the balance equation

The results of Section 3 indicate that significant pressure changes occur as far north as 25N and as far south as 20S (Fig. 4). Thus, setting the limiting latitude at 27.5N and S completely encompasses the equatorial belt, where questionable results are obtained if pressure data are used as input. The theoretical analysis of Section 4 also suggests that 27.5° is a reasonable latitude to separate the equatorial domain, where wind data should be used as input, from regions of higher latitudes, where pressure data are more satisfactory. If the typical lapse rate is considered to be that of the standard atmosphere, the appropriate adjustment wave speed c for the two-layer model is 27.0 m sec^{-1}. This c, together with a dominant length scale of 500 km for the adjustment motions, gives approximately 27.5° for the separation latitude. (Note the lower crossed circle in Fig. 6.) The length scale of 500 km is reasonable for adjustment motions (ageostrophic motions) and is resolvable by the numerical model. Thus, with justification from both Sections 3 and 4, division latitudes of 27.5N and S were used in the mixed solution.

The balanced solution is obtained as follows. The solution for Ψ from 30S to 90S and from 30N to 90N is obtained as in Section 3, using Eq. (3.1) with the observed pressure, i.e., the F in (3.3) is given by

$$F=-f\pm\left[f^2+\frac{2}{\rho}\nabla^2p-2\frac{\beta}{a}\frac{\partial\Psi}{\partial\varphi}+A^2+B^2+2K^2\right]^{\frac{1}{2}}. \tag{5.1}$$

For the belt 25S to 25N the stream function is derived from the observed wind using the identity relationship

$$\nabla^2\Psi=\zeta, \tag{5.2}$$

i.e., the F in (3.3) is given by

$$F=\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a}\frac{\partial u}{\partial\varphi}. \tag{5.3}$$

In this manner Eq. (3.3) can be defined as before for the whole sphere and solved for Ψ as described in Section 3.

With this completed, the pressure in the belt 25S to 25N must be determined from the Ψ field to satisfy the balance equation. We write (2.3) with p as the unknown and assume the density ρ to be the same as the observed, i.e.,

$$\nabla^2p=\rho f\nabla^2\Psi-2\rho J\left(\frac{1}{a}\frac{\partial\Psi}{\partial\varphi},\frac{1}{a\cos\varphi}\frac{\partial\Psi}{\partial\lambda}\right)+\frac{\rho\beta}{a}\frac{\partial\Psi}{\partial\varphi}-\frac{\rho}{a^2}\left(1+\tan\varphi\frac{\partial}{\partial\varphi}\right)\times\left[\left(\frac{1}{a}\frac{\partial\Psi}{\partial\varphi}\right)^2+\left(\frac{1}{a\cos\varphi}\frac{\partial\Psi}{\partial\lambda}\right)^2\right]. \tag{5.4}$$

Eq. (5.4) is of the form

$$\nabla^2p=F(f,\rho,\Psi),$$

and is much simpler to solve than Eq. (3.2) since the forcing function does not depend on the unknown and the equation is always of the elliptic type. Eq. (5.4) is solved for p from 25S to 25N using the observed values of p at 30S and 30N as the boundary values.

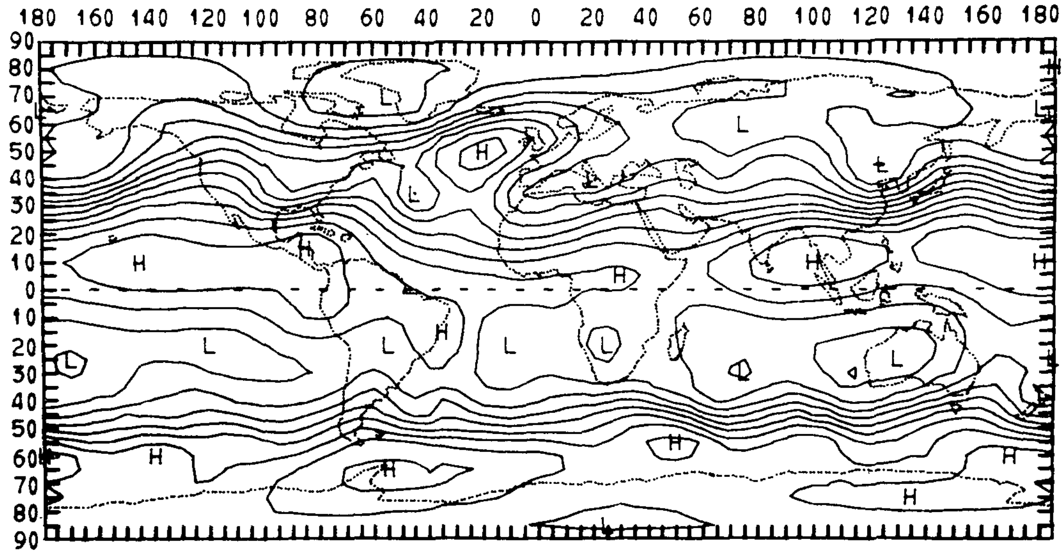

Fig. 7. Solution for the stream function field obtained by the mixed method. Contours are drawn at intervals of 1×10^{11} cm^2 sec^{-1}.

The mixed method of solution was performed using the same pressure field as in the case of the standard solution for middle and high latitudes (Fig. 1), but using the observed vorticity field for the equatorial belt. The solution for Ψ is shown in Fig. 7. The results in middle and high latitudes are very similar to those shown in Fig. 2. However, the easterly jet at the equator in Fig. 2 is absent from Fig. 7, and the velocity magnitudes in the latter case are now typically 1–15 m sec^{-1}, instead of $\sim$30 m sec^{-1} as in the earlier case. The final pressure field obtained by the mixed method is shown in Fig. 8. The pressure change from the initial state (Fig. 1) is shown in Fig. 9. While pressure changes obtained by the standard solution ranged from 0 to -9 mb (Fig. 4), the pressure changes using the mixed method now vary from $+1$ to -6 mb (Fig. 9). The magnitude of the pressure changes suggests that special consideration will be needed to determine the initial static stability, especially in a many layered model.

6. Remarks

This study indicates that a mixed method of solution of the balance equation is consistent with atmospheric

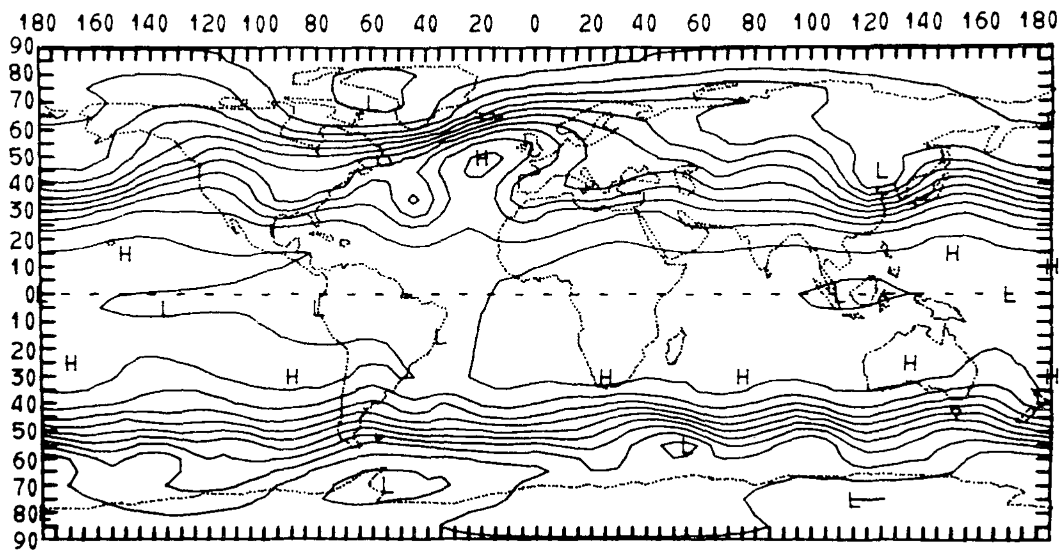

Fig. 8. The final pressure field, after solution by the mixed method. Contours are drawn at 5-mb intervals.

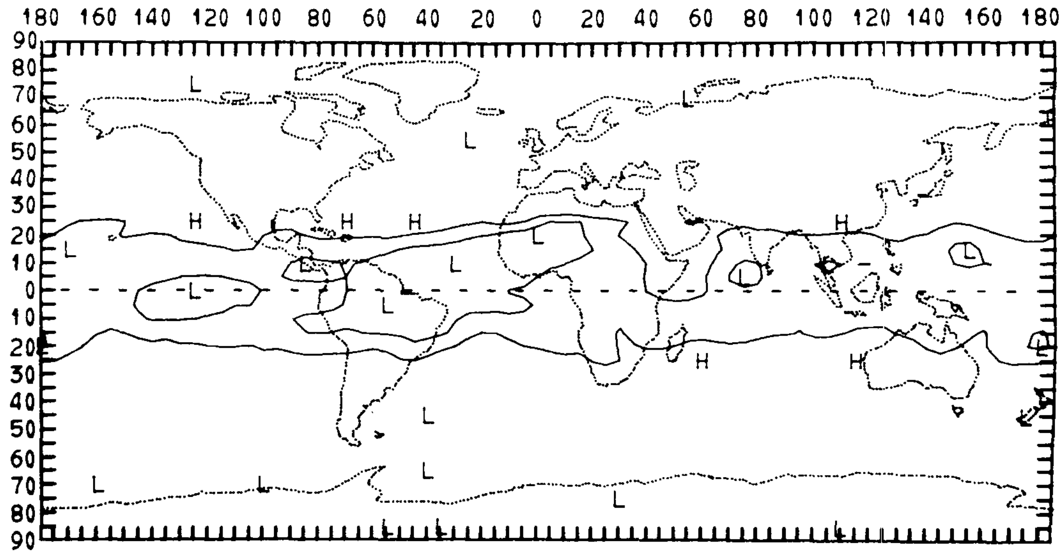

FIG. 9. The pressure change effected by the solution by the mixed method. Contours are drawn at 2-mb intervals, starting at ± 2 mb.

adjustment considerations and gives reasonable pressure and nondivergent velocity fields as final results. Of course, the most important test is to use the solutions in a prognostic general circulation model to see if the small-scale motions are suitably controlled and the large-scale motions accurate. This has been done with the NCAR general circulation model using not only results from the mixed method of solution presented here, but also other variants of the balance relationship. These experiments will be discussed in a future publication.

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