Reduction of Analysis Error Through Constraints of Dynamical Consistency

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ABSTRACT

A method is proposed whereby incorrect analyses at two successive times may be optimally adjusted to maintain dynamical consistency with a given prediction model. It is shown that, in the ensemble average, the mean-square error of the adjusted analyses is half that of the original analyses.

1. Introduction and general remarks

With the gradual realization that an effective but economically feasible global observing system will probably not provide measurements at geographically fixed locations or at standard synoptic times, increasing attention and effort have been given to the development of "dynamical" methods of data processing. Described generically, these are methods that capitalize on the information from a time sequence of data, coordinated by a dynamical prediction model. Among the various techniques that have been proposed, the most straightforward is a process of real-time prediction, in which the "prediction-analysis" is updated or modified to assimilate new data wherever and whenever it becomes available. It has already been shown, in fact, that such a procedure is capable of accurately reconstructing the current state of the atmosphere in large data-free "holes," if the surrounding areas are well covered with a long time-sequence of observations and if the prediction model is physically realistic.

There are clearly at least three questions related to the problem of data assimilation at a *single* time stage. One is that of smoothing out random errors of observation, which can be dealt with very well by the application of the Wiener rms filter and its later generalizations in two and three dimensions. Another is that of "fairing in" a cluster of new observations with a prediction made from earlier data. Finally, there are fundamental and still unresolved questions as to how the wind and pressure fields should be artifically balanced or adjusted in order to suppress the spurious growth of gravity-inertial oscillations.

Let us assume for the moment, however, that all of these questions will be satisfactorily resolved, and that we can reconstruct the current state with fair accuracy from earlier predictions modified by current data.

In instances when "patches" of new data agree with the prediction made from earlier data, one would have no basis for modifying the prediction, considered as an approximation to the current state. Moreover, if the difference between prediction and current observation were *less* than the expected error of prediction, one would certainly conclude that the model is deficient and alter the prediction to conform with the data. On the other hand, if that difference were substantially *greater* than the expected error of prediction, one would strongly suspect that the observations or the analysis based on new data were in error and alter the analysis. This is not a novel point of view, since highly skilled analysts try to impose time continuity between successive analyses on the basis of earlier expectations, particularly in cases when the analysis may be very inaccurate—either because the observations are systematically in error, or because the analysis is based on fragmentary data from a large area.

Stated in rather general terms, the purpose of this paper is to propose a method by which a time-sequence of analyses may be optimally adjusted to maintain dynamical consistency, in instances when two successive analyses are clearly at odds with the prediction model. This procedure is designed to give perfect agreement at the time midway between two successive analyses, but with minimum adjustment of both analyses. Under the assumption that the error of prediction is negligible in comparison with the error of analysis, the mean square error of the adjusted analysis is half that of the original analysis in the ensemble average.

2. A criterion of dynamical consistency

For the sake of illustration, we shall start by formulating a rather simple but nontrivial problem. We suppose that the motions of the atmosphere are governed by the vorticity equation for nondivergent barotropic flow, a tractable and fairly realistic model of

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true atmospheric flows, of the form

$$\frac{\partial \zeta}{\partial t}+\mathbf{V}\cdot\nabla\zeta+\mathbf{V}\cdot\nabla f=0, \quad (1)$$

where $\mathbf{V}$ is the true horizontal velocity vector (with components u and v in the x and y directions, respectively), ζ the true vertical component of relative vorticity, $\partial v/\partial x-\partial u/\partial y$, ∇ the horizontal vector gradient, and f the Coriolis parameter. In addition, owing to the condition for nondivergence,

$$v=\frac{\partial\psi}{\partial x},\quad u=-\frac{\partial\psi}{\partial y},\quad \zeta=\nabla^2\psi,$$

where ψ is the stream function. One important implication of the relationships stated immediately above is that $\mathbf{V}$ and its components are essentially Green's integrals of ζ, and are therefore less subject to observational or analysis error than ζ itself. To simplify matters, in fact, we shall assume that $\mathbf{V}$ is free of error in the first approximation (but only where it enters undifferentiated) and that it does not vary appreciably over the interval between successive analyses.

Let us next imagine that the "prediction-analyses" at two successive times $(n-1)\Delta t$ and $(n+1)\Delta t$ yield vorticity fields Z_{n-1} and Z_{n+1}, respectively. If Δt were sufficiently small, and if Z_{n-1} and Z_{n+1} were without error, then our hypothesis would require that the finite-difference form of Eq. (1) be satisfied, i.e.,

$$\frac{Z_{n+1}-Z_{n-1}}{2\Delta t}+\mathbf{V}\cdot\nabla\left(\frac{Z_{n+1}+Z_{n-1}}{2}\right)+\mathbf{V}\cdot\nabla f=0. \quad (2)$$

In general, of course, Z_{n-1} and Z_{n-1} *are* in error, so that Eq. (2) is not satisfied. To the extent that Eq. (2) is *not* satisfied, Z_{n-1} and Z_{n+1} are not dynamically consistent.

Let us next suppose that the error-contaminated fields Z_{n-1} and Z_{n+1} are adjusted by amounts δ_{n-1} and δ_{n+1}, in such a way that the forecast for $t=n\Delta t$ made from the adjusted analysis at $t=(n-1)\Delta t$ agrees with the hindcast for $t=n\Delta t$ made from the adjusted analysis at $t=(n+1)\Delta t$. This is the condition of dynamical consistency. For small Δt the finite-difference formulation of the consistency condition is

$$Z_{n-1}+\delta_{n-1}-\Delta t\mathbf{V}\cdot\nabla(Z_{n-1}+\delta_{n-1})-\Delta t\mathbf{V}\cdot\nabla f$$
$$=Z_{n+1}+\delta_{n+1}+\Delta t\mathbf{V}\cdot\nabla(Z_{n+1}+\delta_{n+1})+\Delta t\mathbf{V}\cdot\nabla f,$$

or, rearranging terms

$$\delta_{n-1}-\delta_{n+1}-\Delta t\mathbf{V}\cdot\nabla(\delta_{n-1}+\delta_{n+1})$$
$$=Z_{n+1}-Z_{n-1}+\Delta t\mathbf{V}\cdot\nabla(Z_{n+1}+Z_{n-1})+2\Delta t\mathbf{V}\cdot\nabla f.$$

Thus, by defining

$$D=\delta_{n+1}-\delta_{n-1},\quad S=\delta_{n+1}+\delta_{n-1},$$

we have

$$D=-(E+\Delta t\mathbf{V}\cdot\nabla S), \quad (3)$$

in which

$$E=2\Delta t\left[\left(\frac{Z_{n+1}-Z_{n-1}}{2\Delta t}\right)+\mathbf{V}\cdot\nabla\left(\frac{Z_{n+1}+Z_{n-1}}{2}\right)+\mathbf{V}\cdot\nabla f\right].$$

For small Δt it will be noted that E is a measure of the degree to which Z_{n+1} and Z_{n-1} are dynamically inconsistent [see Eq. (2)]. Clearly, if $E=0$, Z_{n+1} and Z_{n-1} are dynamically consistent, and no adjustment is required.

If E is not zero, however, the two adjustment fields δ_{n-1} and δ_{n+1} must be non-zero, but can be constructed in infinitely many ways under the single constraint stated in (3). The remaining question, therefore, is what additional constraint should be placed on the adjustments δ_{n-1} and δ_{n+1}, adjustments, which, up to this point, are not completely determined. In accordance with synoptic practice, it is reasonable to require that δ_{n-1} and δ_{n+1}, be the *least* adjustments that are necessary to satisfy the condition for dynamical consistency. In the case when Z_{n-1} and Z_{n+1} are expected to be equally subject to error, we simply impose the condition that the integral

$$I=\int_A(\delta_{n-1}{}^2+\delta_{n+1}{}^2)dA$$

be minimized over the areal domain A in question. The problem is now properly posed. We seek two fields δ_{n-1} and δ_{n+1} such that (3) is satisfied everywhere and such that the integral I is simultaneously minimized.

3. The variational problem

The condition for joint minimization of the adjustments δ_{n-1} and δ_{n+1} is more conveniently stated by noting that the integral I may also be written as

$$I=\frac{1}{2}\int_A(S^2+D^2)dA,$$

or, in view of the simple relation between S and D given by Eq. (3),

$$I=\frac{1}{2}\int_A[S^2+(E+\Delta t\mathbf{V}\cdot\nabla S)^2]dA.$$

We now require that S be such that I is minimized.

If S is, indeed, the function for which I is the least, then replacement of S by any other function must give values of I that are greater than the minimum value of I. In particular, let us replace S by the function $(S+\epsilon\eta)$, where ϵ is a variable parameter (either positive or negative), and η a continuous function of x and y that vanishes on the boundary of the domain under consideration, but is otherwise chosen arbitrarily. Thus, for any fixed choice of η, the integral

$$I(\epsilon)=\frac{1}{2}\int_A [(S+\epsilon\eta)^2+(E+\Delta t\mathbf{V}\cdot\nabla S+\epsilon\Delta t\mathbf{V}\cdot\nabla\eta)^2]dA$$

is a function of the parameter ϵ; moreover, since $I(\epsilon)$ attains its minimum value when $\epsilon=0$, $\partial I/\partial\epsilon=0$ when $\epsilon=0$. The latter is the necessary condition for an extremal of I.

Differentiating $I(\epsilon)$ with respect to ϵ and setting $\epsilon=0$, we find that

$$\frac{\partial I}{\partial\epsilon}(0)=\int_A [S\eta+(E+\Delta t\mathbf{V}\cdot\nabla S)(\Delta t\mathbf{V}\cdot\nabla\eta)]dA=0.$$

We next note that $\mathbf{V}\cdot\nabla\eta=\nabla\cdot\eta\mathbf{V}$, since $\nabla\cdot\mathbf{V}=0$. Thus, integrating by parts and making use of the fact that η vanishes on the boundary, we may put the equation above in the form

$$\int_A \eta[S-\Delta t\mathbf{V}\cdot\nabla(E+\Delta t\mathbf{V}\cdot\nabla S)]dA=0.$$

The integrand of the integral above consists of two distinct factors, one of which is the arbitarily specified function $\eta(x,y)$ and the other independent of η. There is only one way in which the integral above can vanish for all possible choices of $\eta(x,y)$; namely, when the factor in square brackets is zero. Thus,

$$S-(\Delta t)^2\mathbf{V}\cdot\nabla(\mathbf{V}\cdot\nabla S)=\Delta t\mathbf{V}\cdot\nabla E. \qquad (4)$$

This equation, taken together with appropriate boundary conditions, determines the distribution of S that minimizes I and at the same time satisfies the condition for dynamical consistency in the sense defined earlier. Once S has been determined by solving Eq. (4), D is easily found from Eq. (3), and the adjustments δ_{n-1} and δ_{n+1} are readily calculated from the definitions of S and D.

4. Solution of the extremal equation

On the face of it, Eq. (4) is a partial differential equation, linear but with variable and nonanalytic coefficients. We observe, however, that (4) may be written as

$$\ell\frac{\partial}{\partial s}\left(\ell\frac{\partial S}{\partial s}\right)-S=-\ell\frac{\partial E}{\partial s}, \qquad (5)$$

where $\ell(s)=\Delta t|\mathbf{V}|$, and a partial derivative with respect to s denotes differentiation with respect to distance along a streamline. Thus, (5) is an ordinary differential equation, in that it involves only derivatives along a single known set of coordinate lines, i.e., the streamlines.

Let us first consider the related homogeneous equation

$$\ell\frac{\partial}{\partial s}\left(\ell\frac{\partial\phi}{\partial s}\right)-\phi=0, \qquad (6)$$

or, in operator notation

$$\left(\ell\frac{\partial}{\partial s}+1\right)\left(\ell\frac{\partial}{\partial s}-1\right)\phi=0.$$

Thus, two solutions of Eq. (6) are given by the two linear first-order equations

$$\ell\frac{\partial\phi_1}{\partial s}+\phi_1=0, \qquad (7a)$$

$$\ell\frac{\partial\phi_2}{\partial s}-\phi_2=0, \qquad (7b)$$

which have solutions of the form

$$\left.\begin{aligned}\phi_1^{-1}(s)&=\exp\left[\int_0^s\frac{ds'}{\ell}\right]\\ \phi_2(s)&=\exp\left[\int_0^s\frac{ds'}{\ell}\right]\end{aligned}\right\}.$$

Having found solutions of the homogeneous equation (6), it is now an easy matter to find the general solution of the nonhomogeneous equation (5). Representing a solution S_1 of Eq. (5) as $S_1(s)=f_1(s)\phi_1(s)$, we see that

$$\ell\frac{\partial S_1}{\partial s}=f_1\left(\ell\frac{\partial\phi_1}{\partial s}\right)+\phi_1\left(\ell\frac{\partial f_1}{\partial s}\right),$$

and

$$\ell\frac{\partial}{\partial s}\left(\ell\frac{\partial S_1}{\partial s}\right)=\ell\left[f_1\frac{\partial}{\partial s}\left(\ell\frac{\partial\phi_1}{\partial s}\right)+2\ell\frac{\partial f_1}{\partial s}\frac{\partial\phi_1}{\partial s}+\phi_1\frac{\partial}{\partial s}\left(\ell\frac{\partial f_1}{\partial s}\right)\right].$$

Thus, in view of Eqs. (6) and (7a),

$$\ell\frac{\partial}{\partial s}\left(\ell\frac{\partial f_1}{\partial s}\right)-2\left(\ell\frac{\partial f_1}{\partial s}\right)=-\frac{\ell}{\phi_1}\frac{\partial E}{\partial s}. \qquad (8a)$$

Similarly, representing a second and independent solution of Eq. (5) as $S_2(s)=f_2(s)\phi_2(s)$, we have

$$\ell\frac{\partial}{\partial s}\left(\ell\frac{\partial f_2}{\partial s}\right)+2\left(\ell\frac{\partial f_2}{\partial s}\right)=-\frac{\ell}{\phi_2}\frac{\partial E}{\partial s}. \qquad (8b)$$

Finally, we note that both (8a) and (8b) are first-order linear ordinary differential equations in which the dependent variables are $\ell\partial f_1/\partial s$ and $\ell\partial f_2/\partial s$ and which can be solved in terms of quadratures. Since ϕ_1 and ϕ_2 are known, the general solution of Eq. (5) can be constructed explicitly from f_1 and f_2.

5. A special case

In order to gain some insight into the distinguishing features of this method, let us suppose that $\ell(s)$ is constant along a streamline. In that event,

$$\ell^2\frac{\partial^2 S}{\partial s^2}-S=-\ell\frac{\partial E}{\partial s}. \tag{9}$$

In general, we may represent Z as the sum of the true vorticity ζ and an error ϵ, i.e.,

$$\left.\begin{aligned} Z_{n+1}&=\zeta_{n+1}+\epsilon_{n+1}\\ Z_{n-1}&=\zeta_{n-1}+\epsilon_{n-1}\end{aligned}\right\},$$

in which ζ_{n+1} and ζ_{n-1} are the "true" values of vorticity at $t=(n+1)\Delta t$ and $t=(n-1)\Delta t$, and thus satisfy the finite-difference form of Eq. (1) for small Δt. Accordingly, the errors ϵ_{n+1} and ϵ_{n-1} are related to E as follows:

$$E=\epsilon_{n+1}-\epsilon_{n-1}+\ell\frac{\partial}{\partial s}(\epsilon_{n+1}+\epsilon_{n-1}).$$

In particular, let us examine the optimal adjustments corresponding to the error fields

$$\left.\begin{aligned} \epsilon_{n+1}&=A\ \sin ks\\ \epsilon_{n-1}&=B\ \sin ks\end{aligned}\right\}.$$

In this case

$$E=\sqrt{1+\ell^2k^2}[A\ \sin(ks+p)-B\ \sin(ks-p)],$$

where $p=\arctan\ (\ell k)$.

Solving Eq. (9) for S, we find that

$$S=\frac{\ell k}{\sqrt{1+\ell^2k^2}}[A\ \cos(ks+p)-B\ \cos(ks-p)].$$

Moreover, Eq. (3) implies that

$$D=-\frac{1}{\sqrt{1+\ell^2k^2}}[A\ \sin(ks+p)-B\ \sin(ks-p)].$$

Thus, recalling that $S=\delta_{n+1}+\delta_{n-1}$ and $D=\delta_{n+1}-\delta_{n-1}$, we have

$$\left.\begin{aligned} \delta_{n-1}&=\ \frac{A}{2}\sin(ks+2p)-\frac{B}{2}\sin ks\\ \delta_{n+1}&=-\frac{A}{2}\sin ks+\frac{B}{2}\sin(ks-2p)\end{aligned}\right\}.$$

Finally, the *residual* errors $\hat{\epsilon}$ *after* adjustment are

$$\left.\begin{aligned} \hat{\epsilon}_{n-1}&=\frac{A}{2}\sin(ks+2p)+\frac{B}{2}\sin ks\\ \hat{\epsilon}_{n+1}&=\frac{A}{2}\sin ks+\frac{B}{2}\sin(ks-2p)\end{aligned}\right\}.$$

Comparing these expressions for the residual error $\hat{\epsilon}$ with the original error ϵ, we see, for $\ell k \ll 1$, that the adjustment procedure proposed here tends to average the errors of analysis at times $t=(n+1)\Delta t$ and $t=(n-1)\Delta t$ and to assign equal residual errors at both times.

In general, the mean square residual error is

$$\overline{\hat{\epsilon}_{n-1}{}^2}+\overline{\hat{\epsilon}_{n+1}{}^2}=\frac{A^2+B^2}{4}+\left(\frac{AB}{2}\right)\cos 2p,$$

where the bar denotes the area average. Thus, assuming that the frequency distribution of error amplitudes A (or B) is symmetric around zero error, the *ensemble* mean square of the residual errors is

$$\langle\overline{\hat{\epsilon}_{n-1}{}^2}+\overline{\hat{\epsilon}_{n+1}{}^2}\rangle=\frac{\langle A^2+B^2\rangle}{4},$$

in which $\langle\ \rangle$ stands for the ensemble average. The ensemble mean square of the *original* error, on the other hand, is just

$$\langle\overline{\epsilon_{n-1}{}^2}+\overline{\epsilon_{n+1}{}^2}\rangle=\frac{\langle A^2+B^2\rangle}{2}.$$

We conclude, therefore, that the overall effect of imposing the optimal condition of dynamical consistency is to reduce the ensemble mean square of the residual error to one half that of the original error.

More generally, one may represent the error fields as Fourier series of the form

$$\left.\begin{aligned} \epsilon_{n+1}&=\sum_i A_i \sin k_i s+\alpha_i \cos k_i s\\ \epsilon_{n-1}&=\sum_i B_i \sin k_i s+\beta_i \cos k_i s\end{aligned}\right\}.$$

By simple extension of the arguments outlined above, one can easily show that the conclusions which hold for a single Fourier component also apply to a Fourier series, owing to the linearity of the system of equations and the orthogonality of its solutions. The mean square error is halved by imposing dynamical consistency.

6. Extensions and generalizations

So far, we have considered only a linear dynamical system which, although fairly realistic, lacks certain aspects of nonlinear systems. Upon examination of the corresponding nonlinear problem, it becomes apparent that allowance for simultaneous and compatible adjustments of the velocity field leads to more complicated equations for the optimal adjustments. The basic mathematical problem, however, is still linear for small Δt. The resulting equation for the extremal S has variable coefficients, but can be solved by purely numerical methods. The variational part of the problem presents no difficulty.

In some respects, the problem considered here is slightly unrealistic, in that one might expect that Z_{n-1} would be less in error than Z_{n+1}. Thus, it may be desirable to weight the adjustments at $(n-1)\Delta t$ more heavily than at $(n+1)\Delta t$ and, accordingly, to optimize the condition for dynamical consistency by minimizing an integral of the form

$$I=\int_A [(1+a)\delta_{n-1}^2+(1-a)\delta_{n+1}^2]dA,$$

where a is some positive constant (less than unity), to be determined empirically. This complicates the problem a little, but not in any crucial way.

7. Summary

By imposing the condition that the adjustments of two successive analyses be the least required to maintain dynamic consistency with a given prediction model, it has been shown that the residual errors of adjusted analysis are significantly less than the errors in the original analyses. The principles of dynamical constraint and optimization are easily generalized to more realistic systems.