A Test for the Scale Parameters of Two Gamma Distributions using the Generalized Likelihood Ratio

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(Manuscript received 15 September 1969, in revised form 17 October 1969)

ABSTRACT

The method of maximum likelihood is used to develop a statistical test for the scale parameters of two gamma distributions with common shape factors. A simple method of determining the power of the test using the non-central chi square distribution is also presented. The results of applying the test to gamma-scale parameters are compared with results obtained by applying the "*t*" test to normal and log-normal means. The likelihood ratio test for differences in gamma-scale parameters is more powerful than the "*t*" test applied to log-normal means. The power is nearly equal for the likelihood test of gamma-scale parameters and the "*t*" test for non-transformed means, although the latter test may not be as robust as is the likelihood ratio test. Since many meteorological variables are known to be gamma distributed, this test should have several applications in meteorology.

1. Introduction

It is generally agreed that the F test (Student t^2) is robust for testing scale parameters. However, application of the "*t*" test to data from skewed distributions can result in incorrect decisions. In this regard, Scheffé (1959) states that violation of the assumption of normality has little effect on inferences concerning means, but serious effects on inferences about variances. He also indicates that serious errors can result in the confidence coefficient, significance level, or power calculated from normal theory of variances, if kurtosis is present in the data. In the "*t*" ratio the denominator is a chi square variable involving variances which may be in error due to the presence of non-normality. Thus, it is possible that although two tests may have equal power, one test may not be as robust (i.e., inferences are not seriously invalidated by the violation of assumptions) as the other test. A direct test between the best fitting distributions should be more exact than the inappropriately applied "*t*" test, which in this case involves a dependent numerator and denominator as well as heterogeneity of variance.

The underlying purpose of this paper is to develop a direct test between skewed distributions which are commonly found in meteorological data. One such distribution is the gamma distribution. This distribution has been used extensively in fitting rainfall distributions by Barger and Thom (1949), Thom and Vestal (1968), and Mooley and Crutcher (1968). Schickedanz (1967), Neyman and Scott (1967), and Schickedanz and Decker (1969) have demonstrated its utility in the verification of rainfall modification experiments. The dollar value of paid claims for yearly crop losses due to hail damage can be fitted by the gamma distribution (Schickedanz *et al.*, 1969). In particular, the objective of this paper is to develop a test between the scale parameters of two gamma distributions with common shape factors and to compare its power with that obtained by applying the "*t*" test to non-transformed and transformed data.

2. Development of the test

The density function of the gamma distribution is

$$f(x)=\beta^{-\gamma}\Gamma(\gamma)^{-1}x^{\gamma-1}e^{-x/\beta},\quad x>0,\quad \gamma>0,\quad \beta>0, \tag{1}$$

where $\Gamma(\gamma)$ is the complete gamma function and the symbols β and γ are the scale and shape parameter, respectively, and can be estimated by the method of maximum likelihood (Thom, 1958).

Mood and Graybill (1963) define the generalized likelihood ratio as the quotient

$$\lambda=L(\hat{\omega})/L(\hat{\Omega}), \tag{2}$$

where $L(\hat{\Omega})$ is the maximum of the likelihood function in the entire parameter space Ω with respect to the parameters, and $L(\hat{\omega})$ is the maximum of the likelihood function in the subspace ω with respect to the parameters. The quantity λ is non-negative and varies from 0 to 1. The quantity $-2\ln\lambda$ will be approximately distributed as chi square with one degree of freedom for the case discussed below.

The test will be derived for scale parameters between two gamma distributions with common shape factors. The parameter space Ω is three-dimensional with coordinates $(\beta_1, \beta_2, \gamma)$, while ω for the null hypothesis $\beta_1=\beta_2$ is two-dimensional with coordinates (β, γ). For the parameter space ω (under the null hypothesis of $\beta_1=\beta_2$) γ_1 is assumed equal to γ_2. The procedure is to find the maximum likelihood estimators for the parameter spaces described above.

If there are n_1 observations $(x_{11},x_{12},\cdots,x_{1n_1})$ in the sample from the first population and n_2 observations $(x_{21},x_{22},\cdots,x_{2n_2})$ from the second, then the likelihood function under ω is

$$L(\omega)=\left[\beta_1^{-n_1\gamma_1}\Gamma(\gamma_1)^{-n_1}\prod_{i=1}^{n_1} x_{1i}^{(\gamma_1-1)}\exp\left(-\sum_{i=1}^{n_1} x_{1i}/\beta_1\right)\right] \times\left[\beta_2^{-n_2\gamma_2}\Gamma(\gamma_2)^{-n_2}\prod_{j=1}^{n_2} x_{2j}^{(\gamma_2-1)}\exp\left(-\sum_{j=1}^{n_2} x_{2j}/\beta_2\right)\right]. \quad (3)$$

The log likelihood function under ω is then

$$\ln L(\omega)=-N\gamma\ln\beta-N\ln\Gamma(\gamma)+(\gamma-1) \times\left(\sum_{i=1}^{n_1}\ln x_{1i}+\sum_{j=1}^{n_2}\ln x_{2j}\right)-\beta^{-1}\left(\sum_{i=1}^{n_1}x_{1i}+\sum_{j=1}^{n_2}x_{2j}\right), \quad (4)$$

where $N=n_1+n_2$, $\gamma_1=\gamma_2$, and $\beta_1=\beta_2$. Upon taking the partial derivative of (4) with respect to β, equating it to zero, and solving for β, the maximum likelihood estimate of β is found to be

$$\hat{\beta}=\bar{\bar{x}}/\hat{\gamma}, \quad (5)$$

where

$$\bar{\bar{x}}=N^{-1}\left(\sum_{i=1}^{n_1}x_{1i}+\sum_{j=1}^{n_2}x_{2j}\right).$$

The partial derivative of (4) with respect to γ equated to zero is

$$\ln\bar{\bar{x}}-\ln\hat{\gamma}+\Psi(\hat{\gamma})-N^{-1}\left(\sum_{i=1}^{n_1}\ln x_{1i}+\sum_{j=1}^{n_2}\ln x_{2j}\right)=0, \quad (6)$$

where $\ln\hat{\beta}=\ln\bar{\bar{x}}-\ln\hat{\gamma}$ from (5), and $\Psi(\hat{\gamma})=\partial[\ln\Gamma(\gamma)]/\partial\gamma$ which is the digamma function. Thom (1958) used $\Psi(\hat{\gamma})=\ln\hat{\gamma}-(2\hat{\gamma})^{-1}-(12\hat{\gamma}^2)^{-1}$ as an approximation to the digamma function when estimating γ and β. Substituting this approximation into (6) gives $(2\hat{\gamma})^{-1}+(12\hat{\gamma}^2)^{-1}-A=0$, or upon simplification

$$12A\hat{\gamma}^2-6\hat{\gamma}-1=0, \quad (7)$$

where

$$A=\ln\bar{\bar{x}}-N^{-1}\left(\sum_{i=1}^{n_1}\ln x_{1i}+\sum_{j=1}^{n_2}\ln x_{2j}\right).$$

Solving (7) for $\hat{\gamma}$, one gets

$$\hat{\gamma}=(4A)^{-1}[1+(1+4A/3)^{1/2}]. \quad (8)$$

Similarly, under the parameter space Ω, where the alternative hypothesis (H_a) is $\beta_1=\beta_2$ and $\gamma_1=\gamma_2$, it can be shown that $\hat{\beta}_1=\bar{x}_1/\hat{\gamma}'$, $\hat{\beta}_2=\bar{x}_2/\hat{\gamma}'$, and that $\hat{\gamma}'=(4A')^{-1}[1+(1+4A'/3)^{1/2}]$, where

$$A'=N^{-1}\left(n_1\ln\bar{x}_1+n_2\ln\bar{x}_2-\sum_{i=1}^{n_1}\ln x_{1i}-\sum_{j=1}^{n_2}\ln x_{2j}\right).$$

The log of the maximum likelihood ratio may then be written as

$$\ln\lambda=\ln L(\hat{\omega})-\ln L(\hat{\Omega})=N[\ln\Gamma(\hat{\gamma})'-\ln\Gamma(\hat{\gamma})-\hat{\gamma}\ln\hat{\beta}] +\hat{\gamma}'(n_1\ln\hat{\beta}_1+n_2\ln\hat{\beta}_2)+(n_1\overline{\ln x_1}+n_2\overline{\ln x_2})(\hat{\gamma}-\hat{\gamma}') +n_1\bar{x}_1(\hat{\beta}_1^{-1}-\hat{\beta}^{-1})+n_2\bar{x}_2(\hat{\beta}_2^{-1}-\hat{\beta}^{-1}), \quad (9)$$

where

$$n_k\overline{\ln x_k}=\sum_{m=1}^{n_k}\ln x_{km},\ n_k\bar{x}_k=\sum_{m=1}^{n_k}x_{km}.$$

Wilks (1938) has shown for large samples that the statistic $-2\ln\lambda$ is approximately distributed as chi square with one degree of freedom. For large samples, an exact test of significance is made by consulting a table of the chi square distribution.

3. Power of the test

The approximate power of the generalized likelihood ratio test against a specific alternative is given by Fix (1954) as

$$\text{Power}=P(\Delta)=\text{Prob}[\Delta>\chi_f^{2\prime}(\alpha)], \quad (10)$$

where $\chi_f^{2\prime}$ is the value of the non-central chi square $\chi^{2\prime}$ corresponding to the α level of significance. The power obviously depends on Δ, the non-centrality parameter. The degrees of freedom f are the same as those associated with the likelihood ratio test. Therefore, Δ is estimated, noting that for large samples it is approximately equivalent to the ratio of the joint probability density of the samples evaluated for the estimates of the parameters specified under ω and Ω (Lehmann, 1959). Fix has computed tables of the non-central chi square for the 0.05 and 0.01 size of the test. In these tables, Δ is the tabled value corresponding to values of $P(\Delta)$ and f. In order to obtain the approximate power of the likelihood ratio test, it is sufficient to enter $-2\ln\lambda$ for Δ in the tables, and the values of $P(\Delta)$ for specified degrees of freedom can be obtained by interpolation. If sample size is large, power found this way is exact.

4. An example

The normal, log-normal, and gamma distributions were fitted to weekly rainfall data from Springfield, Ill.,

TABLE 1. Distribution parameters for the normal, log-normal and gamma distributions.

Distribution parameter	Summer	Fall	Winter
$\bar{X}$ Normal mean	0.9040	0.7635	0.3684
S Normal standard deviation	0.9831	0.9129	0.3577
$\bar{X}_L$ Log-normal mean	−0.8471	−1.0417	−1.5850
S_L Log-normal standard deviation	1.5196	1.4671	1.2675
$\hat{\gamma}_s$ Gamma shape factor	0.8207	0.7986	1.0047
$\hat{\beta}_s$ Gamma scale parameter	1.1015	0.9560	0.3667

TABLE 2. Chi square "goodness of fit" test for the normal, log-normal and gamma distributions.

Distribution	Summer	$P(\chi^2 > \chi_0^2)$ Fall	Winter
Normal	<0.01	<0.01	<0.01
Log-normal	0.04	0.30	0.19
Gamma	0.15	0.54	0.43

for the seasons of summer, fall and winter for the period 1960–64. All three distributions and their corresponding frequency histograms are shown in Fig. 1. In Table 1 there is a listing of the parameters of the various distributions. The sample estimates of $\hat{\gamma}_s$ and $\hat{\beta}_s$ were obtained by the method of maximum likelihood (Thom, 1958). The results of the chi square "goodness of fit" test are shown in Table 2. The gamma distribution was the best fitting distribution, and the normal distribution was the poorest fitting distribution. If the often cited 0.05 significance level is used for the chi square criteria, the log-normal distribution will fit the fall and winter data, but does not fit the summer data.

The "t" test was then applied to test for differences in the means of the normal and log-normal distributions and the "λ" (likelihood ratio) test was applied to test for differences in the gamma-scale parameters, for the combinations of summer-fall and fall-winter. The resulting test statistics and the power of the test are tabulated in Table 3. The power of the test for the log-transformed data is noticeably less than that of the non-transformed data and for the "λ" test. Although the normal distribution does not fit the data, the "t" test for the non-transformed data and the "λ" test are very similar in power. It would appear from Table 3 that there is a loss in power when one transforms the data and applies the standard tests for differences in means. The fact that

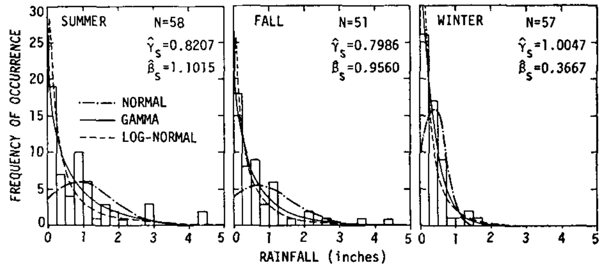

FIG. 1. Theoretical frequency distributions and histograms for weekly rainfall data from Springfield, Ill., for the seasons of summer, fall and winter.

the "t" test for differences in the non-transformed means and the "λ" test for difference scale parameters have nearly the same power is a very interesting result. Since the normal distribution does not fit the data (Table 2), it would appear that the "t" test of the non-transformed means may be robust (not affected by departures from assumptions, i.e., the property of remaining practically valid over a large range of real conditions) and yields nearly the same results as the "λ" test.

Certainly, in most applications, we would be very fortunate if all the assumptions involved in a particular statistical test were *exactly* satisfied. Therefore, the degree of error associated with different degrees of violation of assumptions becomes important. For example, it would appear that the inappropriate use of the "t" test on very skewed data is more serious than for the skewed data used here. In particular, for the case of very skewed daily rainfall data generated by Monte Carlo methods (Schickedanz and Decker, 1969), the "t" test of transformed data was less powerful than the "t" test of non-transformed data and the "λ" test of gamma-scale parameters. However, the power of the "t" test involving non-transformed data was less than the power of the "λ" test involving gamma-distributed data. Since daily rainfall data used were more skewed than the weekly rainfall data used here, it would appear that the greater skewness had an effect on the tests.

5. Conclusions

It would appear that the generalized likelihood ratio test as presented here should be very useful in the atmospheric sciences since many meteorological variates are known to be gamma distributed. However, further research into the topic of robustness and power of the test for different values of skew and kurtosis would indeed be very useful.

TABLE 3. Test statistics and power of the test for the Springfield data.

Test comparison	Test statistic "t" Normal	Test statistic "t" Log-normal	Test statistic $-2\ln\lambda$ Gamma	Power $P(\Delta)$ Normal	Power $P(\Delta)$ Log-normal	Power $P(\Delta)$ Gamma
Summer-fall	0.81	0.45	0.59	0.13	<0.10	0.12
Fall-winter	4.45	1.51	11.86	0.99	0.33	>0.90

Acknowledgments. This work was performed under the general supervision of Glenn E. Stout, Head of the Atmospheric Sciences Section of the Illinois State Water Survey. A portion of this research was performed while Paul T. Schickedanz was a graduate student at the University of Missouri and appreciation is expressed to Wayne L. Decker for his suggestions and comments during that time. Helpful suggestions by Mrs. J. Loreena Ivens, Survey Technical Editor, are appreciated. This research was supported by the National Science Foundation Grants GA-557 to the University of Missouri, and GA-1360 to the Illinois State Water Survey at the University of Illinois.

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