Phase Velocity and Structure Calculations for Unstable Quasi-Geostrophic Waves in a Simple Baroclinic Westerly Current

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(Manuscript received 26 September 1969, in revised form 24 December 1969)

ABSTRACT

A simple numerical method is used to calculate the vertical structure and the phase velocity, $c = c_r + c_i i$, for small, sinusoidal, quasi-geostrophic, unstable perturbations in a baroclinic zonal current which is independent of latitude and is bounded above by a barotropic and isothermal stratosphere.

Values of c for waves of a given effective length are determined as complex eigenvalues by assuming trial values, integrating the perturbation equations numerically between 1000 mb and the tropopause, and evaluating residuals which measure the extent to which the boundary conditions are not satisfied. The residuals are then used to obtain new estimates, which under favorable conditions converge to stable values if the first guess is sufficiently close. Some progress has been made toward automating the selection of first guesses. For middle latitudes (25° – 75°) and moderate-to-strong vertical shears, convergence can usually be obtained for the range of effective wavelengths in which c_i is a substantial fraction of its maximum value. Each eigenvalue for c has $c_i > 0$ and is paired with a conjugate value. In simple cases the value found for c seems to be the only eigenvalue in a large region of the complex plane above the axis of reals. Sample calculations are given for latitude 45° .

1. Introduction

Many studies have been made of the dynamics of small, sinusoidal, quasi-geostrophic perturbations in a baroclinic zonal current independent of latitude. Analytic solutions for particular types of current were obtained in pioneering studies by Charney (1947), Eady (1949) and Kuo (1952, 1953). A useful compendium of analytic solutions for simple cases has recently been published by Derome and Wiin-Nielsen (1966). Green (1960) obtained numerical solutions for a number of more complicated types of zonal wind and stability profiles by subdividing the height range, setting up a system of linear equations for the pressure distribution at discrete heights, and solving a determinantal equation for the phase velocity. Wiin-Nielsen (1967) has studied the dependence of wave stability on the vertical profile of the zonal wind, for the case of zero hydrostatic stability. Other studies have extended the theory to take into account ageostrophic effects (Arnason, 1963; Derome and Wiin-Nielsen, 1966) and lateral variations in the basic current (e.g., Charney and Stern, 1962; Haltiner, 1963; Pedlosky, 1964a, b, 1965; Egger, 1968; Brown, 1969a, b; Stone, 1969).

The present study is concerned only with the separable, purely baroclinic, quasi-geostrophic case in which 1) the zonal wind velocity and hydrostatic stability depend only on the vertical coordinate (pressure) and not on the north-south coordinate, and 2) the perturbation is assumed to be frictionless, adiabatic and geo-

strophic. For simple currents of this type, in which a westerly flow increases with height, the destabilizing effect of vertical shear is opposed by two stabilizing effects: i) hydrostatic stability (important for short waves), and ii) the beta effect, or variation of the Coriolis parameter with latitude (important for long waves). The phase velocity, $c = c_r + c_i i$, for a sinusoidal perturbation, depends on its effective wavelength, $\lambda_e = (\lambda_x^{-2} + \lambda_y^{-2})^{-1/2}$, (Kuo, 1952). Analytic solutions for simplified model currents (Derome and Wiin-Nielsen) indicate that for certain ranges of λ_e all waves are unstable and all values of c form complex conjugate pairs. Thus, if static stability is assumed zero, all waves are unstable when λ_e is less than a longwave limit determined by the beta effect, whereas if the current is statically stable but the beta effect is neglected, waves are unstable if λ_e exceeds a shortwave limit. As will be seen in Section 4, the results of the present study suggest, for model currents with both a beta effect and a realistic profile of static stability, that waves are in general unstable when λ_e lies between an upper and a lower limit, and $|c_i|$ and $|c_i|/\lambda_e$ have maximum values for intermediate values of λ_e ; however, the lower limit may be absent for certain types of current. For values of λ_e outside this range (which we shall call the primary unstable range), neutral waves, with $c_i = 0$, are mathematically possible, but a number of recent studies suggest that amplifying solutions are also possible for most wavelengths (e.g., Burger, 1962, 1966; Miles, 1964a–d, 1965; Huppert and Miles, 1966; Hirota, 1968).

The purpose of the present paper is to point out, for the separable quasi-geostrophic case, that solutions of the perturbation equations can often be obtained by a numerical method which is simpler than that used by Green, and does not involve the solution of a determinantal equation derived from a large tri-diagonal matrix. Attention is focused on the primary unstable range of wavelengths in which all eigenvalues of the phase velocity c occur as conjugate complex pairs. Each value of c is obtained (as one member of such a pair, with $c_i > 0$) by an iterative process, the convergence of which depends on a good choice of the first estimate. Some progress has been made toward automating the selection of these estimates. For middle latitudes and moderate-to-strong vertical shears, convergence can usually be obtained for effective wavelengths not too close to the ends of the primary unstable range. In this way a good picture can often be obtained of the structure and properties of the most unstable waves. In simple cases the pairs of eigenvalues obtained for c seem to be unique.

2. The perturbation equations

In large-scale atmospheric motions which are approximately frictionless, adiabatic and geostrophic, the thermal equation (first law of thermodynamics) and the vorticity equation can be approximated as

$$z_{Pt} + \sigma\omega = (g/f)J(z_P, z), \quad (2.1)$$

$$\nabla^2 z_t - (f^2/g)\omega_P = (g/f)J(\nabla^2 z, z) - \beta z_x. \quad (2.2)$$

Here the x axis is directed eastward, and ∇^2 and J are isobaric Laplacian and Jacobian operators. Subscripts x, y, P and t indicate differentiation, except when used with wavenumbers (k) or wavelengths (λ). The rest of the notation is standard, except that $P = p/(1000 \text{ mb})$, where p is pressure, the vertical motion variable ω is defined as dP/dt (rather than dp/dt) following the motion, and the "stability length" σ is defined as $(R/g)P^{-(1-\kappa)}(-\partial\theta/\partial P)$, where $\kappa = R/c_p \approx 2/7$ and $\theta = TP^{-\kappa}$ is potential temperature, and f and β are evaluated for a mean latitude ϕ . A list of symbols is given in Appendix 2.

We assume a basic westerly current in which the zonal wind velocity $\bar{u} = -(g/f)\bar{z}_y$ and the stability length $\bar{\sigma}$ depend only on P and y , and in which $\omega = 0$, as implied by (2.1). A small perturbation superimposed on this basic current will then be governed by linearized equations derived from (2.1) and (2.2), namely

$$z'_{Pt} + \bar{\sigma}\omega = -\bar{u}z'_{Px} + \bar{u}_P z'_x, \quad (2.3)$$

$$\nabla'^2 z'_t - (f^2/g)\omega_P = -[\bar{u}\nabla'^2 z'_x + (\beta - \bar{u}_{yy})z'_x], \quad (2.4)$$

where $z' = z - \bar{z}(y, P)$. Eqs. (2.3) and (2.4) can be satisfied by perturbations of the form

$$z' = \hat{Z}(y, P) \exp[ik_x(x - ct)], \quad (2.5)$$

$$\omega = \hat{W}(y, P) \exp[ik_x(x - ct)], \quad (2.6)$$

FIG. 1. Typical profiles assumed for the scaled zonal wind velocity, $B(P) = (\bar{u} - u_1)/u_T$, and for the stability parameter, $\tau(P) = P^2 \bar{\sigma}/\sigma_0$, for $P_2 = 0.250$, $\tau_1 = \tau_2 = 0.38271$, $a_u = 0.5$.

provided that

$$\bar{\sigma}\hat{W} = ik_x[\bar{u}_P \hat{Z} - (\bar{u} - c)\hat{Z}_P], \quad (2.7)$$

$$(f^2/g)\hat{W}_P = ik_x[(\bar{u} - c)(\hat{Z}_{yy} - k_x^2 \hat{Z}) + (\beta - \bar{u}_{yy})\hat{Z}]. \quad (2.8)$$

In the present study we further restrict the problem to currents in which $\bar{u}$ and $\bar{\sigma}$ depend only on P and not on y , so that $\bar{u}_{yy} = 0$, and we treat f and β as constants. In this case (2.7) and (2.8) are satisfied by $\hat{Z}(y, P) = Y(y)Z(P)$, $\hat{W}(y, P) = Y(y)W(P)$, provided that $Y_{yy} = -k_y^2 Y$, where k_y is a constant; this implies that Y is $\cos(k_y y)$, $\sin(k_y y)$, or some linear combination of these.

In simplifying (2.7) and (2.8), it is convenient to introduce an effective wavenumber, $k_e = (k_x^2 + k_y^2)^{1/2}$, a nondimensional stability variable, $s(P) = \bar{\sigma}(P)/\sigma_0$, where σ_0 is a standard value, and a new dependent variable, $X = i(\sigma_0/k_x)W$, to replace W . We then obtain

$$dZ/dP = (\bar{u} - c)^{-1}[sX + (d\bar{u}/dP)Z], \quad (2.9)$$

$$dX/dP = a(\bar{u} - c - c_R)Z, \quad (2.10)$$

where $c_R \equiv \beta k_e^{-2}$ may be regarded as a "retrograde" velocity component arising from the beta effect, and $a \equiv g\sigma_0 k_e^2 / f^2$ is nondimensional.

In general, we shall assume that the basic current is divided at $P = P_2$ (evaluated for mean latitude ϕ) into 1) an isothermal and barotropic stratosphere, in which $\bar{u} = u_2$ and $\bar{T} = T_2$, independent of P and y , and 2) a baroclinic troposphere, in which $\bar{u}(P)$ increases monotonically from u_1 at 1000 mb ($P = 1$) to u_2 at $P = P_2$. We define $B(P) = (\bar{u} - u_1)/u_T$, where $u_T = u_2 - u_1$ is the thermal wind between 1000 mb and the tropopause in the basic current (Fig. 1). In the stratosphere, we have $\bar{\sigma}(P) = (\kappa RT_2/g)P^{-2}$; we therefore take the standard value σ_0 to be $\kappa RT_2/g$, and we define $\tau(P) = P^2 s(P)$

$=P^2\bar{\sigma}/\sigma_0$. Then τ is 1 in the stratosphere and an arbitrary function of P (typically <1 and roughly constant) in the troposphere.

As boundary conditions for the system (2.9)–(2.10) we assume

$$X=0 \text{ (i.e., } W=0), \quad \text{at } P=0, \quad (2.11)$$

$$Z \text{ and } X \text{ continuous,} \quad \text{at } P=P_2, \quad (2.12)$$

$$X=0, \quad \text{at } P=1, \quad (2.13)$$

with Z and X bounded everywhere. In view of (2.10), (2.12) implies that dX/dP is also continuous at $P=P_2$.

Corresponding to the wavenumbers k_x , k_y and k_e , we define a zonal (longitudinal) wavelength, $\lambda_x=2\pi/k_x$, a meridional (transverse) wavelength, $\lambda_y=2\pi/k_y$, and an effective wavelength, $\lambda_e=2\pi/k_e$, which is in general less than either λ_x or λ_y . The perturbation amplitudes $Z(P)$, $W(P)$ and $X(P)$, and the phase velocity, $c=c_r+c_i$, are in general complex quantities. If $c_i \neq 0$ the scalar amplitudes vary with time as $\exp(Gt)$, where $G \equiv k_x c_i$ is an amplification factor or growth rate. However, it is more convenient to discuss wave stability in terms of the quantity, $G^*=k_e c_i$, which for any given basic current depends only on the effective wavenumber k_e . We then have $G=(\lambda_e/\lambda_x)G^*$, which is in general less than G^* .

Any solution of the system (2.9)–(2.13) remains a solution 1) if the amplitudes are all multiplied by an arbitrary constant, 2) if an arbitrary constant is added to the basic current velocity $\bar{u}$ at each level, and also to the phase velocity c , or 3) if c , Z and X are all replaced by their complex conjugates. Thus, any complex phase velocities must appear as conjugate pairs. Except in the relations between G and c_i , and between X and W , the equations involve only the effective wavenumber k_e , and not k_x and k_y individually.

By integrating (2.10) between $P=0$ and $P=1$, and using the boundary conditions (2.11) and (2.13), we obtain the relation (Charney, 1947)

$$\left. \begin{aligned} c &= \left[\int_0^1 \bar{u} Z dP \right] \left[\int_0^1 Z dP \right]^{-1} - c_R \\ c &= u_1 + u_T \left[\int_0^1 B(P) Z dP \right] \left[\int_0^1 Z dP \right]^{-1} - c_R \end{aligned} \right\} \quad (2.14)$$

This indicates that $c+c_R$ is a mean value of $\bar{u}$ weighted by the perturbation amplitude Z , and that complex values of c are possible only if the phase of the perturbation changes with height.

Eqs. (2.9) and (2.10) can be combined to give an equation for Z , by eliminating X , or for X , by eliminating Z . As (2.9) and (2.10) are more convenient for numerical integration in their original form, we shall not have occasion to use these derived equations. We note, however, that their form indicates that the influence of hydrostatic stability on wave structure and behavior

depends on the distribution in the vertical of the quantity, $q^2=as=\kappa RT_2\tau(2\pi/Pf\lambda_e)^2$. Apart from its direct dependence on τ , q^2 tends to be large for high levels and short waves and small for low levels and long waves. It also tends to be larger in low latitudes.

The limitations of the assumed model should be kept in mind. It does not apply to jet streams, laterally curved troughs and ridges, large-amplitude perturbations, waves which are too short or in too low a latitude to be essentially geostrophic, or waves which are strongly influenced by adiabatic and terrain effects, including all very long waves.

3. Solution of the perturbation equations

Simple analytic solutions of the system (2.9)–(2.13) are known for certain limiting cases. If $\bar{\sigma}=0$ at all levels (Holmboe, 1959; Wiin-Nielsen, 1967), (2.9) implies that

$$Z=A(\bar{u}-c)=A[u_1+u_TB(P)-c], \quad (3.1)$$

where A is independent of P . In this case (2.14) becomes a quadratic for $c-u_1$, and gives

$$c=(u_1+I_1u_T-\frac{1}{2}c_R)\pm(\frac{1}{4}c_R^2-I_2u_T^2)^{\frac{1}{2}}, \quad (3.2)$$

where

$$I_1=\int_0^1 B dP \quad \text{and} \quad I_2=\int_0^1 (B-I_1)^2 dP.$$

As $c_R=\beta(\lambda_e/2\pi)^2$, the phase velocity c has a pair of distinct real values if $\lambda_e>\lambda_b$, and a pair of conjugate complex values if $\lambda_e<\lambda_b$, where λ_b is a longwave limit for instability given by

$$\lambda_b=2\pi I_2^{\frac{1}{2}}(2|u_T|/\beta)^{\frac{1}{2}}. \quad (3.3)$$

Although (3.1)–(3.2) apply only if $\bar{\sigma}$ is zero at all levels, the dependence of static stability effects on the factor q^2 implies, for *long* waves in middle and high latitudes, even with normal static stabilities, that (3.1) will be a fairly good approximation in the troposphere. Eq. (3.2) will be somewhat in error because the integrals in (2.14) involve the stratosphere, where τP^{-2} is large, but (3.2) should still give a *rough* indication of the variation of c_r and c_i for large values of λ_e .

In the opposite case of very *short* waves, the factor q^2 will be large and c_R small. A simple analytic solution is obtained if there is no stratosphere ($P_2=0$), if $\bar{\sigma}$ and $d\bar{u}/dP$ are independent of P , and if c_R is assumed to be negligible (Derome and Wiin-Nielsen, 1966; see also Eady, 1949). In this case we have

$$c=u_m\pm u_TD^{\frac{1}{2}}, \quad (3.4)$$

where $u_m \equiv u_1 + \frac{1}{2}u_T$ is the value of $\bar{u}$ for $P=\frac{1}{2}$ (i.e., for 500 mb), $D=\frac{1}{4}+q^{-2}-(q \tanh q)^{-1}$, and $q^2=as=k_e^2 f^{-2} \kappa RT_2 s$. The term D varies from a limiting value of $+\frac{1}{4}$ for very short waves (q large), through zero for $q=2.399$, to a limiting value of $-1/12$ for very long waves (q small).

Thus, c has a pair of distinct real values for $\lambda_e < \lambda_a$, and a pair of conjugate complex values, with $c_r = u_m$, for $\lambda_e > \lambda_a$, where λ_a is a shortwave limit for instability, independent of u_T and given by

$$\lambda_a = (2\pi/2.399f)(s\kappa RT_2)^{1/2}. \quad (3.5)$$

The present study considers the more general case in which neither of the above limiting approximations is valid. In general, we assume a barotropic and isothermal stratosphere, and consider only the intermediate range of wavelengths in which values of c occur as conjugate complex pairs. Given the latitude ϕ , the tropopause pressure P_2 , the stratospheric temperature T_2 (which determines σ_0), and the wind and stability profiles $\bar{u}(P)$, $s(P)$, we wish to solve the system (2.9)–(2.13) for Z and X as functions of P . For waves of a given effective length λ_e , the phase velocity c must be determined as an eigenvalue.

In the stratosphere ($0 \leq P \leq P_2$) we have $\bar{u} = u_2 = \text{constant}$ and $s = P^{-2}$. By eliminating X from (2.9) and (2.10) we obtain

$$(d/dP)(P^2 dZ/dP) = bZ, \quad (3.6)$$

where $b \equiv a(u_2 - c_R - c)/(u_2 - c)$ is a complex constant and a is real and positive. The general solution of (3.6) can be written

$$Z = A_1 P^{n_1} + A_2 P^{n_2},$$

where the A 's are arbitrary constants and $n_1 = -\frac{1}{2} + N$, $n_2 = -\frac{1}{2} - N$. Here $N^2 = b + \frac{1}{4}$, and N is defined so that its real part N_r is positive or zero. Then the real part of n_2 is negative, and that of n_1 also negative unless $N_r \geq \frac{1}{2}$. In order to ensure that the modulus of Z remains bounded as P approaches zero, we must have $A_2 = 0$ and $N_r \geq \frac{1}{2}$. Dropping the subscripts on A_1 and n_1 , we thus obtain

$$Z = AP^n, \quad (3.7)$$

$$X = n(u_2 - c)AP^{n+1} = n(u_2 - c)PZ, \quad (3.8)$$

where the complex exponent, $n = (b + \frac{1}{4})^{1/2} - \frac{1}{2}$, involves the as yet unknown phase velocity c .

In the troposphere the variation of Z and X is determined by (2.9) and (2.10), together with the boundary conditions $X = 0$ at $P = 1$ and $X = n(u_2 - c)PZ$ at $P = P_2$. If c were known, one could start at one end of the range $P_2 \leq P \leq 1$, with the appropriate boundary condition satisfied, and integrate (2.9) and (2.10) numerically as a "marching" problem to the other end. One would then find that the boundary condition at that end was also satisfied. In practice, one starts with an approximate value for c , and after integrating from one end of the range to the other, one obtains a residual quantity E (complex if c is complex) which measures the extent to which the boundary condition at the far end is not satisfied. By noting the changes in E as the assumed value of c is varied, one can estimate the value

of c which would reduce E to zero. Under favorable conditions, successive estimates will converge to the true value. When E has been reduced to a sufficiently small value, the calculated tropospheric profiles of Z and X will be fairly accurate, and stratospheric profiles can then be calculated from (3.7) and (3.8), together with (2.12) and the known value of n .

In such calculations, one can without loss of generality take $u_1 = 0$, as a solution for this case is also a solution for the more general case in which a real constant is added to the value of $\bar{u}$ at each level, and also to the phase velocity c . Similarly, one can without loss of generality take $c_i > 0$ (amplification), knowing that a damped solution can be obtained by replacing c , Z , X and W by their complex conjugates. The amplitude of the Z profile is arbitrary; values can be calculated with arbitrary scaling, but are best rescaled for tabulation by dividing by Z_m , the largest value of $|Z|$ encountered in the range of integration.

The convergence of the iterative improvement process for obtaining c depends critically on the use of good approximation formulas to give starting values for c_r and c_i . Details of the schemes used for calculating initial estimates of c_r and c_i , for improving these estimates, and for integrating (2.9) and (2.10) numerically, are given in Appendix 1.

4. Examples of calculation

Profiles of the zonal wind velocity $\bar{u}(P)$ and of the stability factor $s(P) = \bar{\sigma}/\sigma_0$ were assumed to be of the form

$$\bar{u}(P) = u_1 + B(P)u_T, \quad (4.1)$$

$$s(P) = P^{-2}\tau(P), \quad (4.2)$$

with $B(P) = 1$ and $\tau(P) = 1$ in the stratosphere ($0 \leq P < P_2$). In most calculations, tropospheric profiles of B and τ were of the form

$$B(P) = Q - a_u Q(1 - Q), \quad (4.3)$$

$$\tau(P) = \tau_1 + (\tau_2 - \tau_1)Q^2(3 - 2Q). \quad (4.4a)$$

Here, $Q = (1 - P)/(1 - P_2)$, a_u is a parameter which determines the curvature of the wind profile, and τ_1 and τ_2 are values of τ for 1000 mb and for the top of the troposphere. Eq. (4.3) implies a parabolic wind profile, which becomes linear with respect to P if $a_u = 0$, while (4.4a) implies a monotonic variation of τ , with $d\tau/dP = 0$ at the tropopause and at 1000 mb; in most calculations, however, τ_1 and τ_2 were equal, so that τ was constant and $\bar{\sigma}$ varied as P^{-2} in the troposphere. As an alternative to (4.4a), it was also possible to specify the tropospheric profile of s by an equation of the form

$$s(P) = s_1 + (s_2 - s_1)Q^2(3 - 2Q). \quad (4.4b)$$

Expressions for the factors I_1 and I_2 in (3.2) and (3.3), in terms of P_2 and a_u , are readily obtained from (4.1)

TABLE 1. Parameters used in test calculations ($\phi=45^\circ$, $u_T=40 \text{ m sec}^{-1}$, $u_1=0$, $a_u=0$, $T_2=216.66\text{K}$).

Case	P_2	β/β_{45}	$\tau(P < P_2)$	$\tau_1=s_1$	τ_2 or s_2
A	0.000	1	—	0	0
B	0.000	0	—	0.38271	$s_2=0.38271$
C	0.000	1	—	0.38271	$s_2=0.38271$
D	0.250	1	0	0	0
E	0.250	1	1	0	0
F	0.250	1	1	0.38271	$s_2=0.38271$
G	0.250	1	1	0.38271	$\tau_2=0.38271$
H	0.250	1	1	0.05	$\tau_2=0.30$
I	0.250	1	1	1.50	$\tau_2=0.50$

and (4.3). Calculations were restricted to the range of λ_e in which all waves are unstable.

In order to get some indication of the accuracy of the numerical scheme used, and of the influence of different factors on phase velocity and wave structure, calculations were carried out for a series of test cases, all having $\phi=45^\circ$, $u_T=40 \text{ m sec}^{-1}$, $u_1=0$, $a_u=0$ and $T_2=216.66\text{K}$, but with other parameters varying as in Table 1. In cases A and B the hydrostatic stability profiles are quite unrealistic, but simple analytic solutions are known. In case G the zonal wind and static stability profiles are fairly realistic, whereas cases C through F represent situations intermediate between cases A and B and case G. Cases H and I represent situations with low and high static stability. In case B, the Rossby parameter β was arbitrarily set equal to zero; otherwise, it was given its true value β_{45} . With $P_2=0.250$, a constant tropospheric value of 0.38271 for τ , as assumed in case G, would give the same ratio of absolute temperatures at the tropopause and at 1000 mb as is found in the ICAO Standard Atmosphere.

Cases A, B and C assume that there is no tropopause ($P_2=0$), and that $d\bar{u}/dP$ and $\bar{\sigma}$ are both independent of P . Fig. 2a shows the numerically calculated variation of c_r-u_1 and c_i with the effective wavelength λ_e . For comparison, values were calculated analytically from (3.2) in case A ($\bar{\sigma}=0$) and from (3.4) in case B ($\beta=0$, $\bar{\sigma}=\text{constant}$). Values of c_r and c_i calculated by the two methods agreed very closely, the differences being too small to show up in Fig. 2a. For c_r , this close agreement (to 0.002 m sec^{-1}) would be expected from symmetry. For c_i , values calculated numerically were slightly less than those calculated analytically. The largest difference (0.044 m sec^{-1}) occurred in case A for $\lambda_e=7500 \text{ km}$, which is very close to the longwave limit λ_b ; otherwise, the differences in c_i were in the range $0.002\text{--}0.012 \text{ m sec}^{-1}$. Though the errors due to the use of finite differences must inevitably be larger in more complicated cases involving a tropopause discontinuity, these figures suggest that the numerical scheme used is generally sufficiently accurate for meteorological purposes. In case C ($\bar{\sigma}=\text{constant}$, $\beta=\beta_{45}$), c_r was slightly less than in case A, and c_i slightly less than in either case A or case B.

Fig. 2b shows the calculated variation of c_r-u_1 and c_i with λ_e for cases D–I, all of which assume a tropo-

pause at 250 mb. The curves marked D are calculated from (3.2) for a basic current with $\bar{u}=u_2$ for $P \leq P_2$, but with $\bar{\sigma}=0$ at all levels. Cases E–I have $\tau=1$ in the stratosphere and various stability distributions in the troposphere, and c_r and c_i are calculated numerically. In case E, with $s=0$ in the troposphere, c_i is slightly less, and c_r considerably less, than in case D, the differences being largest for short waves. Case F has $s=0.38271$ throughout the troposphere, a value which is realistic at low levels but much too small in the high troposphere. As compared with case E, case F has c_r slightly less for all wavelengths, and c_i about the same for long waves, but much less for short waves.

Case G has the first more or less realistic stability distribution considered so far; in the troposphere $\tau=0.38271$, so that $\bar{\sigma}$ is 16 times as large just below the

FIG. 2. Variation of c_r-u_1 and c_i with λ_e in cases A to C ($P_2=0$), a., and in cases D to I ($P_2=0.250$), b., both for $\phi=45^\circ$, $u_T=40 \text{ m sec}^{-1}$, $a_u=0$, $T_2=216.66\text{K}$. See text.

tropopause as it is at 1000 mb. (For still greater realism the wind profile curvature parameter a_u should be positive instead of zero.) In comparison with case F, values of c_r and c_i are slightly less for long waves and substantially less for short waves. As will be shown later, the reduction in c_r results from the fact that the large values of $\bar{\sigma}$ in the upper troposphere reduce perturbation amplitudes at upper levels relative to lower levels, and in accordance with (2.14), the lower levels, where $\bar{u}$ is less, are given greater weight in determining c_r . Thus, in a basic current with a normal increase in both $\bar{u}$ and $\bar{\sigma}$ from 1000 mb to the tropopause, one can expect the graph of c_r to have a maximum for intermediate wavelengths, where neither c_R (proportional to λ_e^2) nor the modifying effect of the tropospheric stability distribution (proportional to λ_e^{-2}) is too large. In the shortwave range, both c_i and the ratio, $c_i/\lambda_e = G^*/2\pi$, decrease as λ_e decreases; consequently, G^* , like c_r , has a maximum value for intermediate wavelengths.

Case H, with reduced hydrostatic stability in the troposphere, especially at low levels, is intended to simulate the destabilizing effect of latent heat released in precipitation. The model is admittedly imperfect, as such effects would be largely confined to areas with upward motion. At any rate the main differences from case G are a substantial increase in c_i for the shorter waves, and a slight increase in c_r .

Case I is intended to simulate the very high static stabilities found at low levels in continental arctic air in winter. In comparison with case G, both c_r and c_i are substantially reduced, especially for short waves.

The main series of calculations was carried out for various latitudes ϕ , with $u_1=0$, $a_u=0.5$, and climatologically reasonable values for P_2 , T_2 , τ_1 and τ_2 . Results will be illustrated for $\phi=45^\circ$, $P_2=0.250$, $T_2=216.66\text{K}$, and $\tau_1=\tau_2=0.38271$. Graphs of $B(P)$ and $\tau(P)$ for this case are shown in Fig. 1; $I_1=0.5626$, $I_2=0.12734$, and the mean tropospheric wind shear is $u_T/(10.76\text{ km})$.

Fig. 3a shows the real part of the phase velocity c_r (relative to the 1000-mb current velocity u_1) as a function of the thermal wind parameter u_T and the effective wavelength λ_e . Fig. 3b shows the amplification factor or growth rate, $G^*=k_e c_i = 2\pi c_i/\lambda_e$, for perturbations of infinite lateral extent ($k_y=0$, $\lambda_e=\lambda_x$). The growth rate for waves of finite lateral extent is, in general, somewhat less, being given by $G=k_x c_i = (\lambda_e/\lambda_x)G^*$; for example, $G=0.707G^*$ if $\lambda_y=\lambda_x$. The graphs for both c_r and G^* all exhibit a maximum in the intermediate wavelength range. Figs. 3a-b may be compared with Figs. 3 and 4 of Kuo's 1952 paper, bearing in mind that the basic current assumed in his calculations had no stratosphere and continued to increase indefinitely with height. Wind shear and hydrostatic stability in the lowest 10.76 km of his model and the present one are approximately the same if his results for $d\bar{u}/dz=4\text{ m sec}^{-1}\text{ km}^{-1}$ are compared with ours for $u_T=43\text{ m sec}^{-1}$. Apparently the in-

FIG. 3. Variation of $c_r - u_1$, a., and $G^* = k_e c_i$, b., with λ_e and u_T for $\phi=45^\circ$, $P_2=0.250$, $a_u=0.5$, $\tau_1=\tau_2=0.38271$, $T_2=216.66\text{K}$.

clusion of a barotropic stratosphere in the model increases the maximum values of c_r and G^* somewhat, but makes the curves drop off more rapidly with increasing wavelength.

The vertical structure of the perturbation for the case $u_T=40\text{ m sec}^{-1}$ is shown in Figs. 4a-c for growing waves with $\lambda_e=7000$, 4000 and 2000 km. (Fig. 4d is discussed below.) The complex amplitudes Z and X are plotted in polar form, each point representing a particular level. Each amplitude is expressed as a fraction of its maximum modulus. In each figure the meridional wind perturbation is proportional to Z and the vertical motion variable W is proportional to X , but with a phase difference of 90° in each case. The scalar amplitude of the height perturbation, $|Z|$, has its maximum value at the tropopause for the longer waves, and at 1000 mb for the shorter. In the stratosphere $|Z|$ and $|X|$ both de-

FIG. 4. Vertical variation of the complex amplitudes Z and X [in growing waves with effective lengths 7000 km, a., 4000 km, b., 2000 km, c., for $\tau_1 = \tau_2 = 0.38271$; and 2000 km, d., for $s_1 = s_2 = 0.38271$ ($u_T = 40$ m sec $^{-1}$; other parameters as in Fig. 3).

shorter waves $|Z|$ decreases monotonically with height. The phase of the Z perturbation in growing waves is later at the tropopause than at 1000 mb; for the fastest-growing waves (roughly, $\lambda_e = 4000$ km for $u_T = 40$ m sec $^{-1}$, as in Fig. 4b), the lag is about 85° . The polar plot of X forms a wide loop in the complex plane for the longer waves and a narrower loop for short waves. Thus, for waves of moderate length there is no tropospheric level at which the isobaric divergence, $D = -dW/dP$, comes close to zero, but for short waves it has a relatively sharp minimum. The level where $|X|$ (and hence $|W|$) is a maximum descends from about 480 mb for $\lambda_e = 7000$ km to about 680 mb for $\lambda_e = 2000$ km. Most of the features illustrated in Figs. 4a-c can be predicted qualitatively from the governing equations.

In the stratosphere, Z and D vary as P^n and W varies as P^{n+1} , where the real part of n is positive and the imaginary part is negative for growing waves and positive for damped waves. Values of n_r and n_i for growing waves are given in Table 2 as a function of λ_e for $u_T = 40$ m sec $^{-1}$. The real part n_r increases steadily as λ_e decreases, and becomes large for short waves, as would be expected from the relations, $n = (b + \frac{1}{2})^{\frac{1}{2}} - \frac{1}{2}$, $b = a[1 - c_R(u_2 - c)^{-1}]$, where a varies as λ_e^{-2} . The imag-

TABLE 2. Values of the exponent, $n = n_r + n_i i$, for $\phi = 45^\circ$, $P_2 = 0.250$, $u_T = 40$ m sec $^{-1}$, $a_u = 0.5$, $\tau_1 = \tau_2 = 0.38271$, $T_2 = 216.66$ K, as a function of the effective wavelength λ_e (km).

λ_e	7000	6000	5000	5000	3000	2000
n_r	0.40	0.62	0.92	1.36	2.07	3.48
n_i	-0.13	-0.12	-0.11	-0.08	-0.04	-0.01

crease with height, rather slowly for long waves but rapidly for short waves. In the troposphere, the locus of Z for the longer waves is relatively straight, and $|Z|$ has a minimum value at intermediate levels, but for the

inary part n_i has its maximum magnitude for relatively long wavelengths and is generally rather small.

As indicated earlier, the effect of hydrostatic stability on wave structure depends on the distribution in the vertical of the quantity, $q^2 = \kappa RT_2 s (2\pi/f\lambda_e)^2 = \kappa RT_2 \tau (2\pi/Pf\lambda_e)^2$.

For long waves, large values of q^2 occur only in the stratosphere. In the troposphere, Eq. (3.1) for the case $\bar{\sigma}=0$ will hold approximately; the complex height amplitude Z at different levels will be roughly proportional to $\bar{u}-c$, so that the polar plot of Z will be nearly straight, and $|Z|$ will have a minimum value near the "steering" level where $\bar{u}=c_r$. The variation of c with λ_e will be given approximately by (3.2). As λ_e increases, the increase in c_R will reduce both c_r and $|c_i|$; the steering level will become lower, and the ratio $|Z_2/Z_1|$ larger, where subscripts 1 and 2 refer to conditions at 1000 mb and at the tropopause. The calculated variation of c_i and G^* suggests that they become zero for a value of λ_e slightly less than the longwave limit λ_b given by (3.3).

For short waves, values of q^2 are larger, and the effect of the static stability distribution becomes quite important. In the troposphere, τ is normally more or less independent of P , so that q^2 is much larger in the upper troposphere than at 1000 mb. This increase of q^2 with height (roughly as P^{-2}) tends to reduce perturbation amplitudes at upper levels relative to lower levels, as can be seen by comparing Figs. 4c and 4d. In both figures λ_e is 2000 km and u_T 40 m sec⁻¹, but the tropospheric variation of $\bar{\sigma}(P)$ is given by $\tau=0.38271$ ($\bar{\sigma} \propto P^{-2}$) in 4c and by $s=0.38271$ ($\bar{\sigma}$ constant) in 4d. In 4c, the upper troposphere has large values of q^2 and small values of $|Z|$ and $|X|$; in 4d, it has abnormally small values of q^2 and abnormally large values of $|Z|$ and $|X|$. As indicated above, the reduction in upper troposphere amplitudes, caused by large upper troposphere values of $\bar{\sigma}$, reduces c_r in accordance with (2.14). When τ is sufficiently large at all levels, both c_i and c_r/λ_e tend to decrease with λ_e when λ_e is sufficiently small. Thus, with normal static stabilities, both c_r and G^* tend to have maximum values for intermediate wavelengths. In such cases the calculated variation of G^* with λ_e suggests that there is a shortwave cutoff for the present type of instability, roughly independent of u_T (see Appendix 1). However, this critical value of λ_e seems to be smaller, by a factor of almost 2, than the value calculated from (3.5) by using the smallest value of $s(P)$ occurring at any level in the troposphere.

Each complex eigenvalue obtained for c by the present scheme is a member of a conjugate pair, with equal positive and negative values for c_i . In any given case, the question of the uniqueness of a particular eigenvalue in a finite region of the complex plane can be studied by calculating the residual, $E=E_r+E_i i$, for trial values of c ranging over the region. Reversing the sign of c_i merely reverses the sign of E_i , but as (2.4) normally has a singularity at some level for $c_i=0$, it is necessary to restrict the calculation to a region in which c_i exceeds some positive minimum value. Calculations were per-

formed for $u_T=40$ m sec⁻¹ and for values of λ_e ranging from 2000–7500 km. For each wavelength, values of E_r and E_i were calculated for trial values of c_r and c_i forming a 9×9 grid in the complex plane, and ranging from 0.2–1.8 times the normal "first guess" values. In each case the results gave no indication of the existence of any other eigenvalue for c in the region scanned. It seems likely that this is the normal situation in the relatively simple cases considered here. However, the writer would not wish to speculate as regards cases involving wavelengths close to the ends of the primary unstable range, or complicated wind and stability profiles.

For intermediate wavelengths, values of c_r and c_i obtained by starting from different first guesses were usually reproducible to 0.001 or 0.002 m sec⁻¹. However, absolute errors arising from the use of finite differences are considerably larger, and in most cases almost certainly exceed the 0.002–0.012 m sec⁻¹ found in cases A and B of Table 1. In general, the largest errors will be those due to the imperfections of the physical model, e.g., the assumption that perturbations are geostrophic. Such errors become important for short wavelengths, strong vertical shears and low latitudes.

Calculations were carried out for various latitudes, using reasonable values for P_2 , a_u , T_2 , τ_1 and τ_2 . In each case u_T was varied over a wide range (e.g., 15–80 m sec⁻¹), and λ_e in steps of 500 km between 1500 km and the limiting value λ_b given by (3.3), using approximation formulas given in Appendix 1 to provide first guesses for c_r and c_i . For a given value of u_T , solutions were usually obtained for the intermediate range of wavelengths in which G^* was a substantial fraction of its maximum value, but not for shorter or longer wavelengths. Calculations involving weak thermal winds or effective wavelengths $\lesssim 2000$ km were usually unsuccessful, especially for low latitudes.

In interpreting results of calculation, it should be borne in mind that values of c_r and $G^*=k_z c_i$ depend on the effective wavelength, $\lambda_e=(\lambda_x^{-2}+\lambda_y^{-2})^{-1/2}$, which is less than either λ_x or λ_y . In case A, for example, waves will be unstable, no matter how large λ_x is, provided λ_y is small enough to make $\lambda_e < \lambda_b$. A similar situation exists in more complicated cases. Moreover, the amplification factor, $G=k_z c_i=(\lambda_e/\lambda_x) G^*$, involves λ_x and is less than G^* .

In any case, the existence of lower and upper ranges of λ_e in which neutral waves are mathematically possible is probably of limited meteorological interest. As indicated earlier, recent studies suggest that, for most such values of λ_e , slowly amplifying solutions are also possible.

5. Conclusions

The purpose of the present study was to develop a method of carrying out phase velocity and wave structure calculations for quasi-geostrophic perturbations in simple baroclinic westerly currents. The scheme described has to a considerable extent achieved this objective for the range of wavelengths giving the largest

amplification rates. For a given effective wavelength in this range it yields a complex eigenvalue for c with $c_i > 0$, as one member of a conjugate pair. In simple cases this pair of eigenvalues seems to be unique.

The usefulness of the calculations depends mainly on the adequacy of the physical model, which assumes a simple basic current and a small perturbation which is geostrophic, adiabatic, and unaffected by the terrain. Errors due to the use of the geostrophic approximation will, in general, be important in calculations involving short waves, strong vertical shears, or low latitudes. The numerical accuracy of the computation scheme itself seems to be adequate.

For a given basic current, the scheme yields solutions only in the range of effective wavelengths in which c_i is a substantial fraction of its maximum value. Within this range, its success depends on the use of good approximation formulas to provide first guesses for c_r and c_i . In general, convergence can be obtained without too much difficulty for middle latitudes, moderate-to-strong vertical shears, normal static stabilities, and effective wavelengths not too close to the ends of the primary unstable range. In the absence of any one of these conditions, convergence is usually difficult to obtain. Further experimentation with approximation formulas and with parameters used in them may in time extend the range of convergence.

Acknowledgements. The writer is indebted to Dr. J. Clodman for the opportunity to carry out this study, to Dr. A. D. Christie for reading the manuscript, to Miss T. C. Smith and Mrs. A. D. Belnavis for typing the manuscript, and to Mr. J. MacPhail and Miss M. Karnath for assistance in drafting the diagrams. Discussions with Dr. W. L. Godson and Dr. M. Shabbar, and the comments of the reviewers on an earlier version of the paper, were most helpful. This report is published with the approval of the Director, Meteorological Service of Canada.

APPENDIX 1

Numerical Computation Procedures

The following notes on numerical procedures reflect the current state of the computer program used, and may be of interest to prospective users and modifiers of the scheme.

a. Numerical integration

Eqs. (2.9) and (2.10) were integrated between 1000 mb and the tropopause in steps of 10 mb ($\Delta P = 0.01$), using approximations of the type

$$Z_{j+1} = Z_{j-1} + 2\Delta P (dZ/dP)_j,$$

and similarly for X . The accuracy so obtained was considered adequate for the limited purposes of this study, but could no doubt have been increased by the use of a more elaborate integration scheme. The characteristic defect of the scheme used, the gradual divergence of the

curves representing values for even and odd values of j , could just be detected on large-scale plots for certain cases (see Fig. 4c).

b. Iterative solution for phase velocities

Values of the phase velocity, $c = c_r + c_i i$, corresponding to a given effective wavelength λ_e , were found by iterative improvement of starting values. In each improvement cycle, integration was started at $P=1$, with $X=0$ and $Z=1$, and the complex residual at the tropopause (subscript 2) was calculated as $E = [X_2 - n(u_2 - c)P_2 A_2]/Z_m$, where Z_m was the largest value of $|Z|$ encountered in the range of integration. The improvement process for c was based on the assumption that $E_r = A\xi + B\eta$, $E_i = C\xi + D\eta$, where ξ and η were the unknown departures of the current estimates of c_r and c_i from the true values. The coefficients A , B , C and D were estimated by making small changes, of the order of 1%, in the assumed values of c_r and c_i , and noting the resulting changes in E_r and E_i . The corrections to be applied to the assumed values were then given by $-\xi = K(BE_i - DE_r)$, $-\eta = K(CE_r - AE_i)$, where $K = (AD - BC)^{-1}$, except that estimates of $c_r - u_1$ and c_i were not allowed to become zero or change sign. Values of A , B , C , D , K were recalculated in each cycle as long as $E_T \equiv |E_r| + |E_i|$ exceeded a coarse limit $\epsilon_1 (\sim 0.1 \text{ m sec}^{-1})$, but were left unchanged thereafter. The iteration was successfully terminated when E_T no longer exceeded a fine limit $\epsilon_2 (\sim 0.002 \text{ m sec}^{-1})$. The fine limit was set low, in spite of the modest accuracy of the calculation, in order to obtain consistent values of c_r and c_i which could be used in recalculating parameters in approximation formulas.

The iteration was unsuccessfully terminated if the total number of cycles exceeded a pre-set maximum value, or if E_T failed to decrease in any cycle. In such cases an attempt was made to find a better starting value for c by evaluating E_T for values of c forming a rectangular grid in the complex plane about the original starting value. The iteration process was then re-started from the value of c which had given the lowest of value E_T . If this second attempt also failed, the program abandoned that case and went on to the next.

c. Starting values for c

Formulas to provide initial estimates of c_r and c_i as functions of λ_e and the basic current parameters were gradually evolved as results of successful calculations accumulated. In the two cases A and B mentioned earlier, theoretical values based on (3.2) and (3.4) were used. In the main series of calculations, a_u was 0.5 and P_2 , T_2 , τ_1 , and τ_2 were given realistic values for a series of latitudes ϕ . It was found that, for each latitude, values of $c_r - u_1$ and c_i for a wide range of values of u_T and λ_e could be represented fairly well by empirically modified versions of the formulas, $c_r - u_1 + \frac{1}{2}c_R = I_1 u_T$, $c_i^2 + \frac{1}{4}c_R^2 = I_2 u_T^2$, which apply when $\bar{\sigma} = 0$ at all levels (case A). The versions last used in the program will be included for reference, though they could doubtless be

further improved. Different estimation formulas, continuous at a critical wavelength λ_c , were used in different ranges of λ_e , as follows:

For $\lambda_e < \lambda_c$:

$$c_r - u_1 + \frac{1}{2}c_R = H(I_1 u_T - c_1), \quad (A1)$$

$$c_i^2 + \frac{1}{4}c_R^2 = r^2(I_2 u_T^2 + c_2^2), \quad (A2)$$

where r is a linear and H a quadratic function of λ_e , and c_1 and c_2 are constants.

For $\lambda_e > \lambda_c$:

$$c_r - u_1 + \frac{1}{2}c_R = I_1 u_T - c_4, \quad (A3)$$

$$c_i = (I_2 u_T^2 - \frac{1}{4}c_R^2)^{\frac{1}{2}} - c_5 r^2 - c_3, \quad (A4)$$

where c_3 is a constant, and $r_2 = (\lambda_b - \lambda_e)/(\lambda_b - \lambda_c)$. Here λ_b is the longwave limit defined by (3.3) for the given value of u_T , and c_4 and c_5 are chosen to ensure continuity at $\lambda_e = \lambda_c$. Phase velocity and wave structure calculations were attempted only if the calculated first guess for c_i was real and positive.

For given values of the basic current parameters ϕ , P_2 , etc., the estimation parameters used were eight in number: c_1 , c_2 , c_3 (velocities); λ_s , λ_A (lengths); and H_0 , H_1 , H_2 (nondimensional). The factor r in (A2) was calculated as $(\lambda_c - \lambda_A)/\lambda_s$, where λ_s is a characteristic or "scaling" wavelength, and λ_A is roughly analogous to, but not identical with, the shortwave limit λ_a defined by (3.5) in case B. The factor H in (A1) was calculated as $H_0 + H_1 r - H_2 r^2$. An additional parameter r_c (~ 0.6 – 0.8) was used to specify the critical wavelength λ_c as $\lambda_A + r_c \lambda_s$.

In practice, the program read in from data cards values of ϕ , P_2 , T_2 , τ_1 , τ_2 , a_u , together with estimated values for λ_A , $\dots$, c_3 . These cards were followed by one specifying either a single value for u_T or high and low limits for u_T . Calculations were then performed either for the single value of u_T or for a series of values, starting at the high limit and working down in steps of 10 or 5 m sec⁻¹. This procedure could be repeated for other sets of basic current and estimation parameters.

For each value of u_T , the program first evaluated c_r and c_i for three equally spaced values of λ_e lying well within the range λ_A to λ_c (or λ_A to λ_b , if λ_b was less than λ_c). In this range, the quantity $(c_i^2 + \frac{1}{4}c_R^2)^{\frac{1}{2}}$ is assumed to be a linear function of λ_e . If calculations for all three wavelengths were successful, the values of c_r and c_i were used to calculate revised values for λ_A and λ_s (using the two outside wavelengths) and also for H_0 , H_1 and H_2 . If calculations were successful only for the two longest wavelengths, revised values were calculated for λ_A , λ_s , H_0 and H_1 , but not for H_2 . Values of c_1 , c_2 , c_3 were not revised. The program then performed calculations for a series of effective wavelengths λ_e in steps of 500 km, starting at 1500 km but considering only values in the range λ_A to λ_b for which values of c_i estimated from (A2) or (A4) were real and positive. Although revised values of λ_A , $\dots$, H_2 were calculated for each value of

u_T , the values so obtained normally fell within a fairly narrow range.

In each calculation for a given set of values of ϕ , P_2 , etc., successfully calculated values of c_r and c_i for different values of λ_e and u_T were entered on punched cards, for input to a separate regression program which computed revised values for c_1 , c_2 , c_3 .

The approximation scheme just described has been fairly successful for the more normal types of profiles of $B(P)$ and $\tau(P)$. For unusual profiles it is sometimes less suitable; thus, with stability profile H it gave a revised value of λ_A which was negative.

The approximation formula (A2) implies that $c_i = 0$ for an effective wavelength λ_0 slightly greater than λ_A . For given values of ϕ , P_2 , a_u , τ_1 , τ_2 , values of λ_0 calculated from revised values of λ_A and λ_s showed relatively little variation with u_T . For example, in the calculations illustrated in Figs. 3a–b, λ_0 varied from 1123 km for $u_T = 15$ m sec⁻¹ to 1237 km for $u_T = 80$ m sec⁻¹. These estimates of a minimum value of λ_e for instability should not be taken too seriously, as they assume the validity of (A2), and depend to some extent on the wavelengths used in obtaining revised estimates of λ_A and λ_s . The value of λ_a obtained from (3.5) by using the smallest tropospheric value of s (0.38271 at 1000 mb) was 2094 km.

APPENDIX 2

List of Symbols

R	gas constant for unit mass of dry air
c_p	specific heat of air at constant pressure (in dynamical units)
κ	$R/c_p \approx 2/7$
p	pressure
P	$p/(1000 \text{ mb})$
T	absolute temperature
θ	$TP^{-\kappa}$, the potential temperature
t	time
ω	dP/dt (not dp/dt) following the motion
z	geopotential height above sea level
g	acceleration of gravity used in calculating z (980 cm sec ⁻²)
x, y	Cartesian coordinates, x eastward, y northward from a standard latitude
ϕ	latitude
Ω	angular velocity of the earth's rotation
f	$2\Omega \sin\phi$, the Coriolis parameter
β	df/dy , the Rossby parameter
i	unit imaginary number
$()_r$	real part of a complex quantity
$()_i$	imaginary part of a complex quantity
$(-)$	mean value around a latitude circle
$()'$	$() - (-)$, the departure from mean value for latitude
$\nabla^2 F$	$F_{xx} + F_{yy}$ for a scalar variable F
$J(F, G)$	$F_x G_y - F_y G_x$ for scalar variables F, G
P_2	value of P at the tropopause
$\bar{u}(P)$	zonal wind velocity in the basic current
u_1	value of $\bar{u}$ at 1000 mb

u_2	value of $\bar{u}$ at and above the tropopause
u_T	$u_2 - u_1$, the thermal wind between 1000 mb and the tropopause
σ	$(R/g)P^{-(1-\kappa)}(-\partial\theta/\partial P)$, the hydrostatic stability factor
T_2	absolute temperature in the stratosphere
σ_0	$\kappa RT_2/g$ ($\bar{\sigma} = \sigma_0 P^{-2}$ in the stratosphere)
$s(P)$	$\bar{\sigma}(P)/\sigma_0$ (nondimensional static stability)
$\tau(P)$	sP^2 ($\tau=1$ in the stratosphere)
τ_1, τ_2	values of τ at 1000 mb and just below the tropopause
λ_x	$2\pi/k_x$, the zonal (longitudinal) wavelength of a sinusoidal perturbation
λ_y	$2\pi/k_y$, the meridional (transverse) wavelength
λ_e	$2\pi/k_e = 2\pi(k_x^2 + k_y^2)^{-1/2} = (\lambda_x^{-2} + \lambda_y^{-2})^{-1/2}$, the effective wavelength
$Y(y)$	latitude factor for perturbation amplitudes
$\hat{Z}(y, P)$	$Y(y)Z(P)$, the complex amplitude for isobaric height perturbation
$\hat{W}(y, P)$	$Y(y)W(P)$, the complex amplitude for vertical motion
X	$i(\sigma_0/k_x)W$, a dependent variable used in place of W
c	$c_r + c_i i$, the phase velocity of a sinusoidal perturbation
G	$k_x c_i$, the amplification factor (growth rate) for an unstable perturbation
G^*	$k_e c_i$, the limiting value of G for $k_y \rightarrow 0$
c_R	βk_e^{-2} , the retrograde phase velocity component
a	$g\sigma_0 k_e^2/f^2 = \kappa RT_2(2\pi/f\lambda_e)^2$
q^2	$as = \kappa RT_2\tau(2\pi/Pf\lambda_e)^2$
b	$a[1 - c_R(u_2 - c)^{-1}]$
n	$(b + \frac{1}{4})^{1/2} - \frac{1}{2}$, the complex exponent in the relation $Z = AP^n$ for the stratosphere
$B(P)$	$(\bar{u} - u_1)/u_T$, the scaled zonal wind velocity
I_1	$\int_0^1 B dP$
I_2	$\int_0^1 (B - I_1)^2 dP$
λ_b	$2\pi I_2^{1/2}(2 u_T /\beta)^{1/2}$, the minimum value of λ_e for stability if $\bar{\sigma} = 0$
λ_a	$(2\pi/2.399f)(s\kappa RT_2)^{1/2}$, the maximum value of λ_e for stability if $\beta = 0$, $\bar{\sigma} = \text{constant}$ and $d\bar{u}/dP = \text{constant}$
a_u	coefficient determining curvature of the zonal wind profile $\bar{u}(P)$
Q	$(1 - P)/(1 - P_2)$
E	$E_r + E_i i$, the residual quantity indicating the extent to which the tropopause boundary condition is not satisfied
ξ, η	deviations of trial values of c_r, c_i from true values
A, B, C, D, K (in Appendix 1):	coefficients such that $E_r \approx A\xi + B\eta$, $E_i \approx C\xi + D\eta$, $K = (BC - AD)^{-1}$

r, H_0, H_1, H_2 (nondimensional)
 $\lambda_s, \lambda_A, \lambda_c$ (lengths)
 c_1, c_2, c_3 (velocities)
 λ_0 lower limit of λ_e for instability, estimated from values of λ_A and λ_s revised during the calculation

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