Evaluation of the Simultaneous Multiple Pulse Repetition Frequency Algorithm for Weather Radar

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ABSTRACT
Performance of the simultaneous multiple pulse repetition frequency algorithm (SMPRF) for recovery of mean power and mean Doppler velocity is investigated using simulated weather radar data. Operation and functionality of the algorithm is described; methods to estimate mean power values using statistical inversion and to estimate mean velocity from unevenly spaced autocorrelation function samples are presented and analyzed. A simulation technique for constructing multiple pulse repetition interval data is described and the algorithm performance results are presented for an example SMPRF code using three weather profiles. This leads to the development of an error structure related to factors influencing moment recovery, including finite-length time series effects, the effects of overlaid echoes that create an effective signal-to-noise ratio that limits moment recovery performance, and the effects of spectrum width and radar frequency related to coherence time.

1. Introduction
Range–velocity ambiguity is a long-standing problem of pulsed radar systems and is a major issue in weather radar data quality. For uniformly spaced transmitted pulses, the maximum unambiguous range is given by
\[ r_a = cT/2, \]
where \( c \) is the speed of light and \( T \) is the pulse repetition time (PRT). The maximum unambiguous velocity (or Nyquist velocity) is given by
\[ v_a = \lambda/4T, \]
where \( \lambda \) is the radar wavelength. These two equations are combined to express the so-called range–velocity dilemma as
\[ r_a v_a = \frac{\lambda c}{8}. \] 
Thus, for a fixed wavelength, the product \( r_a v_a \) must be constant. Increasing \( r_a \) necessarily decreases \( v_a \). Also, as seen from the definitions of \( r_a \) and \( v_a \), as \( T \) increases, \( r_a \) increases while \( v_a \) decreases, and vice versa. As a reference point, consider a typical PRT of 1 ms at S-band operation, which yields \( v_a = 25 \text{ m s}^{-1} \) and \( r_a = 150 \text{ km} \). Weather phenomena routinely exceed both of these values. At low elevation angle scans, radar echoes can be present out to about 460 km if 18 km above ground level (AGL) is used to define cloud top (taking the earth’s curvature into consideration). The mean velocity of weather echoes can exceed 40 m s\(^{-1}\), but this value is usually considered a sufficient maximum unambiguous velocity; for example, \( v_a = 40 \text{ m s}^{-1} \) is the Federal Aviation Administration (FAA) requirement for Terminal Doppler Weather Radar (TDWR). Thus, to attain a maximum unambiguous range of 460 km, \( T = 3.07 \text{ ms} \) is required, whereas to attain a Nyquist

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velocity of 40 m s\(^{-1}\), \(T = 0.625\) ms is required. The use of higher transmit frequencies further exacerbates this discrepancy by decreasing \(v_u\).

Several techniques exist to ameliorate the effects of range–velocity ambiguity in weather radar. These techniques can be generally divided into four types: 1) phase coding of transmitted pulses, 2) using dual or staggered PRTs, 3) using polarization-diversity pulse pair processing, and 4) employing multiple pulse-repetition frequencies (PRFs).

Phase coding of transmitted pulses allows for separation of overlaid echoes via spectral processing techniques. Random phase coding has been investigated by Zrnić (1979), Laird (1981), Siggia (1983), and Zrnić and Mahapatra (1985). When cohering the return pulses for a particular trip echo (usually the strongest trip first), the overlaid echo behaves as white noise because of the phase coding, and thus unbiased estimates of the strong trip autocorrelation function (ACF) are obtained (e.g., velocity is calculated from the first lag of the ACF). A spectral notch filter can be used to eliminate some of the strong trip echo, thus providing better separation of strong and weak trip echoes. Sachidananda and Zrnić (1999) and Frush et al. (2002) improved upon the random phase coding technique by using a systematic phase coding of transmitted pulses. As is the case with random phase coding, systematic phase codes also whiten the overlaid weak echo, but better separation of the overlaid echoes is possible, resulting in better moment estimates of the weak trip echo.

The dual (or staggered) PRT technique utilizes two PRTs that alternate in time (Sirmans et al. 1976; Dazhang et al. 1984; Zrnić and Mahapatra 1985). Two first lag complex ACF samples corresponding to each PRT are calculated as \(R_1\) and \(R_2\). Thus, the velocity is calculated as

\[
\hat{v} = \frac{\lambda [\text{arg}(R_1 R_2^*)]}{4\pi (T_2 - T_1)},
\]

with the corresponding unambiguous velocity being equal to

\[
v_u = \frac{\lambda}{4(T_2 - T_1)},
\]

where \(T_2 > T_1\). Thus, the time deference \(T_2 - T_1\) determines \(v_u\). However, if \(T_2 - T_1\) is too small, the variance of the mean velocity estimate becomes large. Zrnić and Mahapatra (1985) have shown that \(T_1/T_2 = 2/3\) gives optimal performance. Theoretically for this scheme, the unambiguous range is

\[
r_a = \frac{c(T_2 + T_1)}{2}.
\]

However, because of overlays, the practical unambiguous range is limited to \(r_a = cT_2/2\) (Zrnić and Mahapatra 1985). Sachidananda and Zrnić (2003) proposed a signal processing technique to separate echoes for the one overlay case, thereby extending the unambiguous range to \(r_a = cT_2/2\). The technique works well for narrow spectrum widths up to overlay power ratios of 40 dB, but good recovery is not always possible for larger spectrum widths.

Doviak and Sirmans (1973) proposed a range-velocity mitigation technique for dual-polarization radars utilizing the property that signals produced by the copolar backscattered fields at horizontal and vertical polarizations are highly correlated. Horizontally (H) and vertically (V) polarized uniform PRT (and equal) pulse trains are lagged by \(\tau = \text{PRT}/2\). The PRT can be selected independently to give a large \(r_a\). The orthogonal polarizations are uncoupled (i.e., the depolarization on backscatter is small) and thus have a range fold based on \(\tau\). Because the two channels are correlated, the velocity can be calculated using both channels, yielding a fold velocity based on \(\tau\). In this case, a small \(\tau\) will yield a high \(v_u\), and vice versa. The technique is limited, however, by the large variance in Doppler estimates that results from small values of \(\tau\) and by the incoherent overlay of return echoes acting as noise, which increases the variance of all estimates. Pazmany et al. (1999) used this technique for W-band radar, while Hubbert and Chandrasekar (2006) proposed phase coding of the H- and V-pulse trains to help separate the cross-polarized overlays and decrease the variance of the velocity estimate.

Another technique for range–velocity ambiguity mitigation is to transmit several blocks of uniform PRT pulses with each block at a different PRF. Cho (2005) has evaluated this technique, termed the multipulse repetition interval (MPRI), for the FAA’s TDWR. Each block of PRTs has a distinct \(r_a\) and \(v_u\), and these estimates are combined to optimize radar moment recovery. Selection of the PRFs for each PRT block is based on a previous long PRT scan that determines echo location, and therefore the possible range aliasing that would occur for each PRF. The block PRF is chosen on a radial-by-radial basis, such that the radar moment recovery is optimized for the first 90 km from the radar.

Recently, a new multistaggered PRT technique has been proposed for a range-velocity ambiguity resolution called the simultaneous multiple pulse repetition frequency (SMPRF; Pirittilä et al. 2005). This study served to investigate the effects of the long spatial extent, large dynamic range, and coherency issues that weather radar applications present to the SMPRF al-
algorithm, originally developed for ionospheric measurement applications, for range–velocity mitigation. In the SMPRF algorithm, several different PRTs are chosen and are concatenated to form a block of PRTs, which is repeated in time. Thus, the time length of the block of PRTs is (neglecting the transmit pulse width) \( T = T_1 + T_2 + \ldots + T_i \), where \( i \) is the number of unique PRTs in the block and \( T \) is typically chosen to be equal to or to exceed the desired maximum unambiguous range. A block of pulses separated by the selected PRTs is repeatedly transmitted according to the desired dwell time. Time series with spacing of \( T \) are constructed for each sample time and typically consist of echoes from several resolution volumes. The SMPRF algorithm then uses a matrix inversion technique to solve the set of simultaneous equations to obtain mean power estimates for each resolution volume. The set of constructed time series is also used to generate unequally spaced samples of the ACF for each resolution volume from which mean velocity is estimated.

The theory of the SMPRF technique is presented in Pirttilä et al. (2005), where the technique is illustrated with experimental C-band radar data. This paper presents a detailed evaluation of the SMPRF-recovered mean power and velocity estimate statistics using simulated data and examines issues limiting their recovery. First, the theory of the SMPRF algorithm is reviewed and the mean power and mean velocity recovery techniques are described. Next, the simulation procedure for constructing the time series is developed. The simulated time series are then used to evaluate the mean error and standard deviation of SMPRF-recovered mean power and mean velocity estimates for a particular SMPRF scheme for three weather profiles. The results of the evaluation are generalized and can be extended to any choice of specific code or weather profile.

The main factors affecting performance of the SMPRF algorithm are examined to establish an error structure for the algorithm; the effects of finite-duration time series relating to dwell time, the effects of multiple trip overlay, and the effects of spectrum width and radar frequency relating to coherence time are studied. The paper concludes with a summary of results and conclusions.

2. The SMPRF algorithm

a. Overview of SMPRF operation and definition of terminology

Operation of the SMPRF algorithm entails transmission of multiple pulses at different PRTs to extend the maximum unambiguous measurement range and velocity. Transmission of this set (or block) of PRTs is repeated over the dwell time of the radar measurements. Equally spaced time series are constructed from the radar echoes from which moment estimates are recovered.

To illustrate the operation of the SMPRF algorithm, consider the example code given in Pirttilä et al. (2005). This particular code is composed of four PRTs: 750, 1200, 1500, and 1050 \( \mu \text{s} \). To simplify notation, these PRTs are normalized by the greatest common divisor of the PRTs, namely \( 150 \mu \text{s} \). The normalized PRTs are then 5, 8, 10, and 7 time units, respectively, and these normalized PRTs are referred to as the SMPRF code annotated as “SMPRF(150); 5, 8, 10, 7.”

A diagram of a block of pulses of the SMPRF(150); 5, 8, 10, 7 code is shown in Fig. 1, where the abscissa is labeled in normalized time units and the first time series sample used for moment recovery occurs at \( t = 1 \). Pulses are transmitted at times \( t = -30, -25, -17, -7, \) and 0. The pulse at \( t = 0 \) marks the beginning of the repetition of the PRT block shown at times \( t = -30 \) through \( t = -1 \). Thus, the block of PRTs then continues with pulses occurring at times \( t = 0 + 30i, 5 + 30i, 13 + 30i, \) and \( 23 + 30i \), where \( i \) is the PRT block index ranging from 0 to \( M - 1 \). The number of transmitted blocks, \( M \), corresponds to the dwell time for measurements and also represents the length of the time series. In this code, there are 30 sample times and thus 30 possible time series corresponding to the nor-

Fig. 1. Illustration and operation of the SMPRF code; 5, 8, 10, 7 PRTs, \( \Delta t = 150 \mu \text{s} \).
malized integer times that occur during one block of PRTs. However, sampling of the returning echoes cannot happen simultaneously with pulse transmission, and therefore only 30 - 4 = 26 total times series are gathered for this SMPRF code. Note that a sample time of 150 μs following a transmitted pulse corresponds to a range of cT/2 = 22.5 km, and therefore the normalized sample times of 20 and 21 occur at ranges 450 and 472.5 km, respectively. When a transmitted pulse is more than 20 normalized time units from a sample time, there are no resulting return echoes because 460 km is considered to be the maximum possible range for weather echo return. The arrows in Fig. 1 show the return echo, which is received at time \( t = 1 \) and is composed of return echoes from the indicated transmit pulses. The pulses transmitted at times \( t = -25 \) and \( t = -30 \) do not contribute because they occurred more than 20 time units before the \( t = 1 \) sample time. Therefore, time series sample at \( t = 1 \) is written as

\[
V_1(1) = \beta_{1}^{(5)}(1) + \beta_{8}^{(7)}(1) + \beta_{18}^{(10)}(1),
\]

where the subscript indicates the sample time in normalized time units corresponding to the range resolution volume and the superscript indicates the pulse PRT in normalized time units, which illuminates the resolution volume. The member \((1)\) indicates the first sample of the time series. For example, \( \beta_{1}^{(5)}(1) \) represents the first time series element in the return time series corresponding to the eighth resolution volume illuminated by the previous pulse with PRT of seven normalized time units. Thus, \( V_1(1) \) is a time series sample gathered at sample time 1 and consists of three time series samples from resolution volumes that correspond in range to normalized times 5, 8, and 18, respectively. As the block of PRTs is repeated, additional time series samples are gathered. For the time series in Eq. (5), additional time series samples are gathered at \( t = 1 + 30i, i = 1 \) to \( M - 1 \). The time series corresponding to the sample times \( t = 1 + 30i, i = 0 \) to \( M - 1 \) is written as

\[
V_1(1 + 30i) = \beta_{1}^{(5)}(1 + 30i) + \beta_{8}^{(7)}(1 + 30i) + \beta_{18}^{(10)}(1 + 30i),
\]

and with the index understood as

\[
V_1 = \beta_{1}^{(5)} + \beta_{8}^{(7)} + \beta_{18}^{(10)}.
\]

Similarly, the time series corresponding to sample time \( t = 2 \) consists of samples collected at \( t = 2 + 30i, i = 0 \) to \( M - 1 \) and is written as

\[
V_1 = \beta_{2}^{(5)} + \beta_{7}^{(7)} + \beta_{10}^{(10)}.
\]

Continuing in this way, a set of time series equations is constructed, given by

\[
V_1 = \beta_{1}^{(5)} + \beta_{8}^{(7)} + \beta_{18}^{(10)}, \quad V_{16} = \beta_{3}^{(10)} + \beta_{11}^{(8)} + \beta_{15}^{(5)}, \quad V_{17} = \beta_{1}^{(5)} + \beta_{12}^{(7)}, \quad V_{18} = \beta_{5}^{(7)} + \beta_{13}^{(8)} + \beta_{18}^{(5)}, \quad V_{19} = \beta_{3}^{(10)} + \beta_{14}^{(8)} + \beta_{15}^{(5)}, \quad V_{20} = \beta_{7}^{(10)} + \beta_{15}^{(8)} + \beta_{18}^{(5)}.
\]

It is possible to solve this set of simultaneous time series equations for the mean power of each of the resolution volumes, that is, \( |\beta_{1}^{(5)}|^2 \).

The range spacing between adjacent resolution volumes is 22.5 km for 150-μs range sampling, and thus the above set of equations represents a subset of the total set of equations needed to solve for all desired resolution volumes. If the transmitted pulse width is 1 μs, range sampling is usually done at 1-μs intervals, which correspond to 0.15-km range resolution. In the above example, the range sample interval is 150 μs because of the spacing between PRTs as defined by the particular SMPRF code. Thus, to solve for all of the powers at 1-μs range-sampling intervals, 150 sets of simultaneous equations such as that shown in Eq. (9) need to be constructed. The 150 sets of time series equations are constructed at the following sets of sample times \( \Gamma_j \):

\[
\Gamma_j = (t|t \in j + 150iμs, i = 1, M - 1),
\]

where \( j \) is the equation set number ranging from 1 to 150. This separation of the inversion problem into 150 mutually independent smaller inversion problems is not necessary because one large matrix containing all ranges could be constructed and solved. For example, given a range resolution of 0.150 km and a maximum unambiguous range of 450 km, the matrix to be inverted would have 450 km/150 m = 3000 columns. Inversion of this larger matrix is less efficient and thus more computationally complex. The computational
savings of this reduction are described in Lehtinen (1999).

As shown in Pirttilä et al. (2005) and following the steps in the derivation of the maximum unambiguous velocity for the staggered PRT algorithm described above (Zrnić and Mahapatra 1985), the maximum unambiguous velocity for the SMPRF algorithm is given by

\[ v_{u,\text{SMPRF}} = \pm \frac{\lambda}{4 \min(\text{PRT}_i - \text{PRT}_j); i \neq j}. \]  

(11)

For the SMPRF(150);5,8,10,7 code with \( \Delta_t = 150 \) \( \mu \)s, this yields \( v_{u,\text{SMPRF}} = 89.33 \) m s\(^{-1}\). This value is more than adequate for the measurement of weather phenomena.

In subsequent sections, the following notation is adopted: \( P \) represents the true power value, \( \hat{P} \) represents the estimated mean power value prior to the inversion process, and \( \hat{P} \) represents the mean power estimate after the inversion process.

b. Mean power estimation using the SMPRF algorithm

The mean estimated power for the \( i \)th volume corresponding to the \( j \)th PRT is given by

\[ \hat{P}_i = \frac{1}{M} \sum_{k=1}^{M} |\beta_i^j(k)|^2; j \in (5, 7, 10, 8), i \in (1, 2, \ldots, 20). \]

(12)

The total estimated power \( \hat{Z}_i \) corresponding to the time series \( V_i \) is written as

\[ \hat{Z}_i = \frac{1}{M} \sum_{k=1}^{M} |V_i(k)|^2 \]

\[ = \frac{1}{M} \sum_{k=1}^{M} |\beta_1^5(k) + \beta_6^7(k) + \beta_8^{10}(k)|^2. \]

(13)

As a consequence of the independence between measurement volumes, all cross products are assumed to be equal to zero (Bringi and Chandrasekar 2001), therefore reducing Eq. (13) to

\[ \hat{Z}_i = \frac{1}{M} \sum_{k=1}^{M} |\beta_1^5(k)|^2 + \frac{1}{M} \sum_{k=1}^{M} |\beta_6^7(k)|^2 + \frac{1}{M} \sum_{k=1}^{M} |\beta_8^{10}(k)|^2 \]

\[ = \hat{P}_1 + \hat{P}_6 + \hat{P}_8^{10}. \]

(14)

Thus, the equation set for the total estimated powers, \( \hat{Z}_n; n \in (1, 2, \ldots, 30) \), corresponding to Eq. (9) can be written in matrix notation as

\[ \hat{Z} = \mathbf{A}\hat{P}. \]

(15)

where \( \hat{Z} \) is the vector of estimated power measurements, \( \mathbf{A} \) is the code matrix describing the signal over-lays, with \( A_{ij} \in \{0, 1\} \), and \( \hat{P} \) being the vector of calculated mean powers from each measurement range. Because the row space of the matrix \( \mathbf{A} \) is greater than its column space, the system of equations is overdetermined. The Moore–Penrose pseudoinverse is used to solve this set of equations, yielding the recovered mean power vector \( \hat{P} \), given by (Lehtinen 1999)

\[ \hat{P} = \Sigma_p \mathbf{A}^T \Sigma^{-1} \hat{Z}, \]

(16)

where \( \Sigma \) is the error covariance matrix and \( \Sigma_p \) is the solution covariance matrix, equal to

\[ \Sigma_p = (\mathbf{A}^T \Sigma^{-1} \mathbf{A})^{-1}. \]

(17)

If the error covariance matrix is chosen to be the identity matrix, Eq. (16) reduces to

\[ \hat{P} = \mathbf{A}^+ \hat{Z}, \]

(18)

where \( \mathbf{A}^+ \) is the pseudoinverse of \( \mathbf{A} \). The error covariance matrix was chosen to be the identity matrix for this study, and empirical studies showed similar performance results using an error covariance matrix estimated from measured data as suggested in Lehtinen (1999). Further details of this inversion technique can be found in appendix A.

As seen in Eq. (9), there are either three or four time series constructed for each resolution volume, depending on the number of PRTs that illuminate a given resolution volume. For example, given the resolution volume corresponding to normalized sample time 1, the four time series are \( \beta_1^5, \beta_6^7, \beta_8^{10}, \) and \( \beta_1^{18} \), that is, one time series corresponding to each PRT in the block of PRTs used. Because these time series are samples from the same assumed statistically stationary medium, mean power estimates become equal in the limit of \( M \) (Bringi and Chandrasekar 2001); that is,

\[ \frac{1}{M} \sum_{k=1}^{M} |\beta_1^5(k)|^2 = \frac{1}{M} \sum_{k=1}^{M} |\beta_6^7(k)|^2 = \frac{1}{M} \sum_{k=1}^{M} |\beta_8^{10}(k)|^2 \]

\[ = \frac{1}{M} \sum_{k=1}^{M} |\beta_1^{18}(k)|^2. \]

(19)

However, when calculating the average powers for finite-length time series, these estimates will not be equal because they are estimated from different samples separated in time.

To illustrate this point, consider the first and sixth equations in the set of measurement equations for the SMPRF(150);5,8,10,7 code (with time series time indices, \( k \), removed for convenience):
equations are measured volumes 1, 6, 14, and 24. The time series to echoes from resolution volume 1. The mean powers one for SMPRF (150); 5, 8, 10, 7 code. The argument of the time series product are 5 series that results from two blocks of SMPRF(150); 5, 8, 10, 7 resolution volume. Figure 2 shows the individual time information of unequally time-spaced lags of the ACF of theous lags of the autocorrelation function of normalized time units.

\[ V_1 = \beta_1^{(5)} + \beta_8^{(7)} + \beta_{18}^{(10)} \]
\[ V_6 = \beta_1^{(5)} + \beta_6^{(5)} + \beta_{13}^{(7)} \quad \text{(20)} \]

where \( \beta_1^{(5)} \) and \( \beta_1^{(8)} \) are two time series composed of echoes from resolution volume 1. The mean powers corresponding to \( \beta_1^{(5)} \) and \( \beta_1^{(8)} \) are assumed equal in the matrix inversion process for recovery of mean power estimates. The difference in estimated mean power values from these two time series will cause errors in the power recovery process. As will be shown, these errors can be quite large and bias the mean power estimates, \( \hat{P} \). The calculation of the mean and variance of the moment estimates under theoretically ideal conditions using the SMPRF algorithm is shown in appendix B.

c. Mean velocity estimation using the SMPRF algorithm

The SMPRF velocity recovery is based on the estimation of unequally time-spaced lags of the ACF of the time series for each resolution volume. Consider \( \beta_1^{(5)} \), \( \beta_1^{(7)} \), and \( \beta_1^{(10)} \), \( \beta_1^{(8)} \), which are the time series for the first resolution volume. Figure 2 shows the individual time series that results from two blocks of SMPRF(150); 5, 8, 10, 7 code. The arguments of the time series are given in normalized time units.

These four time series can be used to calculate various lags of the autocorrelation function of \( \beta_1 \). For example, consider the time series \( \beta_1^{(5)} \) and \( \beta_1^{(7)} \). Part of the ACF of \( \beta_1 \) can be calculated as

\[ R_1(5 + k30) = \sum_{i=0}^{M-1} \beta_1^{(5)}(i30)\beta_1^{(7)}[5 + (i + k)30]. \quad \text{(21)} \]

Thus, the ACF lags that can be calculated from this product are \( 5 + k30 \), where \( k \) can range from \( -(M - 1) \) to \( M - 1 \).

Consider the correlation of the returns from measurements volumes 1, 6, 14, and 24. The time series equations are

\[ V_1 = \beta_1^{(5)} + \beta_8^{(7)} + \beta_{18}^{(10)} \]
\[ V_6 = \beta_1^{(5)} + \beta_6^{(5)} + \beta_{13}^{(7)} \]
\[ V_{14} = \beta_{14}^{(10)} + \beta_6^{(8)} + \beta_{13}^{(5)} \]
\[ V_{24} = \beta_1^{(7)} + \beta_6^{(8)} + \beta_{13}^{(10)} \quad \text{(25)} \]

Correlating these pairwise gives

\[ E(V_1 V_6) = R_1(5 + k30) + \eta_{1,6} \]
\[ E(V_1 V_{14}) = R_1(13 + k30) + \eta_{1,14} \]
\[ E(V_1 V_{24}) = R_1(23 + k30) + \eta_{1,24} \]
\[ E(V_6 V_{14}) = R_1(8 + k30) + \eta_{6,14} \]
\[ E(V_6 V_{24}) = R_1(18 + k30) + \eta_{6,24} \]
\[ E(V_{14} V_{24}) = R_1(10 + k30) + \eta_{14,24} \quad \text{(31)} \]

where \( \eta_{i,j} \) represents zero-mean white Gaussian noise (WGN) with power dependent upon the amount of overlaid power present (Bringi and Chandrasekar 2001). Again, time series from different resolution volumes are independent and their cross correlation is zero in the limit of \( M \). However, for finite-length time series, these cross-correlation products will not be zero and will generate “noise” represented by the \( \eta \) terms above. This noise is a limiting factor for the mean velocity recovery process and will be illustrated later in this paper.

Coherence time \( T_D \) limits the range of ACF lag values used by the SMPRF algorithm to generate mean velocity estimates, is defined as the time for which magnitude of the autocorrelation drops to \( e^{-1} \) (Bringi and Chandrasekar 2001), and is given by

\[ T_D = \frac{\lambda}{2 \sqrt{2 \pi} \sigma_v} \quad \text{(32)} \]

The block length of the SMPRF(150); 5, 8, 10, 7 code is 4.50 ms. For a radar operating at C band (for this study, \( \lambda = 5.6 \text{ cm} \)) and a weather signal spectrum width of 2 ms, the coherence time is 3.01 ms. For a spectrum width of 4 m s\(^{-1}\), the coherence time drops to 1.51 ms. Given these parameters, the limits of ACF lags used in the estimation of the mean velocity are constrained to 15 in normalized time, or 15 \( \times 150 \mu s = 2.25 \text{ ms} \). Empirical SMPRF simulations show that the use of ACF lags greater than about 0.75 of the coherence time significantly degrade the mean velocity recovery statistics. Thus, for the following simulation studies of SMPRF(150); 5, 8, 10, 7, ACF lag times less than 2.25 ms are used for velocity recovery. This point will be discussed in section 4c.

Using the example SMPRF(150); 5, 8, 10, 7 code, the set of
ACF lag values available for each resolution volume is given in Table 1. As seen in Table 1, different resolution volumes will have both different number of available ACF lags as well as different ACF lag values. Note that the lack of radar capability to receive while transmitting accounts for the lower number of recovered ACF lags in certain volumes.

Once the ACF lag values are estimated, the mean velocity is then estimated. Two techniques for velocity estimation are examined in the following sections.

d. Spectral maximum technique

The technique used in Pirttilä et al. (2005) to estimate mean velocity from unevenly sampled ACF lag values uses the discrete Fourier transform (DFT) of the interpolated ACF. Missing values of the ACF are interpolated with zeros and the DFT is then taken, yielding an interpolated spectrum (Roberts and Mullis 1987). The location of the peak of the power spectrum is taken to be the estimate for the mean velocity.

The power spectrum is calculated as

$$S(k) = \sum_{n=0}^{N-1} \tilde{R}(n)e^{-(2\pi j/N)nk}, \quad k = 0, \ldots, N - 1,$$

where $N$ is the number of points used in the DFT calculation.

The following simulation illustrates this velocity-estimation technique. In Fig. 3, the power spectrum (zero padding is used so that $N_{FFT} = 512$) of a simulated 64-point weather signal is shown, with $\sigma_v = 2$ m s$^{-1}$, $\nu = 30$ m s$^{-1}$, SNR = $\infty$, and PRT = 150 ms. Then the ACF of the simulated signal is calculated. The unevenly spaced samples of the ACF calculated are obtained by selecting the desired ACF values according to Table 1, with all other ACF lags set to zero. This is equivalent to multiplying the calculated ACF function by a binary mask, $M_1(k)$, given by

$$M_1(k) = (10000101101001),$$

where $M_1(-k) = M_1(k)$. The members of the ACF can be listed as

$$R_1(k) = [\hat{P}_1 0 0 0 0 R_1(5) 0 R_1(7) R_1(8) 0 R_1(10) 0 0 R_1(13)],$$

where $\hat{P}_1$ is the mean power estimate from the first measurement volume and $R_1(-k) = R_1^*(-k)$. The power spectrum of the binary masking sequence is shown in Fig. 4.

Mathematically, the unevenly spaced ACF, $R_1(k)$, is formed by the multiplication of the weather signal and binary mask spectra as

$$R_1(k) = M_1(k)X(k),$$

where $X^*(-k) = X(k)$ is the ACF of the simulated weather signal. Because multiplication in the time domain corresponds to convolution in the frequency domain, power spectrum of the unevenly spaced ACF

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**Table 1. ACF lag values computed using SMPRF$^{(150),5,8,10,7}$ code.**

<table>
<thead>
<tr>
<th>Volume ($i$)</th>
<th>Lag values ($n$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2, 3, 4, 9</td>
<td>5, 7, 8, 10, 13</td>
</tr>
<tr>
<td>5, 13</td>
<td>8, 10, 12</td>
</tr>
<tr>
<td>6</td>
<td>5, 7, 8, 10</td>
</tr>
<tr>
<td>7, 12</td>
<td>5, 8, 13</td>
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<tr>
<td>8, 18</td>
<td>7, 10, 13</td>
</tr>
<tr>
<td>10, 17</td>
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</tr>
<tr>
<td>11</td>
<td>5, 7, 8, 10, 12</td>
</tr>
<tr>
<td>14, 15, 16, 19</td>
<td>5, 7, 8, 10, 12, 13</td>
</tr>
<tr>
<td>20</td>
<td>5, 8</td>
</tr>
</tbody>
</table>

---

**Fig. 3.** Velocity spectrum of simulated weather signal $v = 30$ m s$^{-1}$.

**Fig. 4.** Power spectrum of binary mask sequence.
spectrum $S_1$ is formed by the convolution of the weather power spectrum $X$ and binary mask spectrum $M$ as

$$\tilde{S}_1 = M_1 \ast X,$$

(37)

where $\ast$ denotes convolution.

Figure 5 shows the power spectrum of the simulated weather signal [i.e., the DFT of $R_1(k)$]. Comparing Figs. 3 and 5, it is seen that the peak of the spectra shown in Fig. 5 gives the velocity estimate of $\hat{v} = 28.61$ m s$^{-1}$ with the true velocity $v = 30$ m s$^{-1}$.

e. Phase-unwrapping technique

The technique described by Rubin (1995) can also be used to estimate the mean velocity from an ACF with unequal time spacing. In this technique, the velocity is determined using a minimum mean-square error (MMSE) fit to the various possible unwrapped velocities. For the algorithm to properly unwrap the phase angles and generate unambiguous mean velocity estimates, the interpulse intervals are chosen such that none is a linear combination or multiple of any other. This eliminates ambiguous estimates of the corresponding ACF values. Furthermore, the true mean velocity should reside in the Nyquist intervals.

The calculated arguments, $\theta_{C,i}$, of the autocorrelation function are unwrapped by adding $2k\pi$, $k = 0, \pm 1, \pm 2, \ldots \pm n$ to the first calculated phase, $\theta_{C,1}$. Here, $\theta_{C,1}$ is obtained for the shortest lag, $T_1$, and determines an integer, $m_{k,2}$, from these quantities. The calculated phase $\theta_{C,2}$, obtained for the next shortest lag $T_2$, is determined from the condition

$$|\left(\theta_{C,2} + 2\pi m_{k,2}\right) - \left(\frac{T_2}{T_1}\right)\theta_{C,1}| < \pi.$$

(38)

Once the integer $m_{k,2}$ is determined to satisfy Eq. (38), the phase corresponding to the second shortest PRT is set to be $\theta_2 = \theta_{C,2} + 2\pi m_{k,2}$. The phase angles $\theta_3, \theta_4, \ldots, \theta_m$ are unwrapped sequentially in a similar manner, with the integer $m_{k,i}$ determined from

$$\left|\left(\theta_{C,i} + 2\pi m_{k,i}\right) - \left(\frac{T_i}{T_{i-1}}\right)\theta_{C,i-1}\right| < \pi.$$

(39)

The root-mean-square error (RMSE) of the phase angle offset from each fitted line is determined by

$$\xi_k = \sqrt{\epsilon_{k,1}^2 + \epsilon_{k,2}^2 + \cdots + \epsilon_{k,m}^2},$$

(40)

where $\epsilon_k$ is the offset error between the calculated ACF phase angle and the corresponding point on the fitted line.

The slope of the line $n_k$ with the minimum RMSE yields the velocity of the target. The mean velocity estimator using this phase unwrapping technique is given by

$$\hat{v}_{pu} = \frac{-n_k \lambda}{4\pi}.$$

(41)

This estimator can be considered a special case of the standard pulse pair estimator, in which the ACF phase angle for only one lag equal to one interpulse period is used.

The accuracy of the mean velocity estimate increases with the number of PRTs used, which provides more velocity points to which to fit a line in the MMSE sense. An example of this fitting procedure is illustrated in Fig. 6, which shows the linear minimum mean-square error (LMMSE) fit for three unwrapped velocities and the origin. The mean velocity estimate is calculated from the slope of the dashed line as shown in Eq. (41). If noise and/or spectrum width cause random variation in the measurement of phase angles, the accuracy of the
estimates is decreased. This is exhibited by greater distance between the unwrapped velocity points and the fitted line.

3. Simulation results of SMPRF algorithm for mean power estimation

a. Simulation overview

The time series simulation technique used in this study is described in Chandrasekar et al. (1986). For each resolution volume, a time series, \( \alpha \), is generated with time spacing equal to the greatest common divisor of PRTs, in this case, \( \Delta_t = 150 \mu s \). The time series simulation length is equal to the length of the SMPRF code times the number of PRT blocks transmitted. In the case of the SMPRF\((150);5,8,10,7\) code with 15 blocks transmitted, the dwell time is \( 30 \times 15 = 450 \times 150 \mu s = 67.5 \) ms. The time series \( \beta_i^j \), corresponding to the \( i \)th volume of the \( j \)th PRT, are selected subsets of the time series \( \alpha \), with one element selected from each block. The time series are thus spaced one code length apart, in this case 4.5 ms. For example, consider \( \beta_6^i \), the time series corresponding to the sixth measurement volume illuminated by the pulse of length seven normalized time units. The members of this time series are \( \beta_6^i = (\alpha_{29}, \alpha_{59}, \alpha_{89}, \ldots, \alpha_{449}) \). Again, the length of these time series is equal to the number of code blocks transmitted. This simulation procedure ensures proper time spacing and correlation between the time series constructed for each resolution volume.

To evaluate the mean power and velocity recovery statistics, three simple input mean power versus range profiles are selected. The recovery statistics vary according to the range profile chosen and the recovery performance of the three profiles demonstrate the expected quality of the recovered power and velocity. The analysis of SZ phase coding (Sachidananda and Zrnić 1999), in which strong versus weak trip ratios are considered, is not feasible using the SMPRF algorithm. For example, using SMPRF\((150);5,8,10,7\), each resolution volume appears 3 or 4 times in the estimated power vector \( \tilde{Z} \), and in each instance the overlaid power is from different resolution volumes. It therefore becomes necessary to use range profiles to simulate and illustrate the performance of the SMPRF algorithm.

The three power profiles used in this study are a flat profile with the reflectivity value at each range set to 20 dB, a single-peak triangular profile with 50-dB power level separation, and a double-peak triangular profile with 15-dB power level separation. The SNR in each case is set to be infinite. It is shown through simulation that the performance of the mean velocity estimation is independent of the input mean velocity value, between \( \pm v_r \), and thus a flat mean velocity profile is selected for input with velocity values set randomly within \( \pm v_r \). A constant spectrum width profile of 2 m s\(^{-1} \) is selected for all ranges, a value shown to represent the median spectrum width value of most weather phenomena (Fang et al. 2004). The radar frequency is set to 5.6 GHz (\( \lambda = 5.36 \) cm), 15 blocks are used for the SMPRF\((150);5,8,10,7\) code, which translates to a dwell time of 67.5 ms, and 200 iterations are run to generate statistics.

In the figures illustrating mean power and mean velocity recovery, the vertical box dimension represents one standard deviation while the horizontal line within the box marks the estimate of the mean. The input power and velocity profiles are displayed in each case, superimposed to illustrate the corresponding power separation between ranges.

b. Simulation results

The simulation results for the estimation of mean power and mean velocity using the SMPRF\((150);5,8,10,7\) code with maximum unambiguous measurement range of 460 km are seen in Figs. 7–12. For the 20-dB flat input power profile, Fig. 7 shows unbiased mean power recovery with standard deviation values around 2 dB for each measurement range. Figure 8 shows biased mean velocity recovery performance with standard deviations between 30 and 40 m s\(^{-1} \). Reasons for this behavior are described in the next section. For the 50-dB triangular input power profile, Fig. 9 shows unbiased mean power recovery and standard deviation values less than 5 dB in those ranges with powers 35 dB and
above. For those ranges with less power, the performance is biased with standard deviation values between 10 and 15 dB. Figure 10 shows unbiased mean velocity recovery and standard deviation values less than 0.1 m s\(^{-1}\) for the ranges with power values 35 dB and above. Measurements in ranges with power values below 35 dB exhibit biasing and standard deviation values between 40 and 50 m s\(^{-1}\). These results suggest that the ranges with lesser power are dominated by the noise generated in the correlation process. For the 15-dB double-triangular input power profile, Fig. 11 shows unbiased performance and standard deviations that increase from less than 1 to around 8 dB as the power separation between measurement ranges increases. Figure 12 shows biasing of the mean velocity estimates as the power separation between measurement ranges increases and standard deviation values increasing from about 10 to about 50 m s\(^{-1}\).

These results show that the SMPRF algorithm using the SMPRF\(_{(150);5,8,10,7}\) code can recover reasonable power estimates only for those resolution volumes with the highest relative powers. When the power of a resolution volume drops to about 15 to 20 dB below the resolution volume in the profile with the highest power, the standard deviation of the power measurements can become unreasonably large. The recovered velocities only look reasonable for the highest power resolution volumes for the triangular power profile shown in Fig. 10. In all other locations in this profile, the recovered velocities have unacceptably large biases and standard deviations. Reasons for these effects are illustrated and discussed in section 4b.

4. Analysis of the factors affecting the performance of the SMPRF algorithm

This section provides description, discussion, and illustration of the factors affecting mean power and ve-
locity estimation performance of the SMPRF algorithm. Factors affecting the performance of mean power and mean velocity estimation include finite-length time series effects, the overlaid echoes present, and the relation of ACF lag spacing to coherence time. Each is discussed in the following subsections.

a. Effects of finite-length time series

The received voltage at the radar is a complex stochastic signal, considered as one realization of the underlying stochastic process. Physically, the stochastic process is driven by the reshuffling of the precipitation particles during the dwell time of the radar (Bringi and Chandrasekar 2001). The stochastic process is considered stationary so that the mean of the time series of the magnitude squared voltage is proportional to power. It is well known that this power estimate has unbiased mean and standard deviation, which depends on the spectrum width. In the SMPRF algorithm, multiple time series representations for a resolution volume are created. For example, considering resolution volume 1 using SMPRF(150);5,8,10,7, the time series are \( \beta^{(5)}_1, \beta^{(8)}_1, \beta^{(10)}_1 \), and \( \beta^{(7)}_1 \). Even though these time series are generated from the same resolution volume and are correlated (depending on the spectrum width), the mean power estimates for the given resolution volume are not equal in general. Because the matrix inversion process is based on the equality of these powers [i.e., Eq. (19) is assumed to hold true for each resolution volume], this power inequality produces errors in the mean power recovery process. Power values for each of the 20 resolution volumes shown using the SMPRF(150);5,8,10,7 code can appear up to a maximum of 4 times in the inversion equation, as shown by Eq. (9). Thus, this assumption of power estimate equality of time series from the same resolution volume is one source of error in the power recovery process.

Another error source in SMPRF power recovery due to finite-length time series is the assumption that the cross products of the power calculation in Eq. (19) are equal to zero. In the following analysis, these two error sources are separated and quantified.

Equations (13) and (14) are written under the assumption that the cross-power terms between the resolution volumes are equal to zero; that is,

\[
\sum_k \beta^i(k)\beta^j(k) = 0; \ i \neq n.
\]  

(42)

For finite-length time series, this will not be true. To illustrate this error source, the simulation procedure is changed. Return time series are still generated, but now the mean power values of each time series are calculated individually before being overlaid and separated by the inversion process. The results for this simulation, using the 50-dB triangular profile described above, are shown in Fig. 13.

The results shown in Fig. 13 are similar to those in Fig. 9, suggesting that the cross-power products are not the dominate source of error in the SMPRF mean power recovery process, at least for the simulation parameters used here. Next, the estimated power of the individual time series for a resolution volume is made equal [i.e., Eq. (19) is forced to be true], thus the sole error source is now power from the cross-product terms.

To create this condition, the time series are scaled such that equality exists among the mean power estimates made from each of the time series according to the following equation:
Fig. 14. Mean power profile recovery for 50-dB triangular profile, SMPRF(50,5,8,10,7) code, using scaled time series. This figure illustrates the effects of eliminating the differences in inversion variables in the mean power recovery process [Eq. (19)].

\[
\beta_i^{\text{rec}} = \sqrt{\frac{P_i}{\tilde{P}_i}} \beta_i^{\text{true}},
\]

where \(P_i\) is the input (true) power at the measurement range \(i\) and \(\tilde{P}_i\) is the estimated power at the measurement range \(i\). The result of this simulation using the 50-dB triangular power profile is shown in Fig. 14. As can be seen, the mean power recovery performance of the SMPRF algorithm has significantly improved as compared to Fig. 9. This indicates that inequality of the mean power estimates of the time series is the dominant source of error in the SMPRF algorithm for the simulation parameters used.

To quantify this estimated mean power difference between the various time series, the time series corresponding to the echoes from resolution volume 1—that is, \(\beta_1^{(5)}(k), \beta_1^{(8)}(k), \beta_1^{(10)}(k),\) and \(\beta_1^{(7)}(k)\)—are simulated using the same simulation parameters as before. For each simulation, the pairwise power ratios are calculated; that is, \(|\langle \beta_i^{(5)}(k) \rangle_{\text{rec}} |^2 / |\langle \beta_i^{(5)}(k) \rangle_{\text{true}} |^2 |\langle \beta_i^{(8)}(k) \rangle_{\text{rec}} |^2 / |\langle \beta_i^{(8)}(k) \rangle_{\text{true}} |^2 |\langle \beta_i^{(10)}(k) \rangle_{\text{rec}} |^2 / |\langle \beta_i^{(10)}(k) \rangle_{\text{true}} |^2 |\langle \beta_i^{(7)}(k) \rangle_{\text{rec}} |^2 / |\langle \beta_i^{(7)}(k) \rangle_{\text{true}} |^2\) for \(i = 5, 8, 10, 7\) (with \(\langle \cdot \rangle\) denoting the sample mean operator). The mean and standard deviation values for each ratio are calculated over 50,000 simulations. The mean values were all approximately 0 dB. The standard deviation values, ranging from 0.6889 to 1.17 dB (an approximate 31% difference), are shown in Table 2.

Both conditions described by Eqs. (19) and (42) can also be enforced simultaneously. The mean power values designated by the input range profile are directly combined according to Eq. (9). In this case, no time series are generated, but rather single values for the input mean power value at each resolution volume are generated and combined according to Eq. (9), thus eliminating the error source due to differences in mean power estimates. Because the powers are directly combined, there are no contributing cross-power terms or finite-time effects. In this case, the recovery is nearly perfect, limited by precision of the computer used.

The power ratio variances can be reduced in two ways: 1) the dwell time for the time series can be increased, thus increasing the number of independent samples used in the estimated mean power calculation, or 2) the modeled spectrum widths can be reduced, thus increasing the correlation between time series. Practically, dwell times for scanning weather radars cannot be made arbitrarily large if reasonable volume scan times are required. Spectrum width values are obviously not controllable variables. Even though both adjustments are not practical, separate simulations were run for much narrower spectrum width values and increased dwell times. In both cases, the statistics of the recovered mean power and mean velocity values improved according to the length of the dwell time increase or spectrum width decrease.

In the next section, the effects of overlaid power on the performance of the SMPRF algorithm for mean power and velocity recovery is examined.

### b. Effects of overlaid echoes

The amount of overlaid echo power affects the accuracy of recovered mean power and mean velocity estimates as can be seen from Figs. 9–12. It is difficult to quantify the amount of overlaid echo power for a particular resolution volume because it is a function of sample time [Eq. (9)]. However, it can be seen in Figs. 9–12 that as power for a given resolution volume decreases, the recovery statistics for a given parameter worsen.

The amount of overlaid echo power can be controlled (for modeling purposes) by reducing the maximum unambiguous measurement range. Simulations are performed using the 50-dB triangular profile, but the algorithm’s maximum unambiguous range is now limited to 150 km. Thus, the matrix \(\mathbf{A}\) shown in Eq. (15) is truncated to six columns giving one overlaid echo of power equal to 15 dB in the first resolution volume. The

### Table 2. Std dev of mean power estimates used in the matrix-inversion process for the first resolution volume in the 50-dB triangular power profile, \(\sigma_v = 2\text{ m s}^{-1}\).

<table>
<thead>
<tr>
<th>(\sigma_{\text{dB}})</th>
<th>(\langle \beta_1^{(5)} \rangle_{\text{rec}}^2)</th>
<th>(\langle \beta_1^{(8)} \rangle_{\text{rec}}^2)</th>
<th>(\langle \beta_1^{(10)} \rangle_{\text{rec}}^2)</th>
<th>(\langle \beta_1^{(7)} \rangle_{\text{rec}}^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.6889</td>
<td>0.9854</td>
<td>0.9578</td>
<td>0.6889</td>
</tr>
<tr>
<td>0.8682</td>
<td>0</td>
<td>1.0186</td>
<td>0.9854</td>
<td></td>
</tr>
<tr>
<td>1.1705</td>
<td>1.0186</td>
<td>0</td>
<td>0.9578</td>
<td></td>
</tr>
<tr>
<td>0.6889</td>
<td>0.9854</td>
<td>0.9578</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
remaining resolution volumes have no overlaid power. Simulations showed that in the first-range resolution volume, the mean power recovery performance was unbiased with the standard deviation approximately equal to 2 dB. In the remaining resolution volumes in which no overlaid echoes were present, performance was unbiased with standard deviation values consistently below 0.5 dB for mean power recovery and less than 0.7 m s$^{-1}$ for mean velocity recovery. For the first resolution volume (in this case, the only resolution volume with overlaid power present), the recovered mean velocity parameters resemble noise, having zero mean and standard deviation approximately equal to 45 m s$^{-1}$. This then illustrates the negative effect of overlaid power on the performance of the SMPRF algorithm to recover mean velocity and the differences in performance of such for this profile.

In the next section, the effects of the choice of PRT and subsequently ACF lag spacing relative to coherence time on the performance of the SMPRF algorithm for mean velocity recovery are examined.

c. Effects of ACF lag spacing relative to coherence time for mean velocity recovery

As shown in Eq. (32), coherence time is a function of wavelength and spectrum width. The standard deviation of mean velocity estimates increases as the ACF lag estimates reduce in magnitude. Therefore, SMPRF velocity estimates that rely on ACF lags greater than or near coherence time will have increased variance.

To justify this point, consider the SMPRF$^{(150);5,8,10,7}$ code with $\Delta t = 150 \ \mu s$ and a maximum unambiguous measurement range equal to 460 km. The PRT lengths are 0.75, 1.20, 1.50, and 1.05 ms, respectively. In this study, analysis of mean velocity recovery performance to the strongest range (i.e., 247.5 km), in the 50-dB power profile was performed. The simulation was performed as in section 3. This time, the radar frequency is swept from S to X band with $\sigma_v = 2 \ \text{m s}^{-1}$ at each frequency. The standard deviation of the mean power estimate for the strongest resolution volume is computed at each frequency. In the case of S-band ($\lambda = 10$ cm) operation, with $\sigma_v = 2 \ \text{m s}^{-1}$, $T_D = 5.62$ ms. For X-band ($\lambda = 3.0$ cm) operation, the value for $T_D$ with $\sigma_v = 2 \ \text{m s}^{-1}$ is 1.69 ms.

Figure 15 shows a comparison of mean velocity estimate accuracy for the highest power resolution volume within the 50-dB triangular profile as a function of coherence time. This figure illustrates a relationship between the performance of the SMPRF algorithm to recover mean velocity, the PRTs selected in the code, and the coherence time. It shows that the accuracy of the mean velocity estimates is reduced sharply as the coherence time drops below about 2.25 ms, or 75% of the code length of the SMPRF$^{(150);5,8,10,7}$ code, and justifies using only those ACF lag values within this constraint for mean velocity estimation.

5. Conclusions

Theoretical and operational background of the SMPRF algorithm for range–velocity ambiguity mitigation in weather radar was presented. This study served to investigate the effects of the long spatial extent, large dynamic range, and coherence issues on the SMPRF algorithm, originally developed for ionospheric measurement applications. A time series simulation technique was described to evaluate the algorithm. Because of the unique nature of the overlaid power structure of the SMPRF algorithm, three simple range profiles were simulated and the SMPRF algorithm was applied to recover the mean powers and mean velocities of each range profile. These recovered parameters were then compared to the true parameters to generate recovery statistics and to quantify algorithm performance. These results and underlying theory show that the performance characteristics of the SMPRF algorithm can be generalized to include different radar frequencies, range profiles, and specific code choices.

The simulation studies showed that there are indeed limitations to the SMPRF algorithm and these limitations can be severe. Factors affecting the performance of mean power and mean velocity estimation include finite-length time series effects, the amount of overlaid power present, and the relationship of ACF lags to coherence time. In general, it was shown that recovery of mean power estimates is only good for power ratios less
than about 15 dB. Good mean velocity recovery is only possible for the highest power resolution volumes in a given power profile when there is significant echo overlay along the radar radial, but even this was shown to be untrue for some of the simulated radar reflectivity profiles presented.

The theory behind the SMPRF algorithm does show that unlimited maximum unambiguous range and velocity can be obtained when the analysis is done with expectations of a random variable. Ensemble statistical properties (e.g., the property that the correlation product of two uncorrelated random processes is zero) are used in the theoretical development; however, for finite-length time series representations of these processes, such correlation products are in general not zero and must be considered. For example, consider the SMPRF matrix inversion process for recovering mean power. For the SMPRF(150);5,8,10,7 code analyzed, there are typically four time series representations of the statistical processes of a common resolution volume corresponding to each of the four PRTs used. Even though these time series are measured in close time proximity and are correlated, their mean powers are in general not equal. The degree of power inequality depends on the spectrum width of the weather echo and the radar time series length. For typical radar measurement parameters with a dwell time of 64 ms and a spectrum width of 2 m s⁻¹, the average mean power difference among the four time series is about 1 dB (or 25%). The SMPRF matrix inversion process operates under the assumption that these powers are equal, which is the primary factor that limits the mean power recovery process. The error in the mean power measurements of stronger resolution volumes overwhelms the power in weaker resolution volumes making acceptable mean power recovery in these weaker volumes impossible.

The SMPRF velocity recovery technique is limited by a similar mechanism. Several “first lag” estimates of the ACF for a resolution volume are possible because each resolution volume has multiple time series representations [e.g., there will be four time series representations if the SMPRF(150);5,8,10,7 code is used]. Correlating these time series yields several unequally spaced estimates of the ACF. Each time series gathered at a particular sample time can also contain echoes from several other resolution volumes. Because the overlaid echoes are from physically separate resolution volumes, the overlaid time series are independent and their cross correlations are zero, at least asymptotically. But because the time series are finite, these cross correlations are not zero and in effect mask the desired “first lag” correlation in noise proportional to the strength of the overlaid echoes. This “correlation noise” then limits the accuracy of the recovered velocities. The extent of the limitation depends on the number of overlaid echoes and the strength of each. For the SMPRF(150);5,8,10,7 code used in this study, if there are significant overlaid echoes throughout the radar range, good recovery of mean velocity is possible nowhere.

An SMPRF code can be selected to minimize the number of overlaid echoes while maintaining a high-maximum unambiguous range and a high Nyquist velocity (i.e., select PRTs that are relatively long to yield a high $r_a$ but with small PRT differences that yield a high $v_u$). This would, however, create a long PRT block time limiting the number of available samples within the dwell time and reduce the number of ACF lags available within the coherence time. Of course, the dwell time can be extended to mitigate these shortcomings. These effects either significantly reduce the quality of the recovered mean power and mean velocity estimates or necessitate restrictively slow radar scan rates.

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APPENDIX A

SMPRF Matrix Inversion

In the presence of noise, an additive noise term would be included in Eq. (15). Then the weighted least squares solution for the power estimates is given by (Lehtinen 1999)

$$\hat{P} = \Sigma_p \Sigma^{-1} \tilde{Z},$$

(A1)

where $\Sigma$ is the measurement error covariance matrix, based on the nonstationarity of the system, the inverse of which serves as the weighting matrix (Scharf 1991), and $\Sigma_p = (\Sigma^T \Sigma)^{-1}$ is the solution covariance matrix. For purposes of illustration in this report and for simplicity, this matrix is taken to be the identity matrix.

For mean power recovery using the SMPRF algorithm, there are more measurements than resolution volumes at which to make mean power estimates, meaning the row space of $\mathbf{A}$ is greater than its column space, creating an overdetermined linear system. This implies that there will be no power estimate values, $\hat{P}_r$. 


that satisfy all measurement equations simultaneously. The Moore–Penrose pseudoinverse determines the values of \( \hat{P} \) that come as close as possible in the least squares sense to satisfying all the measurement equations (Ben-Israel and Greville 1977). The equation for the least squares power estimate, assuming the solution covariance matrix to be the identity matrix, is given by

\[
\hat{P} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{Z} = \mathbf{A}^+ \mathbf{Z}.
\]

The pseudoinverse \( \mathbf{A}^+ \) is guaranteed to exist for any full-rank matrix \( \mathbf{A} \) (Arfken 1985), a condition satisfied by the nature of the SMPRF algorithm. The proof that \( \mathbf{A} \) is full rank based on the nature of the SMPRF algorithm is not given. This proof is straightforward; given the definition of “rank” and the cyclic nature of the \( \mathbf{A} \) based on the time delay of returns from later range volumes, the rows of this matrix will always be linearly independent, yielding a full-rank matrix.

However, the Gram matrix (Scharf 1991) \( (\mathbf{A}^T \mathbf{A}) \) may be ill conditioned, meaning the condition number, a measure describing the inverse problem’s amenability to digital computation, is large. This implies increased sensitivity to perturbations in the measurement vector \( \mathbf{Z} \), meaning that a solution to the inversion problem exists, but it is not unique and depends on the continuity of the data (Arfken 1985). This continuity is affected by the relationship of the measurements, namely, the effects of the variability of the finite-length time series calculations and resulting differences in estimates of the inversion variables used in the inversion process, as discussed and shown in section 4a. In these cases, the least squares estimate amplifies the measurement noise and leads to inaccurate mean power estimates. This may occur even when the pseudoinverse itself can be accurately calculated numerically (Arfken 1985).

The condition number for \( \mathbf{B} = \mathbf{A}^T \mathbf{A} \) is defined to be (Golub and Van Loan 1996)

\[
\kappa(\mathbf{B}) = \|\mathbf{B}^{-1}\|_2\|\mathbf{B}\|_2 = \frac{\sigma_{\text{max}}(\mathbf{B})}{\sigma_{\text{min}}(\mathbf{B})},
\]

where \( \sigma_{\text{max}}(\mathbf{B}) \) and \( \sigma_{\text{min}}(\mathbf{B}) \) are the maximum and minimum singular values of \( \mathbf{B} \), respectively. For \( \mathbf{A} \) related to the SMPRF\(_{150;5,8,10,7} \) code with a maximum unambiguous range of 460 km, the condition number of \( \mathbf{A}^T \mathbf{A} \) is found to be 44.91. For the SMPRF\(_{150;5,8,10,7} \) code, the condition number of the Gram matrix \( \mathbf{A}^T \mathbf{A} \) is shown as a function of the maximum unambiguous range (i.e., amount of overlaid power present) in Fig. A1.

Figure A1 shows that the condition number is directly proportional to the number of overlaid echoes present determined by the SMPRF code. The vertical steps relate to the increase in the number of overlays present in the code and directly relate the variation of mean power measurements in the measurement vector \( \mathbf{Z} \) to the accuracy of the mean power estimates.

**APPENDIX B**

**Mean and Variance Calculations for Mean Power Recovery Using the SMPRF Algorithm**

The first- and second-order statistics of the recovered mean power estimate \( \hat{P} \) give insight into the accuracy of the performance of the SMPRF algorithm and provide a theoretical basis for the selection of the SMPRF algorithm for weather radar applications.

The mean of \( \hat{P} \) is shown to be

\[
E(\hat{P}) = (\mathbf{A}^T \Sigma^{-1} \mathbf{A})^{-1} \mathbf{A}^T \Sigma^{-1} E(\mathbf{Z})
\]

\[
= (\mathbf{A}^T \Sigma^{-1} \mathbf{A})^{-1} \mathbf{A}^T \Sigma^{-1} [\mathbf{A} E(\hat{P})]
\]

\[
= (\mathbf{A}^T \Sigma^{-1} \mathbf{A})^{-1} (\mathbf{A}^T \Sigma^{-1} \mathbf{A}) \mathbf{P}
\]

\[
= \mathbf{P},
\]

where \( \mathbf{P} \) is the vector of true power values, implying an asymptotically unbiased estimator. It is important to note that although the above calculations show the SMPRF power recovery algorithm to be unbiased in the asymptotic sense, in practice, with averaging operations performed on finite-length time series and non-stationary inversion parameters, the SMPRF algorithm will exhibit bias, a claim substantiated by Figs. 9 and 11. [For more details on the matrix pseudoinverse problem, see Strang (1988).]

The variance of the mean power estimates is calculated in the following manner:

\[
\text{Var}(\hat{P}) = \mathbf{B} \text{Var}(\mathbf{Z}) \mathbf{B}^T.
\]
where $B = (A^T \Sigma^{-1} A)^{-1}$ and $\text{Var}(\tilde{Z}) = \text{Var}(\tilde{\mathbf{P}}) = \mathbf{A} \text{Var}(\tilde{\mathbf{P}}) \mathbf{A}^T$. For the $i$th measurement volume, this yields

$$\text{Var}(\tilde{P}_i) = \text{Var}\left( \frac{1}{M'} \sum_j A_{i,j} \tilde{P}_{i,j} \right)$$

$$= \frac{1}{M'} \sum_k \sum_j \text{Cov}(\tilde{P}_{i,k}, \tilde{P}_{i,j})$$

$$= \frac{1}{M'} \sum_{j,k} A_{k,j} A_{i,j} R_{k,j} (k - j), \quad \text{(B3)}$$

where $M' = \Sigma A_{i,j}$ and $1 \leq M' \leq M$, which is based on the decorrelating capability of the code and is a factor of the number of averaged samples and correlation between them. Eq. (B3) is a modification of Eq. (5.189) in Bringi and Chandrasekar (2001), shown here for convenience:

$$\text{Var}(\tilde{P}_i) = \frac{1}{M} \sum_{k=(M-1)}^{M-1} (1 - |k|/M) R_p(k). \quad \text{(B4)}$$

It is seen that Eq. (B3) becomes Eq. (B4) in Bringi and Chandrasekar (2001) if $A$ is a matrix of all ones. Simplifying the representation in Eq. (A5) into a more succinct form gives

$$\text{Var}(\tilde{P}_i) = (A^T \Sigma^{-1} A)^{-1} \text{Cov}(\tilde{\mathbf{P}}) A^T (A^T \Sigma^{-1} A)^{-1}$$

$$= \mathbf{C} \text{Var}(\tilde{\mathbf{P}}) \mathbf{C}^T. \quad \text{(B5)}$$

It is seen that the choice of the specific SMPRF code (and thus $A$) influences the number of samples used for calculation and the sample decorrelation through the selection of sample time spacing. In this sense, the variance in the SMPRF mean power recovery algorithm is a function of the spectrum width in both the individual overlaid time series contributing to $\text{Var}(\tilde{\mathbf{P}})$ and the difference in PRTs, described by $A$.

REFERENCES


