Adaptive Dealiasing for Doppler Velocities Scanned from Hurricanes and Typhoons

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(Manuscript received 9 July 2015, in final form 9 May 2016)

ABSTRACT

By fitting a parametric vortex model directly to aliased radar radial velocities scanned from a hurricane, the maximum tangential velocity and its radial distance from the hurricane vortex center can be estimated by the recently developed alias-robust vortex analysis. This vortex analysis can be refined to produce a suitable reference radial velocity field on each tilt of a radar scan for the reference check in the first main step of dealiasing. This paper presents the techniques developed for the refinements and shows how and to what extent the refined vortex analysis can improve the reference check and thus enhance the dealiased data coverage, especially over severely aliased data areas around the vortex core of a hurricane or typhoon. In addition, stringent threshold conditions are used in the reference check and the subsequent continuity check to ensure the accepted data are free of alias or almost so. The robustness and improved performance of the method are exemplified by the results tested with severely aliased radial velocities scanned by operational WSR-88D radars from hurricanes in the United States and by operational China New Generation Weather Radar (CINRAD) base data format SA radars from typhoons in China.

1. Introduction

It is well known in radar meteorology that there is a maximum velocity, called the Nyquist velocity and denoted by \( v_N \), beyond which the radial velocities measured by a Doppler radar are aliased by an integer multiplied by the Nyquist velocity back into the Nyquist interval between \( \pm v_N \) (Doviak and Zrnić 2006, section 3.6). Such an integer is called the Nyquist folding number, and it can be either positive or negative. The central task in correcting an aliased velocity is to determine the Nyquist folding number. This task, called dealiasing, can be accomplished if the alias can be detected and corrected by comparing the aliased radial velocity (i) with a reference radial velocity and/or (ii) with radial velocities measured by the same radar at the neighborhood data points for their continuities in space (or in space and time). These two types of comparisons, called the reference check and the continuity check, respectively, have been commonly used in combinations, but their applications are often confronted with difficulties caused by the absence of any reliable reference velocity and by the lack of a correctly dealiased velocity to start the checking.

The Nyquist velocities used by operational weather radars in the United States are usually in the range between 20 and 36 m s\(^{-1}\), so radial velocities scanned from a hurricane can easily exceed these Nyquist velocities and become severely aliased. The operationally used dealiasing techniques (Eilts and Smith 1990; Jing and Wiener 1993; Witt et al. 2009) for processing real-time WSR-88D radar data were developed primarily for visual and certain quantitative applications with considerable tolerance for bad or poor-quality data to retain as much as possible the original data coverage. The processed data often do not satisfy the high-quality standard required by radar data assimilation. This problem is not completely solved by the dealiasing techniques so far developed, and difficulties are

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DOI: 10.1175/JTECH-D-15-0146.1

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often encountered that are sometimes insurmountable due to the lack of reliable reference information about the true wind field. The encountered difficulties have been reduced to a certain extent by modifying the velocity–azimuth display (VAD) analysis (Browning and Wexler 1968) for direct applications to raw radial velocities to provide reference winds for dealiasing (Yamada and Chong 1999; Tabary et al. 2001; Haase and Landelius 2004; Gong et al. 2003; Gao et al. 2004; Zhu and Gong 2006, Xu et al. 2011). In particular, by formulating the aliasing-caused zigzag discontinuities into the cost function for the VAD fitting via the unconventional approach based on the Bayesian estimation theory (Xu 2009), an alias-robust VAD (AR-VAD) analysis was developed by Xu et al. (2010, hereafter X10) to estimate the mean horizontal wind directly from raw aliased radial velocity data. This AR-VAD analysis was used to produce a reference radial velocity field for radar velocity dealiasing in the first main step of the AR-VAD-based dealiasing method (Xu et al. 2011, hereafter X11). The AR-VAD-based dealiasing has been used operationally for radar data assimilation applications at the National Centers of Environmental Prediction (NCEP) in the United States (Liu et al. 2009). To satisfy the high-quality standard required by operational data assimilation, stringent threshold conditions have been used in the AR-VAD-based dealiasing to ensure the accepted data are free of alias, but good data are also sometimes rejected in areas where the true wind field becomes too nonuniform to satisfy the VAD uniform-wind approximation. This has limited the dealiased data coverage, as often seen from hurricane (or typhoon) winds scanned by operational radars.

Because aliasing can occur in countless different ways and the aliasing scenarios can be extremely complex, it is very difficult or even impossible to develop a single dealiasing method for all different and difficult scenarios to satisfy the high-quality standard required by radar data assimilation. To overcome the difficulties, efforts have been made in developing various adaptive dealiasing methods (rather than a single method) beyond the AR-VAD-based dealiasing. One of the outcomes from these efforts is the adaptive dealiasing (Xu and Nai 2012) based on the alias-robust variational (AR-Var) analysis (Xu and Nai 2013), which was designed specifically for severe winter ice storms scanned by WSR-88D radars using the volume coverage pattern 31 (VCP31) with the Nyquist velocity reduced below 12 m s$^{-1}$. Computationally, this adaptive dealiasing is marginally efficient for operational radar data assimilation applications because each VCP31 volume contains only five tilts of raw radial velocities scanned in about 10 min. As shown by Xu et al. (2013, hereafter X13), the AR-Var-based dealiasing can be further extended and used in place of the AR-VAD-based dealiasing for scan modes other than VCP31, but the required computational time is increased significantly. The extended AR-Var-based dealiasing is thus not sufficiently efficient for operational radar data assimilation applications, especially for assimilating hurricane winds scanned by using VCP21 (VCP121), because each VCP21 (VCP121) volume contains 19 tilts of raw radial velocities scanned in no more than 5 min. Thus, as another outcome from the aforementioned efforts, a new efficient dealiasing is developed adaptively for hurricanes or typhoons by refining the alias-robust (AR) vortex analysis of Xu et al. (2014, hereafter X14) and using it in place of the AR-VAD analysis for the reference check in the first main step of the AR-VAD-based dealiasing.

The performance of the new adaptive dealiasing was highlighted in X14 by an example without presenting the detailed techniques, because the involved refinements are complicated by partially unphysical needs for dealiasing (as explained later in section 2d) and thus not suitable for other possible applications of the AR vortex analysis. Since the detailed techniques are not published but critical for dealiasing, this paper aims to present the detailed dealiasing techniques and show how and to what extent the refined vortex analysis can improve dealiasing adaptively, especially over severely aliased data areas around the vortex core of a hurricane or typhoon. The new adaptive dealiasing has been tested with severely aliased radial velocities (497 volumes) scanned by operational China New Generation Weather Radar (CINRAD) base data format SA radars from typhoons in China in addition to those listed in X14 that were scanned by operational WSR-88D radars from hurricanes in the United States. According to these tests, the new method not only improves the accepted data coverage over the core area of a hurricane or typhoon but also ensures the accepted data are free of alias or almost so. The paper is organized as follows. The refinements made to the AR vortex analysis for dealiasing are presented in the next section. The new method is presented in section 3 with illustrative examples to show the robustness and improved performance of the method. Conclusions follow in section 4. Variables and symbols listed and defined in the appendix will be used in the paper directly without further explanation.

2. Refined AR vortex analysis for dealiasing

Images of radar-observed radial velocity $v_{r,obs}$ from a hurricane or typhoon often exhibit a narrow near-zero-$v_{r,obs}$ zone [see the zigzag gray stripe marked by the yellow + sign in Fig. 5a] containing the true (nonaliased and unobserved) zero-$v_r$ line between the radar and
vortex center on each tilt of radar scan. As the image pixel moves away from this near-zero-$v_r$ zone, the raw $v_r^{\text{obs}}$ often becomes aliased quickly and severely. Because of this, it is necessary and critical to accurately estimate the true zero-$v_r$ line on each tilt and use it to constrain the AR vortex analysis so the analysis-produced reference radial velocity $v_r^{\text{ref}}$ can be sufficiently accurate for dealiasing, especially on the two sides of the true zero-$v_r$ line on each tilt. To achieve this, the AR vortex analysis presented in sections 2–4 of XJL needs to be refined with complex logic in three key components. The detailed logic steps in these three key components are presented in three subsections after the parametric vortex model and related properties are briefly reviewed in the following subsection.

a. Parametric vortex model and zero-$v_r$ point $\phi_+$

As shown in XJL, the parametric vortex of Vatistas et al. (1991) can be used to describe the tangential wind $V_T$ of a hurricane as a function of the radial distance $R$ from the hurricane vortex center at each vertical level in the following form (see Fig. 1 in XJL):

$$V_T = V_M(R/R_M)[1/2 + (R/R_M)^4/2]^{-1/2},$$  

(1)

where $V_M$ is the maximum tangential velocity at $R = R_M$. The hurricane vortex center location $(r_c, \phi_c)$, in the radar-centered coordinates (see Fig. 2 in XJL), can be pre-estimated from operationally issued hurricane location information. The environmental mean wind is usually much smaller than $V_M$, and it impacts mostly the asymmetric flow retrieval but not much the axisymmetric flow according to Murillo et al. (2011) and Harasti et al. (2004). Thus, the environmental mean wind can be neglected. According to (1), $u_r$ can be modeled by

$$u_r^{\text{mod}} = V_T \sin(\alpha - \phi) \cos \theta',$$

(2)

where $\alpha$, $\phi$, and $\theta'$ are defined in the appendix. The expression in (2) will be used not only to fit $u_r^{\text{obs}}$ for the refined vortex analysis in the next subsection but also to compute $u_r^{\text{ref}}$ for the reference check in the first main step of dealiasing in section 3a.

Note that $R$ and $\alpha$ are functions of $(r, \phi, r_c, \phi_c)$ determined by the cosine and sine formulas in (4b) and (4c) of XJL, respectively. Substituting the latter two equations into (1) and then into (2) gives the modeled $v_r$ in the form of

$$v_r^{\text{mod}} = A[1 + (R/R_M)^4]^{-1/2} \sin(\phi - \phi_c),$$  

(3)

where $A = (2/5)V_M r_c/R_M \cos \theta'$. For fixed $r_c, \phi_c$, $V_M, R_M,$ and $\theta$, $A$ is constant and $R$ is a function of $\phi$, so $v_r^{\text{mod}}$ is also a function of $\phi$ according to (3). Since $A > 0$ and $R \geq 0$, it is easy to see from (3) that $v_r^{\text{mod}}(\phi)$ changes sign twice on a range circle at two zero-$u_r$ points:

$$\phi = \phi_+ = \phi_c$$  

(4a)

and

$$\phi = \phi_- = \phi_c + 180^\circ - 360^\circ \text{Int}[(\phi_c + 180^\circ)/360^\circ],$$  

(4b)

where $\text{Int}[(\cdot)]$ represents the nearest integer of $(\cdot)$. $\phi_+$ ($\phi_-)$ denotes the zero-$u_r$ point associated with $\phi_+$. The last term in (4b) is just for the purpose of keeping $\phi_-$ in the same period as $\phi_+$, say, between $-180^\circ$ and $180^\circ$.

Note that $\phi_+$ ($\phi_-$) is the nearest point to (farthest point from) the vortex center among all the points on a range circle; that is, $\text{Min}\{\phi\} = R(\phi_+) = r_c - r$ and $\text{Max}\{\phi\} = R(\phi_-) = r_c + r$. Applying $\delta_\phi$ to (4b) in XJL and (3) gives

$$\delta_\phi v_r^{\text{mod}} = (\pi/180^\circ)A[1 + (R/R_M)^4]^{-1/2} \sin^2(\phi - \phi_c),$$  

(5a)

$$\text{Max}\{\delta_\phi v_r^{\text{mod}}(\phi)\} = \delta_\phi v_r^{\text{mod}}(\phi_+)$$  

$$= (\pi/180^\circ)A[1 + |r_c - r|^2 R_M^4]^{-1/2},$$  

(5b)

$$\text{Min}\{\delta_\phi v_r^{\text{mod}}(\phi)\} = \delta_\phi v_r^{\text{mod}}(\phi_-)$$  

$$= -(\pi/180^\circ)A[1 + (r_c + r)^4 R_M^4]^{-1/2}. $$  

(5c)

Thus, $\delta_\phi v_r^{\text{mod}}$ reaches the positive (negative) maximum at $\phi_+$ ($\phi_-$) on a range circle, and the positive maximum of $\delta_\phi v_r^{\text{mod}}$ at $\phi_c$ is larger or much larger than the absolute value of the negative maximum of $\delta_\phi v_r^{\text{mod}}$ at $\phi_-$, especially when $|r_c - r| < R_M < |r_c + r|$.

b. Search for $\phi_+$ on each selected range circle

As explained in (1), $\phi_c$ can be pre-estimated by $\phi_+^{\text{op}}$ from operationally issued hurricane location information. According to (4a), $\phi_+$ can be searched from the raw $u_r^{\text{obs}}$ around $\phi_+^{\text{op}}$ on each selected range circle by using the following three properties. (i) There can be more than one $\phi_0^{\text{obs}}$ point (where $|v_r^{\text{obs}}|$ reaches a local minimum below 2 m s$^{-1}$) but only one $\phi_+^{\text{op}}$ point (if it exists among $\phi_0^{\text{obs}}$ points) in the immediate vicinity of the true $\phi_+$ point. (ii) Around $\phi_0^{\text{obs}}$, $v_r^{\text{true}}$ is small and thus $v_r^{\text{obs}}$ is free of alias, so the true $\phi_+$ can be estimated by a local least squares fit to $v_r^{\text{obs}}$ around $\phi_0^{\text{obs}}$. (iii) When the local least squares fit applies to $v_r^{\text{obs}}$ around a $\phi_0^{\text{obs}}$ point other than $\phi_+^{\text{op}}$, the associated $\delta_\phi v_r^{\text{obs}}$ cannot be as large as that at $\phi_+^{\text{op}}$ because the latter should be the largest
point as implied by (5b). This property allows \( \phi_{\text{obs}} \) to be distinguished from all other \( \phi_{\text{obs}} \) points.

The above-mentioned three properties are used to search for \( \phi_1 \) on each selected range circle by performing the following four steps.

1) Search for \( \phi_{\text{obs}} \) points within \( \phi_{\text{c}} \pm 80^\circ \) by performing step 1a in appendix A in X10. This search is more efficient than the search over the entire circle used in XIL.

2) Determine the left (right) half-window width \( \Delta \phi_L \) (\( \Delta \phi_R \)) for each \( \phi_{\text{obs}} \) by performing the continuity check counterclockwise (clockwise), as shown by the flowchart in Fig. 1a.

3) Find the local zero-\( \nu_r \) point and associated \( \partial_\rho \nu_r \) around each \( \phi_{\text{obs}} \) by applying the local least squares fit (described in step 1b of appendix A in X10) to \( \nu_{\text{obs}} \) (must be eight or more nonempty data points) within

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4) Set \( \phi_+ \) to the zero point that has the largest \( \partial_\rho \nu_r \) among all the local zero points found above.

c. Estimate \( \phi_+ \) as a function of \( z \) on each tilt

As explained at the beginning of section 2, it is necessary and critical to accurately estimate \( \phi_+ \) as a function of \( z \) or essentially \( r \) from \( \nu_{\text{obs}} \) on each tilt. This is accomplished by performing the following three steps.

1) Search for \( \phi_+ \) on each range circle (selected nearest to each discrete vertical level along each tilt) and go through the entire volume to generate \( \phi_+ (z_n, u_k) \) as a discrete function of \( (z_n, u_k) \), as shown by the flowchart in Fig. 1b.

2) Construct \( \phi_+ (z_n, u_k) \) from \( \phi_+ (z_n, u_k) \) and estimate \( \phi_{\text{obs}} (z) \) by recursively fitting a piecewise-linear continuous function form of \( \phi_+ (z_n) \) in each vertical layer (from the highest to the lowest layer indexed by \( j = 5, 4, 3, 2, 1 \)), as shown by the flowchart in Fig. 1c.

3) Estimate \( \phi_{\text{obs}} (z, \theta_k) \) by recursively fitting a piecewise-linear continuous function to \( \phi_+ (z_n, \theta_k) \) on each tilt through the entire volume, as shown by the flowchart in Fig. 1d.

FIG. 1. Flowcharts for (a) step 2 in section 2b, (b) step 1 in section 2c, (c) step 2 in section 2c, and (d) step 3 in section 2c. In (a), \( \sigma \) (= 2 m s \(^{-1} \)) is the standard deviation of the observation error (not including alias error).

\( \phi_{\text{obs}} - \Delta \phi_L \leq \phi \leq \phi_{\text{obs}} + \Delta \phi_R \). Here, unlike step 1b in appendix A in X10, the half-window width is not set to \( \Delta \phi \) because \( \nu_{\text{obs}} \) is not always alias free within \( \phi_{\text{obs}} \pm \Delta \phi \).

4) Set \( \phi_+ \) to the zero point that has the largest \( \partial_\rho \nu_r \) among all the local zero points found above.

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2) Construct \( \phi_+ (z_n) \) from \( \phi_+ (z_n, u_k) \) and estimate \( \phi_{\text{obs}} (z) \) by recursively fitting a piecewise-linear continuous function form of \( \phi_+ (z_n) \) in each vertical layer (from the highest to the lowest layer indexed by \( j = 5, 4, 3, 2, 1 \)), as shown by the flowchart in Fig. 1c.

3) Estimate \( \phi_{\text{obs}} (z, \theta_k) \) by recursively fitting a piecewise-linear continuous function to \( \phi_+ (z_n, \theta_k) \) on each tilt through the entire volume, as shown by the flowchart in Fig. 1d.
As an example, the discrete vertical profile $\phi_+(z_{n})$ constructed in step 2 is plotted by the plus and multiplication signs in Fig. 2a. As shown, this constructed $\phi_+(z_{n})$ contains not only densely packed true zero-$v_z$ points (plotted by blue + signs) but also scattered false zero-$v_z$ points (plotted by red × signs). The false zero-$v_z$ points are from severely aliased areas on the two sides of the true zero-$v_z$ points, and they are occasionally selected by the searching algorithm (at the vertical levels of the red × signs) due to various reasons. The recursive fitting in step 2 is designed to filter out these false zero-$v_z$ points and other possible outliers from $\phi_+(z_{n})$ to enhance the accuracy of the estimated $\phi_+(z)$, as shown by the black solid line in Fig. 2a.

The discrete vertical profile $\phi_+(z_{m}, \theta_k)$ on the highest tilt of $\theta_k = \theta_{14} = 19.5^\circ$ is plotted by the blue plus signs in Fig. 2b. As shown, there are just two data points in each of the lowest three layers (i.e., $L_j$ for $j = 3, 2, 1$), so the coefficient $b_j$ in the fitting function $a_j + b_j(j\Delta h - z)$ can be still, but poorly, estimated from the two data points in $L_j$ for $j = 3, 2, 1$ by the initial fitting in step 2. However, the constructed $\phi_+(z, \theta_{14})$ at the lower boundary of $L_j$ is not within $\pm \Delta \phi_f$ of $\phi_+(z)$ for $j = 3, 2, 1$ (not shown), so the previously estimated $b_j$ in step 2 is used here again to reconstruct $\phi_+(z, \theta_{14})$ in $L_j$ for $j = 3, 2, 1$, in step 3. This explains why the final estimated $\phi_+(z, \theta_k)$ (plotted by the black solid line in Fig. 2b) does not closely follow the sparse data points in the lowest three layers but follows the same slantwise pattern as $\phi_+(z)$ in Fig. 2a.

Figure 2c shows that the discrete vertical profile $\phi_+(z_{m}, \theta_2)$ is empty in $L_5$ and contains many false zero-$v_z$ points in $L_4 - L_2$. These false zero-$v_z$ points are filtered out by the initial fitting in step 3, while $\phi_+(z, \theta_2)$ is constructed in the empty layer $L_5$ on $\theta_2$ by the subsequent recursive fitting in step 3. This explains why and how $\phi_+(z, \theta_2)$ is estimated as a piecewise-linear continuous function over the entire vertical range, as shown by the black solid line in Fig. 2c. Figure 2d shows that the discrete vertical profile $\phi_+(z_{m}, \theta_1)$ is not only empty in $L_5$ but also very sparse with false zero-$v_z$ points in $L_4 - L_1$. Nevertheless, by performing the recursive fitting in step 3, $\phi_+(z, \theta_1)$ can be estimated over the entire vertical range, as shown by the black solid line in Fig. 2d.

d. Refined AR vortex analysis constrained by $\phi_+(z, \theta_k)$

As explained at the beginning of section 2, the above-estimated $\phi_+(z, \theta_k)$ should be used to constrain the AR vortex analysis. This is done by setting $\phi_c = \phi_+(z, \theta_k)$ as a refined estimate of $\phi_0$ (to replace the preestimated value $\phi_0^{(p)}$) in the modified cost function [see (6)] for the refined AR vortex analysis at each vertical level. Physically, the variation of $\phi_c$ with $z$ (or $r$) on each tilt can be caused by many factors (especially the nonaxisymmetric and nontangential flow structures neglected by the vortex model) in addition to the variation of $\phi_0$ with $z$—the only factor that can be considered here by setting $\phi_c = \phi_+(z, \theta_k)$. Because only the tangential velocity $V_T$ is considered by the vortex model in (1)–(3), $\phi_c$ cannot vary with $z$ (or $r$) on any tilt in the vortex model unless $\phi_c$ varies with $z$, and the variation of $\phi_c$ with $z$ (or $r$) cannot match $\phi_+(z, \theta_k)$ on every tilt unless $\phi_c$ is set to $\phi_+(z, \theta_k)$ for the AR vortex analysis. This explains why we need to set $\phi_c = \phi_+(z, \theta_k)$ as a mathematical treatment that is partially unphysical. As shown in Figs. 9b and 9c in XJL, the AR vortex analysis and its estimated $(V_M, R_M)$ are not sensitive to small or even finite errors in $(r_c, \phi_c)$, so setting $\phi_c = \phi_+(z, \theta_k)$ will not significantly deteriorate the physical realism of the estimated $(V_M, R_M)$ but improve the accuracy of $v_{z\text{ref}}$ to ensure the reference check is free of false dealiasing, as explained earlier.

For the reference check in section 3a, $v_{z\text{ref}}$ must be adequately accurate (at least within $\pm 7 v_{z/4}$ for $v_{z\text{true}}$ for the threshold value of $v_{z/4}$ used by the reference check) at every observation point to not deflag an aliased $v_{z\text{obs}}$ and thus avoid false dealiasing. Because of this, it is better to estimate $(V_M, R_M)$ by using all available observations from all the range circles (one per tilt) around each selected vertical level, and this can also improve
the robustness of the estimated \((V_M, R_M)\) to errors in \((r_a, \phi_a)\), as shown in section 4b in XJL. The cost function in (7) in XJL is thus modified into the following form:

\[
J(V_M, R_M) = \sum_k \sum_i \left( Z_{r^i}^{\text{mod}}(V_M, R_M; \phi_k, r_k) - v_{r^i}^{\text{obs}}(\phi_k, r_k), v_N \right)^2 / \sum_k m_k \quad \text{for } z = z_n,
\]

where \(Z()\), \(v_N\) = \((-2\nu_N \text{Int}((1/2)\nu_N))\) is the aliasing operator defined in (6) in XJL, \(r_k = r(\theta_k)\) is the range radius of the \(k\)th circle selected from the \(k\)th tilt closest to the vertical level at \(z = z_n\), \(\sum_k\) denotes the summation over \(k\) from 1 to \(K\) (i.e., the number of the highest tilt \(\theta_K\)), \(\sum_i\) denotes the summation over \(i\) from 1 to \(m_k\) and \(m_k\) is the total number of observations on the \(k\)th circle. In (6), \(v_{r^i}^{\text{mod}}(V_M, R_M; \phi_k, r_k)\) is the radial velocity modeled by (3) in which \(R\) is a function of \((\phi_k, r_k; \phi_i, r_i)\) for \(r = r_k\) and \(\phi = \phi_k\) while \(\phi_i\) is set to \(\phi_0(z, \theta_k)\) and \(r_i\) is preestimated from operationally issued hurricane location information and will be fine-tuned as described later.

As shown by the black solid lines in Figs. 2b and 2c, the estimated \(\phi_0(z, \theta_k)\) and \(\phi_0(z, \theta_2)\) decrease rapidly as \(z\) decreases toward and into the boundary layer (below \(1\) km). This feature is seen for every estimated \(\phi_0(z, \theta_k)\) and is caused by the strong inward radial wind in the hurricane boundary layer. Since the hurricane radial wind is not considered in the vortex model, the refined vortex analysis is not applicable to the hurricane boundary layer. Besides, the environmental mean wind can become significant in the middle and upper troposphere but is neglected in the vortex model, so the refined vortex analysis can become inaccurate as \(z\) increases to \(4\) km and beyond. Since the reference check does not have to produce seed data over the entire radial range (and thus the entire depth) on each tilt, it is safe and sufficient to apply the refined vortex analysis only to selected vertical levels between \(1\) km ≤ \(z\) ≤ \(4\) km. Here, “seed data” are termed as those dealiased or deflagged good data that are produced and accepted by the reference check in the first main step and used as “seeds” by the continuity check in the second main step.

At each vertical level (every \(\Delta z = 25\) m from \(z = 1\) to \(4\) km), the global minimum of the cost function \(J\) in (6) can be searched in the space of \((V_M, R_M)\) by using the standard conjugate gradient descent algorithm (see section 3 in XJL). To fine-tune \(r_a\), we simply select a series of 17 uniformly distributed values for \(r_a\) (every \(2.5\) km apart) in the \(\pm 20\)-km vicinity of the preestimated \(r_a\), and then find the global minimum, \(J^{\text{min}}(r_a) = J(V_M^{\text{min}}(r_a), R_M^{\text{min}}(r_a))\), for each selected value of \(r_a\), where \([V_M^{\text{min}}(r_a), R_M^{\text{min}}(r_a)]\) is the global minimum point for the selected \(r_a\). Among the nine selected values of \(r_a\), the value

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Fig. 2. (a) Discrete vertical profile \(\phi_0(z_n)\) plotted by blue + signs for retained zero-\(v_r\) points and by red \(\times\) signs for rejected zero-\(v_r\) points and estimated \(\phi_0(z)\) plotted by the black solid line. (b) Discrete vertical profile \(\phi_0(z_n, \theta_k)\) on the highest tilt \((\theta_k = 14.5^\circ)\) and estimated \(\phi_0(z, \theta_k)\). (c) Discrete vertical profile \(\phi_0(z_n, \theta_2)\) on the second lowest tilt \((\theta_2 = 1.45^\circ)\) and estimated \(\phi_0(z, \theta_2)\). (d) Discrete vertical profile \(\phi_0(z_n, \theta_1)\) on the lowest tilt \((\theta_1 = 0.5^\circ)\) and estimated \(\phi_0(z, \theta_1)\). All the results are produced from a full volume of \(v_r^{\text{obs}}\) scanned from Hurricane Katrina by the KLIX radar over the 5-min period from 1028 to 1033 UTC 29 Aug 2005.
that yields the smallest $f^\text{min}(r_c)$ is used as a refined estimate of $r_c$ at the selected vertical level $z_n$. This refined estimate is denoted by $r_c^e$, and $(V_{Me}, R_{Me}) = [V_{M}^{\text{min}}(r_c^e), R_{M}^{\text{min}}(r_c^e)]$ is the global minimum point for $r_c = r_c^e$.

As explained in section 3a in XJL, a suitable initial guess of $(V_{M}, R_{M})$ can be easily obtained from operationally issued hurricane wind information within the concave area of the cost function. This feature is shown by the example in Fig. 3a (Fig. 3b), where $r_c$ is already fine-tuned to $r_c^e = 179.16$ km for $z_n = 1$ km ($r_c^e = 164.16$ km for $z_n = 4$ km) after $\phi_e$ is set to $\phi_e^a(z, \theta_k)$ for each range circle (selected on each tilt $\theta_k$) around $z = z_n = 1$ km (4 km). As shown in Fig. 3a (Fig. 3b), the initial guess (marked by the $\times$ sign) is at $(V_{Me}, R_{Me}) = (76 \text{ m s}^{-1}, 96 \text{ km})$ for $z_n = 1$ km [or $(V_{Me}, R_{Me}) = (56 \text{ m s}^{-1}, 106 \text{ km})$ for $z_n = 4$ km], while the global minimum point $(V_{Me}, R_{Me})$ found by the descent algorithm (marked by the $+$ sign) is at $(V_{Me}, R_{Me}) = (81.09 \text{ m s}^{-1}, 66.91 \text{ km})$ for $z_n = 1$ km [or $(V_{Me}, R_{Me}) = (51.20 \text{ m s}^{-1}, 99.94 \text{ km})$ for $z_n = 4$ km], which is almost identical to the global minimum point (marked by the open circle) found by the brute-force search (with resolutions of $1 \text{ m s}^{-1}$ in $V_M$ and 1 km in $R_M$). The algorithm converges in 28 (12) iterations in Fig. 3a (Fig. 3b), which largely explains why the convergence requires more iterations in Fig. 3a than in Fig. 3b. As the selected vertical level increases from $z_n = 1$ to 2 km and beyond, the geometry of $J(V_{M}, R_{M})$ changes gradually and becomes very similar to that in Fig. 3b, so the required number of iterations decreases rapidly toward 12. The descent algorithm is thus not only accurate but also very efficient in most cases. This convergence property is similar to that found in XJL but is now further tested and verified with real radar radial velocities.

After $r_c^e, V_{Me}^e$, and $R_{Me}^e$ are obtained for each vertical level as functions of $z_n$, they are treated as input data for the least squares fits to estimate $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$, respectively, as linear functions of $z$ over the vertical range of 1 km $\leq z \leq 4$ km. The input $r_c^e(z_n)$, $V_{Me}^a(z_n)$, and $R_{Me}^a(z_n)$ are plotted with their estimated $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$ in Figs. 4a–c, respectively, for the same radar volume scan as that used in Figs. 2 and 3. Figure 4a shows that $r_c^a(z)$ decreases linearly from 178.6 to 155.6 km as $z$ increases from 1 to 4 km, and this linear decrease fits the general trend of the variation of $r_c^a(z)$ with $z_n$, although the data points of $r_c^a(z_n)$ are plotted with their estimated $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$ in Figs. 4a–c, respectively, for the same vertical level as functions of $z_n$. The input $r_c^a(z_n)$, $V_{Me}^a(z_n)$, and $R_{Me}^a(z_n)$ are plotted with their estimated $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$ in Figs. 4a–c, respectively, for the same radar volume scan as that used in Figs. 2 and 3. Figure 4a shows that $r_c^a(z)$ decreases linearly from 178.6 to 155.6 km as $z$ increases from 1 to 4 km, and this linear decrease fits the general trend of the variation of $r_c^a(z)$ with $z_n$, although the data points of $r_c^a(z_n)$ are plotted with their estimated $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$ in Figs. 4a–c, respectively, for the same radar volume scan as that used in Figs. 2 and 3. Figure 4a shows that $r_c^a(z)$ decreases linearly from 178.6 to 155.6 km as $z$ increases from 1 to 4 km, and this linear decrease fits the general trend of the variation of $r_c^a(z)$ with $z_n$, although the data points of $r_c^a(z_n)$ are plotted with their estimated $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$ in Figs. 4a–c, respectively, for the same radar volume scan as that used in Figs. 2 and 3. Figure 4a shows that $r_c^a(z)$ decreases linearly from 178.6 to 155.6 km as $z$ increases from 1 to 4 km, and this linear decrease fits the general trend of the variation of $r_c^a(z)$ with $z_n$, although the data points of $r_c^a(z_n)$ are plotted with their estimated $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$ in Figs. 4a–c, respectively, for the same radar volume scan as that used in Figs. 2 and 3. Figure 4a shows that $r_c^a(z)$ decreases linearly from 178.6 to 155.6 km as $z$ increases from 1 to 4 km, and this linear decrease fits the general trend of the variation of $r_c^a(z)$ with $z_n$, although the data points of $r_c^a(z_n)$ are plotted with their estimated $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$ in Figs. 4a–c, respectively, for the same radar volume scan as that used in Figs. 2 and 3. Figure 4a shows that $r_c^a(z)$ decreases linearly from 178.6 to 155.6 km as $z$ increases from 1 to 4 km, and this linear decrease fits the general trend of the variation of $r_c^a(z)$ with $z_n$, although the data points of $r_c^a(z_n)$ are plotted with their estimated $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$ in Figs. 4a–c, respectively, for the same radar volume scan as that used in Figs. 2 and 3. Figure 4a shows that $r_c^a(z)$ decreases linearly from 178.6 to 155.6 km as $z$ increases from 1 to 4 km, and this linear decrease fits the general trend of the variation of $r_c^a(z)$ with $z_n$, although the data points of $r_c^a(z_n)$ are plotted with their estimated $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$ in Figs. 4a–c, respectively, for the same radar volume scan as that used in Figs. 2 and 3. Figure 4a shows that $r_c^a(z)$ decreases linearly from 178.6 to 155.6 km as $z$ increases from 1 to 4 km, and this linear decrease fits the general trend of the variation of $r_c^a(z)$ with $z_n$, although the data points of $r_c^a(z_n)$ are plotted with their estimated $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$ in Figs. 4a–c, respectively, for the same radar volume scan as that used in Figs. 2 and 3. Figure 4a shows that $r_c^a(z)$ decreases linearly from 178.6 to 155.6 km as $z$ increases from 1 to 4 km, and this linear decrease fits the general trend of the variation of $r_c^a(z)$ with $z_n$, although the data points of $r_c^a(z_n)$ are plotted with their estimated $r_c^a(z), V_{Me}^a(z)$, and $R_{Me}^a(z)$ in Figs. 4a–c, respectively, for the same radar volume scan as that used in Figs. 2 and 3.
linear increase fits $R_M(z_a)$ roughly. A qualitatively similar decrease of $V_M(z)$ and increase of $R_M(z)$ with the increase of $z$ (from 1 to 4 km) are obtained by the refined vortex analysis from all other radar volume scans so far tested (see section 3b).

3. Adaptive dealiasing tested with velocities scanned from hurricanes and typhoons

a. Adaptive dealiasing

The new adaptive dealiasing consists of the same two main steps, the reference check and continuity check, as the AR-VAD-based dealiasing except that the abovementioned refined vortex analysis is used in place of the AR-VAD analysis to produce $v_r^{\text{ref}}$ for the reference check in the first main step. In particular, after $\phi_r^a(z, \theta_k), r_c^a(z), V_M^a(z)$, and $R_M^a(z)$ are estimated by the refined vortex analysis as described in section 2, they are used as $\phi_r$, $r_c$, $V_M$, and $R_M$, respectively, and substituted back into (3) to compute $v_r^{\text{ref}}$ as a function of $\phi$ and $r(z, \theta_k)$ on each tilt over the vertical range between 1 km $\leq z \leq$ 4 km. The threshold condition for the alias correction in the reference check is the same as that used in X11. This threshold condition and related algorithm for the reference check are reviewed briefly below.

Preceding the reference check, all the raw data points are flagged. The reference check goes through every data point along each range circle on each tilt between 1 km $\leq z \leq$ 4 km. At each data point, the Nyquist folding number is estimated first by

$$N = \text{Int}[(v_r^{\text{ref}} - v_r^{\text{obs}})/(2v_N)],$$

where $\text{Int}[(\ )]$ represents the nearest integer of ( ) and $v_r^{\text{obs}}$ is the observed raw radial velocity. If $N = 0$, then $v_r^{\text{obs}}$ needs no correction, but the data point is deflagged with $v_r^{\text{obs}}$ accepted as a seed datum only if

$$|v_r^{\text{obs}} - v_r^{\text{ref}}| \leq v_N/4. \quad (8)$$

If $N \neq 0$, then $v_r^{\text{obs}}$ is corrected to $v_r^{\text{cob}} = v_r^{\text{obs}} + 2Nv_N$, but $v_r^{\text{cob}}$ is accepted as a new seed datum (in place of the original $v_r^{\text{obs}}$) only if

$$|v_r^{\text{cob}} - v_r^{\text{ref}}| \leq v_N/4. \quad (9)$$

If the threshold conditions (8) and (9) are both not satisfied, then the data point remains to be flagged and the check proceeds to the next data point.

The above-mentioned threshold value of $v_N/4$ implies that $v_r^{\text{ref}}$ must be within $\pm v_N/4$ of $v_r^{\text{true}}$ to unmistakably correct possible aliasing in $v_r^{\text{obs}}$ and must be within $\pm 7v_N/4$ of $v_r^{\text{true}}$ to flag (but not correct) possible alias in $v_r^{\text{obs}}$ at each observation point. Although the original vortex
analysis in XJL can estimate $V_M$ and $R_M$ quite reliably and accurately, its computed $v_r^{ref}$ may occasionally have a large error beyond $\pm 7\nu_N/4$, especially in the vicinity of $\phi_\nu$. With the refinements in section 2, $v_r^{ref}$ produced by the refined vortex analysis can have the required accuracy—that is, within $\pm \nu_N/4$ ($\pm 7\nu_N/4$) of $v_r^{true}$—at every observation point between $1 \leq z \leq 4$ km on every tilt for all the 1099 volume scans of severely aliased $v_r^{obs}$ so far tested (see the next subsection). The satisfactory performance of the refined vortex analysis makes the reference check simple and clean (without needing any additional quality check).

After the above-mentioned check, the continuity check of X11 is performed in the second main step. As described in section 2c in X11, the continuity check uses all available seed data in a properly enlarged area around each flagged data point that is being checked, so it can traverse small data gaps (up to 10 km in the radial direction or $5^\circ$ in the azimuthal direction). Here, the continuity check is performed with the following modification. In the AR-VAD-based dealiasing, the reference check and its produced seed data in the first main step are limited within the cutoff radial range ($r = 30$ km for $\theta < 1^\circ$ or $r = 80$ km for $\theta \geq 1^\circ$) to avoid false dealiasing, and the continuity check proceeds circle by circle outward one way only from the smallest range circle (covered by the seed data) to the largest range circle on each tilt. In the new adaptive dealiasing, however, the reference check and its produced seed data cover a wide radial range (over the depth between $1 \leq z \leq 4$ km) on each tilt, so the continuity check can easily start from the smallest range circle covered by the seed data and go outward to the largest range circle first and then come back inward to the smallest range circle (below $z = 1$ km) and thus complete the entire tilt.

b. Test results with hurricanes

The new adaptive dealiasing described in the previous subsection has been successfully tested with 602 volumes of severely aliased $v_r^{obs}$ scanned from three hurricane cases by four operational WSR-88D radars [Klix (Slidell, Louisiana), KMOB (Mobile, Alabama), KHGX (Houston, Texas), KMHX (Morehead City, North Carolina)]. Among the 602 volume scans, 52 volumes were scanned by the KLIX radar from Hurricane Katrina over the entire period from 0900 to 1400 UTC 29 August 2005; 110 volumes were scanned by the KMOB radar from Hurricane Katrina from KMOB radar over the entire period from 1200 to 2200 UTC 29th August 2005; 235 volumes were scanned by the KHGX radar from Hurricane Ike over the entire period from 0900 to 1900 UTC 13 September 2008; and 205 volumes were scanned by the KMHX radar from Hurricane Irene over the entire period from 0500 to 2100 UTC 27 August 2011. An example is given in Fig. 5.

Figure 5a shows the image of raw $v_r^{obs}$ at the lowest tilt ($\theta_1 = 0.5^\circ$) scanned by the operational KLIX radar at 1028 UTC 29 August 2005 from Hurricane Katrina. This lowest tilt is from the same volume of $v_r^{obs}$ as that used to produce Fig. 2, the spatial resolutions are 250 m in the radial direction and $1^\circ$ in the azimuthal direction, and the maximum detection range of the radar is 175 km in this case. At this time, the hurricane vortex center (marked by the yellow C in Fig. 5a) was about 160 km to the south (with $\phi_e \approx 175^\circ$) of the KLIX radar. Since the Nyquist velocity ($\nu_N = 21.5$ m s$^{-1}$) used by this scan is much smaller than the estimated $V_M^{true}(z) = 66.3$ m s$^{-1}$ at $z = 1$ km) and therefore must be much smaller than the absolute values of the true maximum positive and negative radial velocities, the observed radial velocities are severely aliased in two main areas (marked by the two white A letters) on the two sides of the southern zero-$v_r$ line [estimated by $\phi_\nu^{true}(z, \theta_1)$ with $\nu_r$ viewed as a function of $r$] on $\theta_1 = 0.5^\circ$. The small red area inside the large green area to the southeast of the radar site shows doubly aliased raw $v_r^{obs}$.

Figure 5b shows that $v_r^{ref}$ produced by the refined vortex analysis (between $1 \leq z \leq 4$ km—that is, $75 \leq r \leq 198$ km) is plotted only over the $v_r^{obs}$-covered area can capture the gross structure and magnitude of $v_r^{true}$, including the southern zero-$v_r$ line and the positive and negative maxima of $v_r^{true}$ on its two sides. Figure 5c shows that the seed data produced by the reference check are free of false dealiasing and cover sufficiently broad areas, so the subsequent continuity check can correct most aliases with no false dealiasing, as shown in Fig. 5d. As listed in the first row of Table 1, the rejected data are only 0.17% of the total raw data, so the accepted alias-free data are 99.83% of the total on this lowest tilt. The AR-VAD-based dealiasing (X11) is also alias free as shown in Fig. 5e but its rejected data are 14.76% of the total on this tilt, so it underperforms the new adaptive dealiasing (not only on this lowest tilt but also over the entire volume, as shown in Table 1). The extended AR-Var-based dealiasing (X13) is alias free and its rejected data are 1.04% of the total on this tilt, so it only slightly underperforms the new adaptive dealiasing on the lowest tilt (as well as for the entire volume, as shown in Table 1) but the computational time is increased by nearly 4 times (from 15 to 56 s of CPU time for dealiasing the entire volume on a Dell XII computer). The operationally used dealiasing (Eilts and Smith 1990) rejects no datum but it is not alias-free, as shown in Fig. 5f (also see the last column in Table 1).

As the vortex center of Hurricane Katrina moved northward into the 100-km radial range of the KLIX radar in the later time period (1332–1338 UTC 29 August
FIG. 5. (a) Image of raw $v_r^{\text{obs}}$ on the lowest tilt ($\theta_1 = 0.5^\circ$) scanned with $v_\phi = 21.5$ m s$^{-1}$ from Hurricane Katrina by the KLIX radar at 1028 UTC 29 Aug 2005. (b) Image of $v_r^{\text{ref}}$ produced by the refined vortex analysis. (c) Image of seed data produced by the reference check. (d) Image of dealiased $v_r^{\text{obs}}$ produced by the new adaptive dealiasing. (e) Image of dealiased $v_r^{\text{obs}}$ produced by the VAD-based dealiasing (X11). (f) Image of dealiased $v_r^{\text{obs}}$ produced by the operationally used method (Eilts and Smith 1990). The lowest tilt of raw $v_r^{\text{obs}}$ in (a) is from the same volume of $v_r^{\text{obs}}$ as that used in Fig. 2. In (a), the yellow + and − signs mark the two zero-$v_r$ points $\phi_+\text{ and } \phi_-$, respectively, on the range circle of $r = 50$ km; the yellow C marks the hurricane vortex center, which is at $(r, \phi) = (r_c, \phi_c) = (160 \text{ km, } 175^\circ)$ in the radar coordinates; and the two white A’s mark the two severely aliased areas. In (f), the white letter F marks false dealiasing in the small green area. The color scale on the top of each panel shows red (green) for positive (negative) value, that is, $v_r > 0$ ($v_r < 0$) for an outward (inward) radial component velocity.
Typhoon Hagupit. As shown by the image of raw
data scanned in this time. The Nyquist velocity
$V_N = (26.8 \text{ m s}^{-1})$ used on this tilt is significantly smaller than the estimated
$V_M(z) = (59.5 \text{ m s}^{-1} \text{ at } z = 1 \text{ km})$, so the raw radial ve-
locities are severely aliased in two main areas (marked by
the two white A’s) on the two sides of the southern zero-$v_r$
line [estimated by $\phi_v^a(z, \theta_1)$ with $z$ viewed as a function of $r$]
on $\theta_1 = 0.6^\circ$. By extrapolating $r_v^a(z)$, $V_M^a(z)$, and $R_M^a(z)$ to
$z < 1 \text{ km}$ but with $\phi_v^a(z, \theta_1)$ estimated in $L_z$ for $z < 1 \text{ km}$
according to step 3 in section 2c, the $v_r^\text{ref}$ field is produced on
this lowest tilt by the refined vortex analysis with the radial
range of $68 \text{ km} \leq r \leq 186 \text{ km}$ (associated with the vertical
range of $1 \text{ km} \leq z \leq 4 \text{ km}$ on $\theta_1 = 0.6^\circ$) extended inward
from $r = 68 \text{ km}$ to $r = 5 \text{ km}$ (i.e., downward below $z = 1 \text{ km}$)
for the reason explained later. As shown in Fig. 6b, this $v_r^\text{ref}$
field can roughly capture the structure and magnitude of
$v_r^\text{true}$, including the southern zero-$v_r$ line and the posi-
tive and negative maxima of $v_r^\text{true}$ on its two sides.

Figure 6c shows that the seed data produced by the reference
check are free of alias and cover sufficiently
broad areas, although the coved areas are sparse to the
north and northeast of the radar. Figure 6d shows that the
continuity check using the seed data between
$68 \text{ km} \leq r \leq 186 \text{ km}$ (associated with the vertical range of
$1 \text{ km} \leq z \leq 4 \text{ km}$ on $\theta_1 = 0.6^\circ$) in Fig. 6c can correct
aliases over most data areas except for the small sector
area (marked by the white F) between $150^\circ < \phi < 190^\circ$
within $r < 55 \text{ km}$. This small sector area contains the zero-$v_r$
line (as shown in Fig. 6a), so the true radial velocities are
small and the raw data are not aliased in this small sector
area. However, as shown in Fig. 6a, the raw data are very
noisy not only in this small sector area but also in
the adjacent sector area of aliased raw data between

<table>
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<tr>
<th>Tilt</th>
<th>No. of raw data</th>
<th>New adaptive method</th>
<th>AR-VAD-based method</th>
<th>AR-VAR-based method</th>
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FIG. 6. The first four and last panels are as in Figs. 5a–d and 5f, respectively, but for the lowest tilt ($\theta_1 = 0.6^\circ$) scanned with $v_{\text{ref}} = 26.8$ m s$^{-1}$ from Typhoon Hagupit by the operational Yangjiang (Z9662) radar at 1135 UTC 23 Sep 2008. The $v_{\text{ref}}$ field in (b) and the seed data in (c) are no longer confined in the radial range of $68 \leq r \leq 186$ km (associated with the vertical range of $1 \leq z \leq 4$ km on $\theta_1 = 0.6^\circ$) but extended inward from $r = 68$ km to $r = 5$ km (i.e., downward below $z = 1$ km on $\theta_1 = 0.6^\circ$). (e) The image of the finally dealiased data produced by the continuity check using all the seed data in (c), that is, not only those confined between $68 \leq r \leq 186$ km but also the additional seed data within $r < 68$ km. In (f) the image of dealiased $v_{\text{obs}}$ produced by operationally used dealiasing (Eilts and Smith 1990) is shown. The typhoon vortex center, marked by the yellow C in (a), is at $(r, \phi) = (r_c, \phi_c) \sim (68$ km, $182^\circ)$ in the radar coordinates. The white F in (d) marks the small sector area of false dealiasing between $150^\circ < \phi < 190^\circ$ within $r < 55$ km.
Table 2. As in Table 1, but for the nine tilts of the entire volume scanned by the operational Yangjiang (Z9662) radar at 1936 UTC 23 Sep 2008 from Typhoon Hagupit (see Figs. 6 and 7). The right column under the operational method is omitted because no datum (0%) is rejected. The reduced percentage, 1.69% (0.26%), listed in the parentheses in the first (last) row in the right column under the new adaptive method shows the improved result by using the extended reference check (see Fig. 6e).

<table>
<thead>
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<td>Rejected data (%)</td>
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<td>34.20</td>
</tr>
<tr>
<td>$\theta_2 = 1.6^\circ$</td>
<td>181 291</td>
<td>0</td>
<td>0.76</td>
<td>0</td>
<td>33.40</td>
</tr>
<tr>
<td>$\theta_3 = 2.5^\circ$</td>
<td>173 843</td>
<td>0</td>
<td>0.18</td>
<td>0</td>
<td>30.87</td>
</tr>
<tr>
<td>$\theta_4 = 3.5^\circ$</td>
<td>155 400</td>
<td>0</td>
<td>0.20</td>
<td>0</td>
<td>36.19</td>
</tr>
<tr>
<td>$\theta_5 = 4.4^\circ$</td>
<td>128 091</td>
<td>0</td>
<td>0.43</td>
<td>0</td>
<td>42.24</td>
</tr>
<tr>
<td>$\theta_6 = 6.5^\circ$</td>
<td>90 263</td>
<td>0</td>
<td>0.94</td>
<td>0</td>
<td>47.78</td>
</tr>
<tr>
<td>$\theta_7 = 10$</td>
<td>61 654</td>
<td>0</td>
<td>0.80</td>
<td>0</td>
<td>62.05</td>
</tr>
<tr>
<td>$\theta_8 = 14.7^\circ$</td>
<td>48 620</td>
<td>0</td>
<td>0.87</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>$\theta_9 = 19.6^\circ$</td>
<td>40 367</td>
<td>0</td>
<td>6.19</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>All 9 tilts</td>
<td>1 037 795</td>
<td>0.54 (0.26)</td>
<td>1.09</td>
<td>1.57</td>
<td>43.27</td>
</tr>
</tbody>
</table>

120° < $\phi$ < 152° within 10 km < $r$ < 125 km. These two sector areas of very noisy data are connected (around the radial of $\phi = 150^\circ$) quite homogeneously, which makes it very difficult to exactly identify the boundary between the two (nonaliased and aliased) areas even with human expertise. Thus, as the continuity check goes radially inward (from the seed data points in Fig. 6c), the aliased raw data in the above-mentioned second sector area (between 120° < $\phi$ < 152° within 10 km < $r$ < 125 km) are correctly dealiased first, and then the nonaliased raw data in the above-mentioned first sector area (between 150° < $\phi$ < 190° within $r$ < 55 km) are falsely dealiased due to the very noisy and homogeneous connection between the two sector areas. There are 5653 falsely dealiased data on the lowest tilt only, so the percentage of incorrectly accepted aliased data is 3.57% on this tilt (merely 0.54% for the entire volume) as listed in the first (last) row of Table 2. Such a failure of the continuity check (or any other type of continuity check) is clearly inevitable in this very difficult case, and the difficulty is caused by high data noise and poor data quality in this rarely encountered case.

When the $v_{r|z}^{\text{ref}}$ field in Fig. 6b is used not only between 68 km ≤ $r$ ≤ 186 km but also within $r$ < 68 km, the reference check can produce additional seed data in the aforementioned two sectors of very noisy data as revealed in Fig. 6c. These additional seed data can reduce falsely dealiased data produced by the continuity check in the second main step. With this approach, the dealiased data produced by the continuity check are shown in Fig. 6e, where the number of falsely dealiased data is reduced from 5653 to 2670 and thus the percentage of incorrectly accepted aliased data is reduced from 3.57% to 1.69% on the lowest tilt (from 0.54% to 0.26% for the entire volume) as listed in the first (last) row of Table 2. This extended reference check can improve dealiasing in the above-mentioned rarely encountered difficult case, but it also increases the risk of producing falsely dealiased seed data in the first main step. For all other tilts and volumes so far tested, such an extension is risky and unnecessary. For this very difficult case, the AR-VAD-based dealiasing is not free of alias, its incorrectly accepted aliased data are 10.29% , and its rejected data are 34.20% of the total, as listed in the first row of Table 2. The extended AR-Var-based dealiasing rejects all the data and thus fails to work on this tilt (see the first row of Table 2). The operationally used dealiasing rejects no datum, but its accepted data are severely aliased, as shown in Fig. 6f. Its incorrectly accepted aliased data are 16.36% (43.29%) of the total on the lowest tilt (over the entire volume) as listed in last column of Table 2.

The above-mentioned difficulty is seen at the lowest tilt ($\theta_1 = 0.6^\circ$) of merely one volume scan. For other higher tilts in this volume and any tilts (including the lowest tilts) of the other 496 volumes (scanned by operational CINRAD/SA radars in China) so far tested, the method is free of false dealiasing. As an example, Fig. 7a shows the image of raw $v_{r|z}^{\text{obs}}$ on the second lowest tilt (at $\theta_2 = 1.6^\circ$) for the same volume as in Fig. 6. Again, the Nyquist velocity $v_N = (26.8 \text{ m s}^{-1})$ used on this tilt is significantly smaller than the estimated $V_M(z) = (59.5 \text{ m s}^{-1})$ at $z = 1 \text{ km}$, so the observed radial velocities are severely aliased in two main areas (marked by the two white A’s) on the two sides of the southern zero-$v_r$ line [estimated by $\phi = \theta_2$ with $z$ viewed as a function of $r$] on $\theta_2 = 1.6^\circ$. Figure 7b shows that $v_{r|z}^{\text{ref}}$ produced by the refined vortex analysis (between 1 km ≤ $z$ ≤ 4 km—that is, 34.3 km ≤ $r$ ≤ 116.8 km—but plotted only over the $v_{r|z}^{\text{obs}}$-covered area) can capture the gross structure and magnitude of $v_{r|z}^{\text{true}}$, including the southern zero-$v_r$ line and the positive and
Fig. 7. The first five panels are as in Figs. 5a–e, but for the second lowest tilt at $\theta_2 = 1.6^\circ$ in the same volume scanned from Typhoon Hagupit by the operational Yangjiang (Z9662) radar as that in Fig. 6. (f) The image of dealiased $v_\text{obs}$ produced by the extended AR-Var-based dealiasing (X13).
negative maxima of $v_{r,\text{true}}$ on its two sides. Figure 7c shows that the seed data produced by the reference check are free of false dealiasing and cover sufficiently broad areas, so the subsequent continuity check can correct most aliases with no false dealiasing as shown in Fig. 7d and the rejected data are only 0.76% of the total on this tilt as listed in the second row of Table 2.

The AR-VAD-based dealiasing is alias free on the second lowest tilt as shown in Fig. 7e, but its rejected data are 33.40% (43.27%) of the total on this tilt (over the entire volume), as shown in Table 2. The extended AR-Var-based dealiasing is alias free on the second lowest tilt as shown in Fig. 7f, while its rejected data are 0.99% (30.93%) of the total on this tilt (over the entire volume) and this percentage is lower than that rejected by the AR-VAD-based dealiasing but higher than that rejected by the new adaptive dealiasing, as shown in Table 2.

The new adaptive dealiasing also works well with no false dealiasing for all 60 volumes of severely aliased $v_{r,\text{obs}}$ scanned from this typhoon (Typhoon Hagupit) during the time period of 0700–1400 UTC 23 September 2008 and the 437 volumes from five other typhoons. These 437 volumes include 60 volumes from Typhoon Metsa scanned by the operational Ningbo, China (Z9574), radar during the time period of 0400–1200 UTC 5 August 2005, 10 volumes from Typhoon Chanthu scanned by the operational Zhanjiang, China (Z9759), radar during the time period of 1200–1400 UTC 22 July 2010, 81 volumes from Typhoon Utor by the operational Yangjiang (Z9662) radar during the time period of 0000–0800 UTC 14 August 2013, 105 volumes from Typhoon Usagi scanned by the operational Shanwei, China (Z9660), radar during the time period of 0000–1025 UTC 22 September 2013, and 181 volumes from Typhoon Fitow scanned by the operational Wenzhou, China (Z9577), radar during the time period of 0130–2000 UTC 6 October 2013. For the 1099 volume scans so far tested, the new adaptive dealiasing never fails to work even when the raw data coverage becomes poor (close to or slightly less than 180° on most range circles).

4. Conclusions

The techniques presented in this paper refine the vortex analysis of XJL so it can be used in place of the AR-VAD analysis (X10) to improve the reference check in dealiasing. This upgrades the VAD-based dealiasing (X11) to a new adaptive dealiasing method applicable solely to raw radial velocities scanned from hurricanes and typhoons. This new adaptive method has been tested successfully with 602 volumes of severely aliased radial velocities scanned from hurricanes in the United States. The method has been also successfully tested with 497 volumes of severely aliased $v_{r,\text{obs}}$ scanned from six typhoon cases by operational CINRAD/SA radars in China. The dealiasing results were verified with human expertise by carefully viewing the enlarged radial velocity images pixel by pixel on each tilt for all the tilts in the 1099 volumes. According to these verifications, the new adaptive dealiasing outperforms the operationally used dealiasing (Eilts and Smith 1990) in all the cases so far tested, and it works very well as long as the vortex center is within the 200-km radial range from the radar. It can correct most aliases on each tilt with no false dealiasing (except for a small sector area on one tilt where the raw data are very noisy with poor quality as examined in Fig. 6), so it cannot only satisfy the high-quality standard required by data assimilating but also reject less or much less data than the extended AR-Var-based dealiasing (X13) and the AR-VAD-based dealiasing for hurricanes and typhoons, especially when the hurricane or typhoon is close to the radar (within 100-km radial range). The new adaptive dealiasing is more efficient (by nearly 4 times) than the extended AR-Var-based dealiasing and satisfies the computer time constraint explained in the introduction, and it can be used safely as long as the hurricane or typhoon center is within the 200-km radial range.

In this paper, the AR vortex analysis is refined and used for the reference check in the second main step of dealiasing, but the analysis is confined between 1 km $\leq z \leq$ 4 km (except for one rare case in which the analysis is extended below 1 km on the lowest tilt, as shown in Fig. 6b) for the following two reasons. First, the analysis tends to become unreliable in the middle troposphere and beyond ($z >$ 4 km), where the environmental mean wind is often significant but not considered in the vortex model [see (1) and (2)]. Second, the analysis also tends to become unreliable or even completely inapplicable in the boundary layer, where the radial wind becomes strong toward the hurricane eyewall [as implied by the rapid change of the estimated $u_r$ with $z$ ($\leq$1 km) in each panel of Fig. 2] but not considered in the vortex model. With the first limitation, the reference radial velocity field and subsequent continuity check on a middle tilt ($3^\circ < \theta < 5^\circ$) cannot cover isolated raw data patches (if any), so these patches will remain to be flagged and thus finally rejected. It is possible to extend the analysis beyond $z = 4$ km, and this may improve the dealiased data coverage on middle tilts. Such an extension deserves further investigations.

Multiple pulse repetition frequency (PRF) scans have been used occasionally and optionally (with VCP121 and VCP221) by NOAA operational WSR-88D radars for hurricanes. A multi-PRF dealiasing algorithm has been developed and applied to multi-PRF scans to improve the reliability of dealiasing, but regions with only velocity data from the scan using PRF with the largest...
unambiguous range still can have dealiasing errors, as
the Nyquist velocity is reduced to \( v_N \approx 20 \text{ m s}^{-1} \) (Zittel
et al. 2008). By setting \( v_N \) differently according to dif-
ferent PRF for the cost function in (6) of this paper, the
new adaptive dealiasing presented in this paper can be
extended to improve the multi-PRF dealiasing algo-

**Acknowledgments.** We are thankful to Kang Nai of the
University of Oklahoma (OU) for providing the continuity
check subroutines and plotting for Figs. 5–7, and to the
anonymous reviewers for their comments and suggestions,
which improved the presentation of the paper. The re-
search was supported by the ONR Grant N000141410281
to OU. Funding was also provided by China Meteorolog-
ical Administration Special Public Welfare Research
Fund (GYHY201506004 and GYHY201506021), the Na-
tional Natural Sciences Foundation of China (Grant
91337103), the National Key Technology R&D Program
(2012BAC22B02), and National Program on Key Basic
Research Project (973 Program, 2013CB430106) of China.

**APPENDIX**

**Glossary of Variables and Symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_j )</td>
<td>Coefficient for the constant part of the fitting function in the ( j )th layer</td>
</tr>
<tr>
<td>( b_j )</td>
<td>Coefficient for the linear part of the fitting function in the ( j )th layer (see section 2c)</td>
</tr>
<tr>
<td>( J )</td>
<td>Cost function defined by (6) in the space of ((V_M, R_M))</td>
</tr>
<tr>
<td>( J_{\text{min}}(r_c) )</td>
<td>Global minimum of ( J ) for given ( r_c )</td>
</tr>
<tr>
<td>( L_j )</td>
<td>( j )th vertical layer</td>
</tr>
<tr>
<td>( N )</td>
<td>Nyquist number</td>
</tr>
<tr>
<td>( R )</td>
<td>Radial distance from vortex center</td>
</tr>
<tr>
<td>( R_M )</td>
<td>( R ) where ( V_T ) reaches ( V_M )</td>
</tr>
<tr>
<td>( R_M^{\text{min}}(r_c) )</td>
<td>( R_M ) at the global minimum point of ( J ) for given ( r_c )</td>
</tr>
<tr>
<td>( R_M^{\text{est}}(z_n) )</td>
<td>( R_M^{\text{min}}(r_c) ) for ( r_c = r_c^* ) at ( n )th vertical level</td>
</tr>
<tr>
<td>( R_M^{\text{est}}(z) )</td>
<td>Linear function of ( z ) estimated for ( R_M ) from ( R_M^{\text{est}}(z_n) ) (see section 2d)</td>
</tr>
<tr>
<td>( V_M )</td>
<td>Maximum value of ( V_T )</td>
</tr>
<tr>
<td>( V_M^{\text{min}}(r_c) )</td>
<td>( V_M ) at the global minimum point of ( J ) for given ( r_c )</td>
</tr>
<tr>
<td>( V_M^{\text{est}}(z_n) )</td>
<td>( V_M^{\text{min}}(r_c) ) for ( r_c = r_c^* ) at ( n )th vertical level</td>
</tr>
<tr>
<td>( V_M^{\text{est}}(z) )</td>
<td>Linear function of ( z ) estimated for ( V_M ) from ( V_M^{\text{est}}(z_n) ) (see section 2d)</td>
</tr>
<tr>
<td>( V_T )</td>
<td>Tangential velocity of modeled vortex in (1)</td>
</tr>
<tr>
<td>( \Delta h )</td>
<td>Depth (=1 km) of each vertical layer</td>
</tr>
<tr>
<td>( r )</td>
<td>Radial distance from radar</td>
</tr>
<tr>
<td>( r_c )</td>
<td>Radial distance of vortex center from radar</td>
</tr>
<tr>
<td>( r_c^* )</td>
<td>Refined estimate of ( r_c ) at a selected vertical level</td>
</tr>
<tr>
<td>( r_c^{\text{est}}(z_n) )</td>
<td>Linear function of ( z ) estimated for ( r_c ) from ( r_c^{\text{est}}(z_n) ) (see section 2d)</td>
</tr>
<tr>
<td>( v_N )</td>
<td>Nyquist velocity</td>
</tr>
<tr>
<td>( v_r )</td>
<td>Radial velocity–velocity component along the radar beam (positive for outward)</td>
</tr>
<tr>
<td>( v_r^{\text{mod}} )</td>
<td>Modeled ( v_r ) in (2) and (3)</td>
</tr>
<tr>
<td>( v_r^{\text{obs}} )</td>
<td>Observed ( v_r ) by radar</td>
</tr>
<tr>
<td>( v_r^{\text{ref}} )</td>
<td>Reference ( v_r ) produced by the AR vortex analysis in section 3a</td>
</tr>
<tr>
<td>( v_r^{\text{true}} )</td>
<td>True ( v_r )</td>
</tr>
<tr>
<td>( z )</td>
<td>Height above radar</td>
</tr>
<tr>
<td>( z_0 )</td>
<td>Lowest vertical level (at ( z = 250 \text{ m} )) in dealiasing</td>
</tr>
<tr>
<td>( z_n )</td>
<td>Height of ( n )th vertical level</td>
</tr>
<tr>
<td>( \Delta z )</td>
<td>Vertical interval (=25 m)</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Azimuthal angle of measurement point viewed from vortex center</td>
</tr>
<tr>
<td>( \phi )</td>
<td>Azimuthal angle of measurement point viewed from radar</td>
</tr>
<tr>
<td>( \phi_{+} )</td>
<td>Zero-( v_r ) point with ( \hat{a}_d v_r &gt; 0 ) on a range circle</td>
</tr>
<tr>
<td>( \phi_{-} )</td>
<td>Zero-( v_r ) point with ( \hat{a}_d v_r &lt; 0 ) on a range circle</td>
</tr>
<tr>
<td>( \phi_c )</td>
<td>Azimuthal angle of vortex center viewed from radar</td>
</tr>
<tr>
<td>( \phi_{\text{op}} )</td>
<td>Preestimated ( \phi_c ) from operationally issued hurricane location information</td>
</tr>
<tr>
<td>( \phi_{\text{obs}} )</td>
<td>Observation point where (</td>
</tr>
<tr>
<td>( \phi_{\text{obs}}^{\text{est}} )</td>
<td>( \phi_{\text{obs}} ) point (if exists) in the immediate vicinity of the true ( \phi_{+} ) on a range circle</td>
</tr>
<tr>
<td>( \phi_{+}(z_R) )</td>
<td>( \phi_{+} ) constructed as single discrete function of ( z_n ) from ( \phi_{+}(z_n, \theta_k) ) (see Fig. 2b)</td>
</tr>
<tr>
<td>( \phi_{+}'(z) )</td>
<td>Continuous function of ( z ) estimated for ( \phi_{+} ) from ( \phi_{+}(z_n) ) (see Fig. 2b)</td>
</tr>
<tr>
<td>( \phi_{+}'(z_n, \theta_k) )</td>
<td>Estimated ( \phi_{+} ) at ( n )th vertical level on ( k )th tilt (see Fig. 2a)</td>
</tr>
<tr>
<td>( \phi_{+}'(z, \theta_k) )</td>
<td>Continuous function of ( z ) estimated for ( \phi_{+} ) from ( \phi_{+}'(z_n, \theta_k) ) on ( k )th tilt (see Fig. 2c)</td>
</tr>
<tr>
<td>( \Delta \phi )</td>
<td>Maximum half-window width (=15°) for searching ( \phi_{+} ) around each ( \phi_{0}^{\text{obs}} )</td>
</tr>
<tr>
<td>( \Delta \phi_L )</td>
<td>Left half-window width for searching ( \phi_{+} ) around each ( \phi_{0}^{\text{obs}} )</td>
</tr>
<tr>
<td>( \Delta \phi_R )</td>
<td>Right half-window width for searching ( \phi_{+} ) around each ( \phi_{0}^{\text{obs}} )</td>
</tr>
<tr>
<td>( \Delta \phi_j )</td>
<td>Half-window width for filtering data points in ( j )th layer (see Figs. 2a and 2b)</td>
</tr>
</tbody>
</table>
\[ \theta' \] Slope angle of radar beam relative to Earth's surface beneath measurement point

\[ \theta \] Elevation angle of radar beam at the radar site

\[ \theta_k \] \( \theta \) for \( k \)th tilt

\[ \theta_K \] \( \theta \) for the highest tilt (with \( k = K \))

\[ \delta_{g\nu_r} \] Azimuthal derivative of \( \nu_r \)

REFERENCES


