ON THE THEORY OF CYCLONES*

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ABSTRACT

The "tendency equation" is applied to analyze the pressure changes produced in wave-shaped westerly flow. With sufficiently strong westerlies horizontal divergence and convergence occur in such distribution as to cause eastward displacement of troughs and crests. When the westerlies drop under a "critical speed" the distribution of divergence and convergence is reversed.

In the normal case of supercritical speed and increasing west wind with height, incipient waves will develop thermal asymmetry and will intensify. The same kind of irreversible growth of incipient waves would occur in easterlies which increase with height. Therefore, the temperate westerlies in all seasons, and the subtropical easterlies in summer and fall, are the regions of dynamic instability, where, respectively, the extratropical and the tropical cyclones are generated.

The tendency equation applied to closed circulations shows such distribution of pressure rise and fall as to make the patterns drift westwards unless they are strongly "eccentric." The main reason for the usual eastward drift of closed isobar patterns in the temperate latitudes lies, however, in the fact that the superimposed wave pattern in the upper layers produces overcompensating accumulation of air where the low levels show depletion, and depletion of air where the low levels show accumulation. The behavior of the composite low and high-level depression is discussed from this viewpoint.

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1. Historical. The starting point in any cyclone theory must be to devise the mechanism of cyclogenesis. In the polar-front theory that mechanism is visualized as the development of an unstable wave on a pre-existing, or recently generated front. This led V. Bjerknes to formulate the following program for a mathematical search for the "cyclone wave": Show that among the waves which can appear on the polar-front surface there are also unstable ones which grow spontaneously and which are bound to represent the essential mechanism of the nascent cyclone.

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The theoretical model consisted of two fluid layers both moving zonally on the rotating earth but with different zonal velocity, separated by a sloping surface of discontinuity. In 1928 H. Solberg (9) succeeded in finding *all waves* which are possible in such a system, and among these waves also the cyclone wave was found. The integrations led to very complicated mathematical computations, and many of the formulas were difficult to discuss, but the main results can be summarized in a diagram, as shown in Figure 1 (first devised in that form by C. L. Godske in 1936 (3)).

The diagram purports to show that three separate

FIG. 1. Godske's diagram of stability and instability of atmospheric waves.

dynamical factors are of importance for the generation of the waves:

1°. *Gravity* comes into play whenever the motion has vertical components. Since in the initial system light air overlies heavier air, the effect of gravity permits only stable waves to form.

2°. The *shear* in the motion at the surface of separation is always a dynamical factor of instability. This effect is illustrated in small scale by the transition from laminar (stable) to turbulent (unstable) flow, which occurs when the shear passes a certain critical value. The shear of large-scale atmospheric currents is quite analogous and brings about the formation of unstable waves.

3°. The *rotation* of the earth is a stabilizing factor. That effect is a consequence of the conservation of angular momentum, and is thus of the same nature as the stability of a gyroscope.

The joint effect of these three factors is shown in the diagram, which gives the stability (as indicated by the square of the frequency, negative square of frequency representing unstable waves) as a function of length of waves in the sloping surface of discontinuity.

Short waves are practically uninfluenced by the rotation of the earth, and gravity is the only stabilizing factor. For the very shortest waves the instability due to the shear in the frontal surface overcompensates the gravitational stability, and unstable "billow-cloud waves" result. When a certain critical wave length is exceeded, the waves become stable.

In long waves (of thousands of kilometers wave length) the vertical wave component is very small compared with the horizontal component, and the stabilizing effect of gravity is therefore insignificant. The only stabilizing influence in this range is that due to the earth's rotation, and the waves are therefore called *inertial waves*. If the waves are very long the inertial stability will dominate, but for decreasing wave length the influence from the shear will increase and finally dominate, giving unstable inertial waves in this band. Decreasing the wave length still further finally brings the gravitational influence into play and the waves are again stable.

There are thus two types of unstable waves in the atmosphere: (1) the short gravitational waves, discovered theoretically by Helmholtz, and represented in the atmosphere by billow clouds; (2) the long, unstable inertial waves, discovered by Solberg, and identified in nature as the cyclone waves.

Solberg's theory can only be considered as the first step in the theory of cyclone formation. It has shown the *theoretical necessity* of a long, unstable inertial wave in a frontal surface. It remains to discover the dynamical structure, and in particular the kinematical properties of these waves. In this respect the theory

has been somewhat unsuccessful because of the complexity of the mathematical results.

In 1937 J. Bjerknes (1) started another approach by asking for the explanation of the pressure changes occurring during the development of the cyclone. This led to the theoretical study of the wave motion in the continuous air masses encountered above the surface of discontinuity represented by the polar-front surface. These internal waves (known since the beginning of synoptic aerology) are closely connected with the boundary waves at the polar-front surface; the two develop together and move along together, the upper waves lagging a little behind in phase.

The upper waves have lately been the subject of several theoretical treatments the most important of which, by Rossby and by Haurwitz, are referred to in the sequel.

2. Plan of present paper.* The guiding idea in the present paper is to try to find the mechanism which causes the internal waves in the continuous medium to intensify together with the intensification of the polar-front wave. As shown by Solberg and Godske the initial impulse for wave formation is always available due to the shear at the polar-front surface. This paper purports to show how the baroclinic air mass above the polar-front surface reacts to such a wave impulse. The development of that idea is to be found in Sections 13-15.

Before arriving that far, the tools of analysis must be presented and some preliminary studies of simplified (barotropic) atmospheric models must be carried out.

3. The tendency equation. The new tool of analysis introduced in 1937 was the so-called tendency equation. Its derivation follows.

The hydrostatic equation is assumed to be valid in all aerological computations leading to the construction of the upper-level pressure maps. For their theoretical interpretation we can therefore safely take our start from the hydrostatic equation

$$-\delta p = \rho \delta \phi,$$

where ϕ is the geopotential. The pressure at any level is then obtained by integration of this equation from the level ϕ to the upper limit of the atmosphere, $\phi = \infty$, where the pressure is zero, thus

$$p = \int_{\phi}^{\infty} \rho \delta \phi. \quad (3.1)$$

The pressure is here represented as the weight of the vertical column of air of unit cross section extending from the level ϕ to the top of the atmosphere.

* A considerable portion of this paper, including many of the diagrams, has been reproduced in Chapter 10 of the book: Holmboe, Forsythe, Gustin, *Dynamic Meteorology*, Wiley and Sons, New York, 1945.

The local rate of change of the pressure at the level ϕ is evidently given by the change in weight of the vertical column. It is obtained by partial time differentiation of (3.1), thus

$$\left(\frac{\partial p}{\partial t}\right)_\phi = \int_\phi^\infty \frac{\partial \rho}{\partial t} \delta\phi. \quad (3.2)$$

The local rate of decrease of density is, according to the equation of continuity, equivalent to the mass divergence:

$$-\frac{\partial \rho}{\partial t} = \text{div}(\rho v) = \text{div}_H(\rho v) + \frac{\partial}{\partial z}(\rho v_z). \quad (3.3)$$

The last expression on the right gives the mass divergence as the sum of horizontal and vertical mass divergence, which substituted in equation (3.2) gives

$$\left(\frac{\partial p}{\partial t}\right)_\phi = - \int_\phi^\infty \text{div}_H(\rho v) \delta\phi + (g\rho v_z)_\phi. \quad (3.4)$$

The local pressure change at the level ϕ is here represented explicitly as the change in the weight of the vertical air column. This change is caused in part by horizontal divergence of mass above the level ϕ , and in part by the vertical transport of mass through the base of the column.

Equation (3.4), known as the *tendency equation*, was derived in 1904 by Margules (6), but most of its practical applications date from 1937 or later. The first term on the right will be referred to as the *divergence term*. (We shall in the following refer to the mass divergence, $\text{div}(\rho v)$, simply as the divergence. No confusion will arise from this terminology because the velocity divergence, $\text{div } v$, will not be discussed in this paper.) In a qualitative sense we may visualize positive horizontal divergence as a horizontal spreading of air, and negative divergence (i.e., convergence) as a horizontal crowding of air. This makes the physical interpretation of the divergence integral in the tendency equation quite clear: Horizontal convergence increases the mass of air within the vertical column and shows up as a pressure rise at the base of the column. Horizontal divergence decreases the mass of air present within the column, and makes the pressure fall at the base of the column.

Equally obvious is the meaning of the second term on the right in the tendency equation, which will be called the *vertical motion term*. An influx of air from below into the column increases the weight of air inside and thereby also the pressure at its base. Correspondingly an outflow of air downwards through the base of the column represents a loss of weight of the column and a decrease of pressure at the base.

On a level part of the surface of the earth the vertical motion is zero and the tendency equation re-

duces to

$$\left(\frac{\partial p}{\partial t}\right)_0 = - \int_0^\infty \text{div}_H(\rho v) \delta\phi. \quad (3.5)$$

All large-scale pressure changes observed on the surface map (except special mountain effects) should be explicable in terms of this equation. The problem of finding the distribution of horizontal divergence will be discussed at some length in this paper.

4. Relation between the horizontal divergence and the field of pressure. The equation of motion implicitly contains the relation between the field of motion and the pressure field. By making suitable assumptions it is possible to estimate the distribution of the horizontal divergence directly from the horizontal pressure field.

In the layer of frictional influence near the surface of the earth the winds have a systematic component across the isobars from high to low pressure. Every "low" is then a region of horizontal mass convergence and every "high" a region of horizontal mass divergence within the layer of friction. However, the layer of friction represents only about 1/20 of the weight of the atmosphere, and any horizontal divergence in this layer may easily be over-compensated by horizontal divergence in the remaining 19/20 of the atmosphere.

In the free atmosphere, above the layer of frictional influence, the wind is in the first approximation geostrophic, blowing parallel to the isobars of the pressure map. Furthermore, at any given latitude the geostrophic wind is inversely proportional to the distance between the isobars. We can, therefore, think of the isobars of a constant-level pressure map as representing the horizontal motion of the air in the same level. The air flows along "isobaric channels" covering the strips between successive isobars. Where the channel is wide, the air flows slowly, whereas in the narrow parts of the channel the air flows rapidly. The product of the density ρ , the wind speed v , the channel width δn , and the constant channel depth δz defines what may be called the *transport capacity* δF of the isobaric channel:

$$\delta F = \rho v \delta n \delta z. \quad (4.1)$$

The transport capacity is practically constant for a reasonably straight channel which runs about west-east along a circle of latitude.

If the transport capacity of isobaric channels were constant all over the map, there would be no convergence or divergence to produce pressure changes. Every isobaric channel would be like a well regulated river with constant transport all along, so that no local accumulations or depletions would occur. However, this is fulfilled only approximately in the atmosphere, because the geostrophic wind is only a first approximation to reality. Actually the isobaric air

channels change their transport capacity somewhat from point to point. The places of minimum transport capacity will then be "bottlenecks" in the flow of air. The air approaching a bottleneck will crowd horizontally and cause a *longitudinal* mass convergence in the current, whereas beyond the bottleneck there will be longitudinal mass divergence. To complete the list of shortcomings of the geostrophic wind, the air does not follow strictly along the isobaric channels but overflows slightly from one to the other. It overflows toward lower pressure when the air speeds up, and toward higher pressure when the air slows down. This also may cause horizontal mass divergence, which will be referred to as *transversal* mass divergence. For brevity, we shall hereafter write 'divergence' instead of 'mass divergence.'

In the following the longitudinal and transversal divergence will be evaluated quantitatively, or estimated qualitatively, for selected simple patterns of flow under the two general headings: (i) *wave-shaped flow patterns*, and (ii) *closed flow patterns*. These flow patterns, of course, correspond to wave-shaped isobar patterns and closed isobar patterns, respectively. The former are mainly observed in the upper levels; the latter, in the lower levels of the traveling cyclones. The results from the study of these fundamental patterns will thus ultimately throw some light on the mechanism of the composite cyclonic disturbances.

FIG. 2. Wave-shaped isobar pattern.

5. Longitudinal divergence in wave-shaped isobar patterns. The type of wave-shaped isobar pattern to be studied is shown schematically in Figure 2. The amplitude of the wave disturbance has a maximum in the middle of the pattern and tapers off to the north and to the south of this latitude. This resembles the usual pressure distribution on an upper-level map above a moving cyclone in the latitudes of the westerlies. In order to facilitate the discussion it has been assumed that the isobar pattern is *symmetric with respect to the trough lines and the wedge lines*, and that these lines are straight lines with north-south orientation. It is further assumed that the shape of the isobar pattern remains unchanged.

It follows from the symmetry of the pattern with respect to the trough lines and the wedge lines that these lines are the places of maximum or minimum transport capacity of each isobaric channel. To estimate the distribution of longitudinal divergence, it is thus sufficient to evaluate the transport capacity of the isobaric channels at their southern bends where they intersect the trough lines, and at their northern bends where they intersect the wedge lines. From the place of minimum to the place of maximum transport capacity (along the direction of the current) there will be longitudinal divergence, and from the place of maximum to the place of minimum transport capacity there will be longitudinal convergence.

The trough lines and the wedge lines are not only the places of maximum or minimum transport capacity of the isobaric channels but are also the places of maximum or minimum speed. At such points the flow is directed along the isobars and satisfies strictly the gradient wind equation:

$$Kv^2 + 2\Omega v \sin \varphi = -\frac{1}{\rho} \frac{\delta p}{\delta n}. \quad (5.1)$$

Here δn is the width of the isobaric channel bordered by the isobars with the pressures p and $p + \delta p$. K is the horizontal curvature of the air trajectory.* Multiplying equation (5.1) by $\rho \delta n \delta z$, and introducing the notation δF for the transport capacity from (4.1), we have

$$\delta F(Kv + 2\Omega \sin \varphi) = -\delta p \delta z = \text{const.} \quad (5.2)$$

* The geometric significance of the horizontal curvature is shown schematically in Figure 3. The horizontal levels of the earth may with sufficient accuracy be considered as spherical. Any horizontal curve is therefore a spherical curve. The osculating plane of the spherical curve at the point P intersects the sphere in a circle PQ , which is the circle of curvature at P . The radius

FIG. 3. Definition of horizontal curvature.

of space curvature at P is the radius, r , of this circle, and the space curvature at P is by definition $1/r$. When the spherical curve is projected orthogonally into the horizontal plane HH tangential to the earth at P , the curvature of this projected curve at P is the horizontal curvature of the spherical curve at P . The radius of horizontal curvature has the length $R = a \tan \sigma$, where a is the radius of the earth, and σ is the angle subtended by r or R at the center of the earth. Consequently, the latitude circle in latitude φ has the constant radius of horizontal curvature $a \cot \varphi$, while for a great circle the radius of horizontal curvature is everywhere infinite.

The channel depth δz , and the pressure difference across the channel δp , are both constant all along the isobaric channel. The transport capacity at the places with gradient flow is accordingly proportional to the reciprocal of the quantity $Kv + 2\Omega \sin \varphi$. The distribution of longitudinal divergence is therefore obtained by comparison of the value of this quantity at the places where the isobaric channel intersects the trough line and the wedge line.

FIG. 4. Isobaric channel from wave pattern in Figure 2.

Figure 4 shows an arbitrary isobaric channel in the pattern of Figure 2 bordered by the isobars with the pressures p and $p + \delta p$ ($\delta p < 0$). The channel has its cyclonic bend in the latitude φ where it intersects the trough line, and its anticyclonic bend in the latitude φ' where it intersects the wedge line. Obviously, the relation $\varphi' > \varphi$ holds in all cases. In the following all the quantities at the cyclonic bend will be denoted by unprimed symbols, and the corresponding quantities at the anticyclonic bend by primed symbols.

The transport capacities $\delta F'$ and δF at the two bends must satisfy one of the following three conditions.

$$\delta F' \cong \delta F. \quad (5.3)$$

The upper condition states that the bottleneck in the current is at the cyclonic bend of the isobaric channel. West of this bend, air accumulates and causes longitudinal convergence; east of the bend there will be longitudinal divergence. The middle condition states that the transport capacities are the same at the two bends, so that the longitudinal divergence is zero. The lower condition states that the bottleneck in the current is at the anticyclonic bend, so the flow has longitudinal convergence to the west of this bend and longitudinal divergence to the east of it.

According to equation (5.2) the conditions (5.3) can be written

$$Kv + 2\Omega \sin \varphi \cong K'v' + 2\Omega \sin \varphi', \quad (5.4)$$

or after rearrangement of the terms

$$\begin{aligned} Kv - K'v' &\cong 2\Omega(\sin \varphi' - \sin \varphi) = \\ &= 4\Omega \cos \frac{\varphi' + \varphi}{2} \sin \frac{\varphi' - \varphi}{2}. \end{aligned} \quad (5.5)$$

The angle $(\varphi' + \varphi)/2$ is the *central latitude* $\bar{\varphi}$ around which the isobaric channel winds. The angle $(\varphi' - \varphi)/2$

is half of the difference in latitude between the northern and southern bends of the channel. This angle will be called the *angular amplitude* of the isobar and denoted by σ_p . Thus

$$\bar{\varphi} = \frac{\varphi' + \varphi}{2}, \quad \sigma_p = \frac{\varphi' - \varphi}{2}. \quad (5.6)$$

With these notations introduced, (5.5) takes the form

$$Kv - K'v' \cong 4\Omega \cos \bar{\varphi} \sin \sigma_p. \quad (5.7)$$

This formula contains the horizontal curvatures of the air trajectories (paths) at the southern and the northern bends of the isobaric channel.

The relation between the curvature of the path and the curvature of the streamline at an arbitrary point P is obtained as follows: Let ψ denote the angle, measured counterclockwise from east to the direction of the wind. At a given time ψ is a function $\psi = \psi(s)$ of the arc length s along the streamline through P . If the flow pattern is steady, this function is the same at all times. If the flow pattern is changing, the wind direction at every point is also a function of time. The wind direction along the streamline through P is then

$$\psi = \psi(s, t).$$

Consider now the particle initially at P . During the time element dt this particle moves the infinitesimal distance ds along its path which is tangential to the streamline. The corresponding change of ψ on the particle is then

$$d\psi = \frac{\partial \psi}{\partial s} ds + \frac{\partial \psi}{\partial t} dt. \quad (5.8)$$

The curvature of the path is defined as the angular turn of the wind per unit arc length along the path:

$$K = \frac{d\psi}{ds}.$$

We shall denote the curvature of the streamline by K_s . This curvature is defined as the angular turn of the wind per unit arc length along the streamline at a fixed time

$$K_s = \frac{\partial \psi}{\partial s}.$$

The speed of the particle is $v = ds/dt$.

When (5.8) is divided by ds and the above notations are introduced we find the following relation, first derived by Blaton (2):

$$K = K_s + \frac{1}{v} \frac{\partial \psi}{\partial t}. \quad (5.9)$$

The expression $\partial \psi / \partial t$ is the *local rate of turning* of the

wind at the point P . The local rate of turning is determined by the wind recordings at a fixed station, or approximately by the wind change at the station between consecutive maps.

Equation (5.9) gives the path curvature in terms of the streamline curvature and the local rate of turning of the wind. The two curvatures at a given point are equal when the wind direction at that point does not change. In particular, if the wind direction is independent of time at every point in the field, that is, if the flow is steady, then the path and streamline curvatures are everywhere equal.

In a nonsteady field the local turning of the wind is caused in part by the movement of the flow pattern, and in part by the change in shape of the pattern. In a moving but unchanging wave-shaped pattern of the type discussed above (Fig. 2) the local turning of the wind at the wedge line (or the trough line) is obtained as follows: Let SS (Fig. 5) represent a streamline in

FIG. 5. Determination of the local rate of turning of the wind on the wedge line.

the neighborhood of the wedge line at the time t . During the time element dt the flow pattern moves the distance ds to the position indicated by the streamline $S'S'$. The local turning of the wind $d\psi$ at the station P' during the time element dt is numerically the same as the angular turn of the wind through the arc length ds along the streamline SS from P to P' . Thus

$$\frac{\partial \psi}{\partial t} dt = - \frac{\partial \psi}{\partial s} ds.$$

The speed of propagation of the flow pattern is $c = ds/dt$. The local rate of turning of the wind at the wedge line is accordingly

$$\frac{\partial \psi}{\partial t} = - cK_s.$$

When this expression, which also holds at the trough line, is substituted in the general formula (5.9) we have at any point on the trough line and on the wedge line, respectively,

$$\begin{aligned} Kv &= K_s(v - c) \\ K'v' &= K'_s(v' - c). \end{aligned} \quad (5.10)$$

Let $\bar{v}$ be the mean zonal wind, $(v' + v)/2$, in the isobaric channel (Fig. 4), and let Δv be the half difference $(v' - v)/2$ of the speeds at the northern and

southern bends. Thus

$$\bar{v} = \frac{v' + v}{2}, \quad \Delta v = \frac{v' - v}{2}. \quad (5.11)$$

Introducing these notations in (5.10), we find

$$\begin{aligned} Kv &= K_s(\bar{v} - c) - K_s\Delta v \\ K'v' &= K'_s(\bar{v} - c) + K'_s\Delta v. \end{aligned}$$

The difference between these expressions is

$$Kv - K'v' = (K_s - K'_s)(\bar{v} - c) - (K_s + K'_s)\Delta v.$$

Substituting this in (5.7) and solving for $\bar{v} - c$, we have finally

$$\bar{v} - c \cong \frac{4\Omega \cos \bar{\varphi} \sin \sigma_p}{K_s - K'_s} + \frac{K_s + K'_s}{K_s - K'_s} \Delta v. \quad (5.12)$$

The expression on the right has the dimensions of a velocity. It is completely determined by the location and geometry of the pattern, and by the difference of wind speeds at the northern and southern bends of the isobar. We shall in the following refer to this quantity as the *critical speed* and denote it by v_c , thus

$$v_c = \frac{4\Omega \cos \bar{\varphi} \sin \sigma_p}{K_s - K'_s} + \frac{K_s + K'_s}{K_s - K'_s} \Delta v. \quad (5.13)$$

With this notation introduced the conditions (5.12) take the form

$$\bar{v} - c \cong v_c. \quad (5.14)$$

$\bar{v} - c$ is the mean value of the zonal speed relative to the moving wave pattern, and will be called more briefly the *relative zonal wind*.

The three conditions (5.14) are equivalent to the three conditions (5.3) taken in the same order: Thus, if the relative zonal wind is *supercritical*, that is, greater than the critical speed, the bottleneck in the current is at the cyclonic bend of the isobaric channel, so the flow has longitudinal convergence to the west of the trough and longitudinal divergence to the east of the trough. If the relative zonal wind is *critical*, that is, equal to the critical speed, the transport capacities are the same at the two bends, so the longitudinal divergence is zero. If the relative zonal wind is *subcritical*, that is, less than the critical speed, the bottleneck in the current is at the anticyclonic bend of the isobaric channel, so the flow has longitudinal convergence to the east of the trough and longitudinal divergence to the west of the trough.

The expression (5.13) for the critical speed becomes much simpler when the curvature is numerically the same at the cyclonic and the anticyclonic bends, that is, when $K_s = -K'_s$. The second term on the right side of equation (5.13) is then equal to zero, and the

critical speed is

$$v_c = \frac{2\Omega \cos \bar{\varphi} \sin \sigma_p}{K_s}. \quad (5.15)$$

So far no analytical expression has been specified for the streamlines or the isobars. The only restriction in the choice of isobaric pattern prescribes that the isobars should be symmetrical with respect to the north-south trough lines and wedge lines. The critical speed may be determined directly on the weather map from (5.13). If the curvatures at the two bends are numerically equal, we may use formula (5.15). This formula may be simplified still further when the streamlines are assumed to be sine curves.

6. Critical speed in sinusoidal waves. In order to treat the streamlines as simple sine curves it is necessary to consider the surface of the earth as flat and the latitude circles as straight parallel lines in the region between the two bends. A sinusoidal streamline can then be drawn winding between the latitudes φ and φ' as shown in Figure 6. In a standard system of

FIG. 6. Sinusoidal streamline.

coordinates with the origin at the point where the streamline intersects the central latitude at the time $t = 0$, this streamline has the equation

$$y = A_s \sin k(x - ct). \quad (6.1)$$

Here A_s is the amplitude and c is the speed of the wave. If L_s is the wave length, $k = 2\pi/L_s$ is called the *wave number* and represents the number of waves in the linear interval of 2π length units.

If the latitudinal amplitude of the streamline is denoted σ_s , the linear amplitude A_s on the horizontal plane may be defined by orthogonal projection (see footnote p. 4) as

$$A_s = a \sin \sigma_s. \quad (6.2)$$

The linear wave length L_s can be expressed by the angular wave length of longitude λ_s as follows:

$$L_s = \lambda_s a \cos \bar{\varphi}.$$

The angular wave length defines an *angular wave number* $n = 2\pi/\lambda_s$, which gives the number of waves along the total circumference of the latitude circle. The rela-

tion between the linear wave number k and the angular wave number n is

$$k = \frac{2\pi}{L_s} = \frac{2\pi}{\lambda_s a \cos \bar{\varphi}} = \frac{n}{a \cos \bar{\varphi}}. \quad (6.3)$$

The curvature of the streamline is by definition the angular turn of its tangent per unit arc length along the streamline; thus

$$K_s = \frac{\partial \psi}{\partial s}. \quad (6.4)$$

If ψ is the angle between the tangent and the x -axis, then the slope of the streamline (6.1) is

$$\tan \psi = \frac{\partial y}{\partial x}.$$

Differentiating this formula with respect to the arc length s , we have

$$\frac{1}{\cos^2 \psi} \frac{\partial \psi}{\partial s} = \frac{\partial}{\partial s} \left(\frac{\partial y}{\partial x} \right). \quad (6.5)$$

At the southern and northern bends, where the streamline is parallel to the x -axis, $\cos^2 \psi = 1$ and $\delta s = \delta x$. Therefore from (6.4) and (6.5) we have at such points

$$K_s = \frac{\partial^2 y}{\partial x^2}.$$

When (6.1) is differentiated twice with respect to x , we find for the curvature of a sinusoidal streamline at the two bends

$$K_s = -k^2 A_s \sin k(x - ct) = \pm k^2 A_s. \quad (6.6)$$

In the last expression on the right the positive sign should be used at the southern bend and the negative sign at the northern bend.

By substituting in (6.6) the values for A_s from (6.2) and for k from (6.3), the curvature at the southern bend becomes

$$K_s = \frac{n^2 \sin \sigma_s}{a \cos^2 \bar{\varphi}}.$$

When this expression for K_s is introduced in (5.15) we find for the critical speed in a sinusoidal flow pattern the expression

$$v_c = \frac{2\Omega a \cos^3 \bar{\varphi} \sin \sigma_p}{n^2 \sin \sigma_s}. \quad (6.7)$$

It can be shown (see Holmboe, Forsythe, Gustin (5), Section 12.06) that the streamline amplitude and the isobar amplitude have the same value. The amplitude factor in (6.7) is therefore equal to unity, and we have

$$v_c = \frac{2\Omega a \cos^3 \bar{\varphi}}{n^2}. \quad (6.8)$$

This formula has been derived by Rossby (7) from the principle of conservation of vorticity.

The critical speed in a sinusoidal flow pattern is thus a function of the latitude and the angular wave number. The critical speed is large for long waves in low latitudes, and small for short waves in high latitudes. Some values of the critical speed, as computed from (6.8) are given in Table 1.

TABLE 1.

$$v_c = 2\Omega a \cos^3 \bar{\varphi} / n^2 (\text{ms}^{-1})$$

$\bar{\varphi}$	n				
	2	3	6	10	20
70°	9.3	4.1	1.0	0.4	0.1
60°	29.0	12.9	3.2	1.2	0.3
50°	61.6	27.4	6.8	2.5	0.6
40°	104.3	46.3	11.6	4.2	1.0
30°	150.7	67.0	16.8	6.0	1.5
0°	232.0	103.1	25.8	9.3	2.3

For long waves it seems unwarranted to consider the earth as flat and to neglect the curvature of the latitude circles. No detailed discussion of this question will be given here. It may suffice to mention that waves spanning over one-half to one-third of the circumference of the earth must be treated by spherical methods. Practical synoptic cases are more likely to deal with wave numbers around $n = 6$ or more (wave lengths 60° of longitude or less), and in such cases the difference between the flat and the spherical treatment seems to be insignificant.

7. Transversal divergence in wave-shaped isobar patterns. We now turn our attention to that part of the horizontal divergence which is caused by the overflow of mass from one isobaric channel to another, which will be referred to as transversal divergence. Figure 7 shows the same idealized pressure pattern which was discussed in Section 5. A qualitative estimate of the transversal divergence in this pressure pattern is obtained by considering the inflow or outflow of mass

FIG. 7. Transversal divergence shown by streamlines (broken lines) intersecting the northern and southern straight isobar of the wave pattern.

across the northern straight isobar $p = p_0 - 4$, and across the southern straight isobar $p = p_0$.

Suppose for a moment that the pressure pattern in Figure 7 is stationary. Particles at the northern edge of the pattern will then, during their eastward motion, have their maximum speed while passing the longitudes of the wedges and minimum speed while passing the longitudes of the troughs. In order to change speed in that rhythm the particle must have a component toward high pressure while it retards, i.e., while it moves from the wedge to the trough, and must have a component toward low pressure while it accelerates, i.e., during the movement from trough to wedge. The flow at the northern edge of the pattern must therefore be as qualitatively indicated by the streamline (dotted line), in the diagram. The analogous analysis along the southern straight isobar leads to a streamline with opposite phase. There is thus outflow from the region east of the trough, both across the northern and the southern edge of the isobar pattern, and therefore transversal divergence occurs in this region. Similarly, there is transversal convergence in the region west of the trough.

If the pressure pattern is moving, the same rule holds, provided that the pattern does not move eastward faster than the mean speed of the air, i.e., provided that the relative zonal wind is from the west. If the pressure pattern moves eastward with the mean zonal speed, the relative zonal wind is zero, and the transversal divergence of the flow pattern is zero. If the pressure pattern moves eastward faster than the air, the relative zonal wind is from the east; the particles at the northern edge slow down east of the trough (while being overtaken by the trough), and they speed up to the west of the trough (while being overtaken by the wedge). The streamlines at the northern and at the southern edges will have the opposite phase from those shown in Figure 7, and consequently give transversal convergence into the region east of the trough and transversal divergence from the region west of it. These rules may be summarized as follows: *The transversal divergence and the relative zonal wind have the same sign in the region to the east of the trough and have opposite signs in the region to the west of the trough.*

The above rule gives the sign of the transversal divergence. The mean divergence over a given area is obtained by dividing the total outflow from that area by the area. The areas of outflow and inflow in the wave pattern of Figure 7 are proportional to the meridional distance between the northern and southern straight isobars. The transversal divergence is therefore proportional to the reciprocal of the width of the pattern, and is zero for a pattern of infinite lateral extent. All the isobars then have the same shape, and

the longitudinal divergence is the total horizontal divergence.

8. Total horizontal divergence associated with wave-shaped isobar patterns. The total horizontal divergence is the sum of the longitudinal and the transversal divergence. The following three cases occur:

(i) $\bar{v} - c > v_c$: The relative zonal wind is from the west and is greater than the critical speed; the longitudinal and transversal divergence are both positive to the east of the trough and are both negative to the west of the trough.

(ii) $\bar{v} - c < 0$: The relative zonal wind is from the east; the longitudinal and transversal divergence are both negative to the east of the trough and are both positive to the west of it.

In both cases (i) and (ii) the two parts of the horizontal divergence have the same sign at any point of the flow pattern.

(iii) $v_c > \bar{v} - c > 0$: The relative zonal wind is from the west and is smaller than the critical speed; the longitudinal divergence is negative to the east of the trough and positive to the west of the trough; the transversal divergence is positive to the east of the trough and negative to the west of the trough. The two parts of the horizontal divergence then counteract each other throughout the field. For a certain value of the relative zonal wind, less than the critical speed but greater than zero, the longitudinal divergence is balanced by the transversal divergence and, the total horizontal divergence is zero.

Rossby (7) and Haurwitz (4) have derived the value for the relative zonal wind for the case of zero horizontal divergence from the principle of the conservation of absolute vorticity. The vorticity theorem for nondiverging flow is

$$\frac{d}{dt}(\zeta + 2\Omega \sin \varphi) = 0, \quad (8.1)$$

where ζ is the vertical component of the vorticity relative to the earth, and $2\Omega \sin \varphi$ is the vertical component of the vorticity due to the rotation of the earth. The theorem (8.1) holds at any level of nondivergence if the small effect due to the solenoids is neglected. The time derivative in (8.1) is the differentiation following the moving particle. When this derivative is separated into local and advective derivatives, we have in the case of horizontal motion

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}.$$

Since $2\Omega \sin \varphi$ depends only on latitude, we have

$$\frac{d}{dt}(2\Omega \sin \varphi) = v_y \frac{\partial}{\partial y}(2\Omega \sin \varphi) = v_y \frac{2\Omega \cos \varphi}{a}.$$

Equation (8.1) can therefore be written

$$\frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} + v_y \frac{2\Omega \cos \varphi}{a} = 0. \quad (8.2)$$

Haurwitz (4) investigated a model on a flat earth similar to Figure 7, having the velocity field

$$\begin{aligned} v_x &= \bar{v} \left[1 + A \frac{2\pi}{D} \sin \frac{2\pi}{D} y \sin \frac{2\pi}{L} (x - ct) \right], \\ v_y &= \bar{v} A \frac{2\pi}{L} \cos \frac{2\pi}{D} y \cos \frac{2\pi}{L} (x - ct), \end{aligned} \quad (8.3)$$

which corresponds to the streamlines

$$y - y_0 = A \cos \frac{2\pi}{D} y \sin \frac{2\pi}{L} (x - ct). \quad (8.4)$$

D represents twice the meridional width of a cell, i.e., twice the meridional distance between two consecutive straight streamlines. It is readily seen that the velocity field (8.3) is always nondiverging, $\text{div}_H \mathbf{v} = 0$. When the relative vorticity $\zeta = (\partial v_y / \partial x) - (\partial v_x / \partial y)$ is evaluated from (8.3) and substituted in (8.2) we find after a simple calculation

$$\frac{4\pi^2}{L^2} (\bar{v} - c - v_c) + \frac{4\pi^2}{D^2} (\bar{v} - c) = 0, \quad (8.5)$$

where $v_c = (2\Omega a \cos^3 \varphi) / n^2$ is the critical speed in sinusoidal waves on a flat earth (6.8). If $\bar{v}$ has the value which satisfies equation (8.5), equations (8.3) represent a moving wave-shaped flow pattern of such a nature that the individual particles retain constant absolute vorticity. Of all the nondivergent flow patterns represented by (8.3) only the special one which also satisfies (8.5) has physical reality. That pattern, at the same time representing nondivergence and conservation of individual absolute vorticity, is found at the level where the west wind relative to the moving wave has the value

$$\bar{v} - c = v_c (1 + L^2/D^2)^{-1} < v_c. \quad (8.6)$$

This result of Haurwitz confirms the qualitative rule derived above that in a wave-shaped flow pattern where longitudinal and transversal divergence cancel, the relative zonal wind must be subcritical.

For a pattern of infinite lateral extent D is infinite, and so from (8.6) the relative zonal wind is critical. That is the condition for zero longitudinal divergence, so the transversal divergence is zero in a wave-shaped flow pattern of infinite lateral extent.

The distribution with height of the horizontal divergence in an actual synoptic situation can be estimated with the aid of the above rules for longitudinal and transversal divergence. In general, the strength of the zonal circulation and also the relative zonal circu-

lation increases with height. In the region to the east of every trough there is longitudinal divergence above the level where the relative zonal wind has the critical speed and longitudinal convergence below that level. In the same region there is transversal divergence above the level where the relative zonal wind is zero, and transversal convergence below that level. Between the level of zero longitudinal divergence and the level of zero transversal divergence the two effects will have opposite signs. At some intermediate level the longitudinal and the transversal divergence are equal and opposite, and the total horizontal divergence is zero.

The distribution both of the longitudinal and the transversal divergence depends upon the *relative* zonal wind and hence also upon the speed of the wave. The speed of the wave prescribes the pressure tendency. The pressure tendency is, from the tendency equation (3.4), a direct consequence of the distribution of horizontal divergence. Thus the horizontal divergence, the speed of the wave, and the pressure tendency are interdependent. *The wave will travel with such a speed that the pressure tendencies arising from the displacement of the pressure pattern are in accordance with the field of horizontal divergence.*

The three-dimensional structure of the wave disturbance may be analyzed from this point of view. The conditions are fundamentally different in a barotropic and in a baroclinic current. In order to gain some experience we shall first examine the rather unreal barotropic waves before proceeding to the qualitative analysis of the much more complicated conditions in the real atmospheric waves, which are always baroclinic.

9. Barotropic waves in a westerly current. To simplify matters we shall assume that the wave pattern has infinite lateral extent, so that all isobars have the same shape. The transversal divergence is then zero and the total horizontal divergence is equal to the longitudinal divergence (see Section 7).

In a barotropic current the isothermal surfaces coincide with the isobaric surfaces, so the dynamic thickness of each isobaric layer is constant throughout. The slope of all isobaric surfaces is then the same along any given vertical, and the geostrophic wind has no variation with height. The strength of the zonal circulation and the shape of the streamline pattern is therefore the same at every level.

The pressure tendency at the ground indicates whether the wave moves or not. If we assume that the earth is flat the tendency at the ground is

$$\left(\frac{\partial p}{\partial t}\right)_0 = - \int_0^\infty \text{div}_H(\rho v) \delta\phi. \quad (3.5)$$

The sign of the divergence is determined from the three conditions

$$\bar{v} - c \begin{cases} > \\ = \\ < \end{cases} v_c. \quad (5.14)$$

The mean zonal wind $\bar{v}$ has the same value at all levels in a barotropic current; the speed c of the wave is characteristic of the entire wave and therefore independent of height; the critical speed is determined by the wave length and so is also independent of height. Thus the divergence has the same sign throughout any vertical column. Figure 8 shows the distribution of divergence in the barotropic wave for each of the three conditions (5.14). The upper diagrams show the iso-

FIG. 8. The barotropic wave.

bars at an arbitrary level. The lower diagrams are east-west vertical cross sections showing the divergence in one column to the east of the trough and in one column to the west of the trough.

The diagrams to the right represent a wave for which the relative zonal wind is supercritical,

$$\bar{v} - c > v_c. \quad (9.1)$$

The bottleneck in the current will then be at the cyclonic bend of every isobaric channel. Air is accumulated at all levels to the west of the trough, and is depleted at all levels to the east of the trough. At the ground the pressure is therefore rising to the west of the trough and falling to the east of it. Thus, the wave moves to the east ($c > 0$), and hence from (9.1), $\bar{v} > v_c$.

The center diagrams represent a wave for which the relative zonal wind is critical,

$$\bar{v} - c = v_c. \quad (9.2)$$

The transport capacities are the same at the two bends of each isobaric channel. The entire flow is non-diverging and the pressure tendency at the ground is zero. So the wave is stationary ($c = 0$), and hence from (9.2) $\bar{v} = v_c$.

The diagrams to the left represent a wave for which the relative zonal wind is subcritical,

$$\bar{v} - c < v_c. \quad (9.3)$$

The bottleneck in the current will then be at the anti-cyclonic bend of every isobaric channel. Air is accumulated at all levels to the east of the trough and is depleted at all levels to the west of the trough. At the ground the pressure is therefore rising to the east of the trough and falling to the west of it. So the wave moves to the west ($c < 0$), and hence from (9.3) $\bar{v} < v_c$.

The above results may be summarized as follows:

$$\bar{v} - c \geq v_c, \quad (9.4)$$

$$c \geq 0, \quad (9.5)$$

$$\bar{v} \geq v_c. \quad (9.6)$$

We have shown that the three conditions (9.4) read from top to bottom imply the conditions (9.5) taken in the same order. The conditions (9.4) and (9.5) combined give the conditions (9.6). It is readily seen that any one of (9.4), (9.5), (9.6) implies the other two. Thus from (9.5, 9.6) the *barotropic wave moves eastward when the mean zonal wind is supercritical; it is stationary when the mean zonal wind is critical; it moves westward when the mean zonal wind is subcritical.*

For waves of finite width the transversal divergence is different from zero. The only modification from the above rules is then that the stationary, nondiverging

wave occurs for a smaller value of the zonal wind than the critical speed.

The tendency at upper levels is given by the complete tendency equation

$$\left(\frac{\partial p}{\partial t}\right)_\phi = - \int_\phi^\infty \text{div}_H(\rho v) \delta\phi + (g\rho v_z)_\phi. \quad (3.4)$$

We have shown that the divergence in the barotropic wave has the same sign throughout any given atmospheric column, so the divergence term has the greatest magnitude at the ground and decreases with height. The vertical motion term is zero at the ground and increases in magnitude with height. In the special case of a homogeneous current (ρ independent of p) the pressure tendency would be the same at all levels. Here the decreasing contribution from the divergence term with height is exactly balanced by the increasing contribution from the vertical motion term. In the more general barotropic case where the density decreases with height both the pressure amplitude and the total tendency decrease upward. In this case the contribution from the divergence term must decrease more rapidly with height than the contribution from the vertical motion term increases.

The zero isallobars at all levels coincide with the trough lines and the wedge lines. This means that the amplitude of the pressure wave does not change with time; in other words, the barotropic wave is a *stable* wave, which moves without any changes in its internal structure. This result is in complete accord with classical wave theory. The only destabilizing factor—shear—is absent in the barotropic wave, so the wave must be stable.

The speed of propagation of the barotropic wave can be obtained only by a complete dynamical analysis of the wave. Such analysis gives (see Holmboe, Forsythe, Gustin (5), Section 12.07)

$$c = \xi^2 \phi_h (\bar{v} - c - v_c). \quad (9.7)$$

ϕ_h is the dynamic height of the homogeneous atmosphere and $\xi = n/(\Omega a \sin 2\bar{\varphi})$. This formula confirms the conditions (9.4, 9.5, 9.6) derived from the preceding qualitative discussion. The wave moves toward the east ($c > 0$) when the zonal wind is supercritical, it is stationary ($c = 0$) when the zonal wind is critical, and it moves toward the west ($c < 0$) when the zonal wind is subcritical.

The formula (9.7) shows further that c is proportional to $\bar{v} - c - v_c$. That means that the speed of the wave is proportional to the horizontal divergence in the current, which is just what the tendency equation predicts in a qualitative sense.

Solving for c from (9.7) we have

$$\frac{c}{\bar{v} - v_c} = \frac{\xi^2 \phi_h}{1 + \xi^2 \phi_h}. \quad (9.8)$$

TABLE 2.
 $c/(\bar{v} - v_c)$

$\bar{\phi}$	n				
	2	3	6	10	20
45°	0.59	0.76	0.93	0.97	0.99
50°, 40°	0.60	0.77	0.93	0.97	0.99
60°, 30°	0.66	0.81	0.94	0.98	0.99
70°, 20°	0.78	0.89	0.97	0.99	1.00
80°, 10°	0.93	0.97	0.99	1.00	1.00

Rossby (7) derived essentially the same formula as (9.8). Some values of $c/(\bar{v} - v_c)$ in (9.8) are given in Table 2. The table has been computed for $\phi_h = 78,400 \text{ m}^2 \text{ s}^{-2}$, which corresponds to a surface temperature of about 273°K . The table shows that the ratio $c/(\bar{v} - v_c)$ is considerably smaller than unity, for long waves in middle latitudes, but it rapidly approaches unity for shorter waves. For waves of the same order of magnitude as those actually observed in the atmosphere (wave lengths 60° of longitude or less) the wave moves approximately in accordance with the simple formula

$$c = \bar{v} - v_c. \quad (9.9)$$

The short barotropic wave moves then approximately as though the flow were nondiverging. The physical reason is that the divergence has the same sign throughout every atmospheric column. Even a very small divergence when added over the whole column is sufficient to produce the pressure change at the ground associated with the movement of the pressure wave.

10. Baroclinic waves in a westerly current. We shall again assume that the wave pattern has infinite lateral extent, so that all isobars have the same shape; the transversal divergence is then zero.

We shall consider the normal case where the temperature within each isobaric layer decreases toward

FIG. 9. Level of nondivergence in a baroclinic wave.

FIG. 10. Determination of level of nondivergence from conditions on 10,000-foot map.

the pole, so that the speed of the zonal current increases with height (Fig. 9). Since the wave moves with the same speed at all levels, the relative zonal wind increases with height from the ground to the tropopause. The critical speed is determined by the wave length and the latitude from Table 1. At the level where the relative zonal wind has this critical value the flow is nondiverging. Above this level of nondivergence the relative zonal wind is supercritical with the bottleneck in the flow at the trough line. Below the level of nondivergence the relative zonal wind is subcritical, and the bottleneck in the current is at the wedge line. For short waves and for strong westerlies the level of nondivergence is low. Vertical columns have then preponderance of divergence to the east of the trough and convergence to the west of the trough and the wave moves toward the east. For weak zonal circulation and for long waves the level of nondivergence is high and the wave moves toward the west. The fact that the waves usually move toward the east indicates that the level of nondivergence as a rule is relatively low.

The commonly used upper-level map in daily weather analysis is the 10,000-foot map. When the relative zonal wind on the 10,000-foot map deviates from the critical speed, the magnitude of the deviation may be used to estimate the height of the level of nondivergence. As an example, consider the 10,000-foot map for Jan. 20, 1944 at 0400Z (Fig. 10) which shows a well developed wave. The mean zonal wind, computed from the mean zonal pressure gradient at the trough and wedge, is $v = 12^\circ$ long. per day. The mean zonal thermal wind, computed from the mean zonal

temperature gradient, is $\Delta v = 8^\circ$ long. per day per 10,000 feet. By linear extrapolation of this thermal wind from the 10,000-foot level we obtain a plot of zonal wind against height, as shown in the lower left diagram.

The mean speed of the trough during the preceding 24 hours was 7° long. per day. According to Table 1, p. 8 the critical speed of the wave at 40° lat. is 12° long. per day. The relative zonal wind has this critical value at about 19,000 feet, which accordingly is the level of nondivergence. Higher up the relative zonal wind is supercritical, so that there the bottleneck of the flow is at the trough and there is convergence of air into the sample column behind the trough. Below the level of nondivergence the relative zonal wind is subcritical. The bottleneck in the flow is at the wedge and the air is being depleted from the sample column east of the wedge.

The analysis in Figure 10 is based exclusively on the 10,000-foot map. We note that the sample column has divergence from the ground up to 19,000 feet, which represents more than one-half of the entire atmospheric column by weight. Although the effect of this divergence is partly counteracted by the convergence higher up we should expect to find falling pressure at the ground. Moreover the horizontal divergence below 19,000 feet must be associated with descending motion which reaches its greatest value at the level of nondivergence, and decreases higher up. We should therefore expect clear skies or only scattered high clouds in the region of the column. The surface map in the same region, Figure 11, shows a weak warm front just formed close to the sample column. As expected, the pressure is falling rapidly both ahead of and behind

the front, and most of the stations have clear sky even ahead of the warm front.

Our experience has been that the position of the level of nondivergence varies considerably from day to day and even from place to place within the same weather map. We have not yet made any systematic statistical study of its behavior.

FIG. 12. Construction of relative streamlines.

11. The relative streamlines. Another important matter to consider in connection with the baroclinic waves is how the flow pattern changes in shape and phase from level to level. The isobars will, in general, intersect the isotherms in any real baroclinic wave. It follows then: (i) that the wind shear (along the isotherm) makes an angle with the wind (along the isobar), so that the shape of the streamlines changes from level to level. (ii) the horizontal advection at each level will move the isotherms and so cause the structure of the wave pattern to change in time as it moves.

The internal structure of a moving wave is deformed by the relative wind, that is, the wind as seen by an observer who moves along with the wave. The streamlines of this relative wind are the relative streamlines.

Let v be the wind and c the velocity of propagation of the wave. The wind relative to the wave is then $v - c$. The relative streamlines are the streamlines of the velocity field $v - c$. The construction of the relative streamlines is shown in Figure 12. At the trough lines and the wedge lines of the streamline pattern both the real wind and the relative wind are zonal. These meridians are therefore also the trough or wedge lines of the relative streamline pattern. Both patterns have consequently the same wave length, but their amplitudes are in general different. The streamlines are by definition everywhere tangential to the velocity. Let δy_S and δy_R be the respective meridional increments on the streamline and on the relative streamline, corresponding to the same zonal increment δx . The differential equations for the streamlines and the relative streamlines are then respectively

$$\frac{\delta y_S}{\delta x} = \frac{v_y}{v_x},$$

$$\frac{\delta y_R}{\delta x} = \frac{v_y}{v_x - c}.$$

FIG. 11. Surface map corresponding to 10,000-foot map in Figure 10.

The ratio of these equations gives

$$\frac{\delta y_S}{\delta y_R} = \frac{v_x - c}{v_x} \quad (11.1)$$

The ratio on the right side in (11.1) is constant in the special case where throughout the level the zonal wind component v_x has the constant value v of the actual wind at the trough line and at the wedge line. Equation (11.1) may then be integrated. If the integration is taken from the inflection point (the central latitude) to the wedge line we find

$$\frac{A_S}{A_R} = \frac{v - c}{v} \quad (11.2)$$

When the wind speed v at the trough line is different from the wind speed v' at the wedge line, the ratio between the two amplitudes is not so simple as indicated by (11.2). However, if the difference $v' - v$ is small compared to the mean speed $\bar{v}$ it can be shown that the amplitude ratio has the approximate value

$$\frac{A_S}{A_R} = \frac{\bar{v} - c}{\bar{v}} \quad (11.3)$$

Thus, in a wave where the variation in the zonal wind is not too large, the ratio of the streamline amplitude to the relative streamline amplitude is the same as the ratio of the mean relative zonal wind to the mean zonal wind (cf. Rossby (8)).

Figure 13 shows the streamline (full line) and the relative streamline (broken line) for three different values of the wave speed. In all three cases the streamline is the same. The relative streamline is obtained from the formula (11.3) or directly by inspection of the velocity vector diagrams in the figure. In the upper diagram the wave is stationary ($c = 0$) and the relative streamline coincides with the streamline, $A_S = A_R$. In the middle diagram the wave moves toward the east with a slower speed than that of the air. The rela-

FIG. 13. Streamlines and relative streamlines for different values of the wave speed.

tive streamline is here in phase with the streamline, but the amplitudes differ in the sense $A_S < A_R$. In the lower diagram the wave moves to the east with a greater speed than that of the air. In this case the relative zonal wind is negative, so that the amplitudes A_S and A_R have opposite signs. The relative streamline is 180° out of phase with the streamline.

For a wave moving toward the east we have then the following simple rules for the relation between the streamlines and the relative streamlines: (i) If the wave moves toward the east with a slower speed than that of the air ($\bar{v} - c > 0$) the two patterns are in phase, with the relative streamlines having the larger amplitude. (ii) If the wave moves toward the east faster than the air ($\bar{v} - c < 0$) the two patterns are 180° out of phase.

The wave will always appear as a stationary wave for an observer moving along with the wave. Therefore the relative streamlines are also the relative paths and will indicate the true meridional displacements of the air particles.

12. Stable baroclinic waves. A wave whose shape and internal structure do not change during its propagation will be called stable. We shall investigate whether stable waves can exist in a baroclinic current, and study the three-dimensional structure of such waves.

We shall assume that the motion is nearly horizontal and that the temperature of individual particles is conserved. If the relative streamlines were to intersect the isotherms, the temperature field relative to the moving wave would be deformed and the wave would not be stable. Therefore in a stable baroclinic wave *the relative streamlines must coincide with the isotherms*.

This rule implies that the isotherms have the same shape at all levels if the wind is assumed to be geostrophic. By the thermal wind relationship the shear of the geostrophic wind is directed along the horizontal isotherms. Since, now, a stable wave moves with the same speed at all levels, the shear of the wind is also the shear of the relative wind. But when both the relative wind itself and its shear with height have the same direction, the relative wind can not turn with height. And so *the isotherms, along which the relative wind blows, are in phase and have the same shape at all levels* (cf. Rossby (8)).

The main features of the stable baroclinic wave are shown schematically in Figure 14. The central part of the diagram represents horizontal maps at four selected levels with one streamline (full line) and one relative streamline (broken line) drawn in each map. In keeping with the above rule, relative streamlines have been given the same shape at all levels. On account of the baroclinic temperature field (warm to the south and cold to the north) the zonal wind component increases with increasing height, as indicated in the

FIG. 14. Analysis of stable baroclinic wave.

left part of the diagram. The wave is assumed to move toward the east with a certain speed c . At some level the speed of the mean zonal current must be equal to the speed of the wave. Above that level the relative zonal wind is positive and the streamlines are in phase with the relative streamlines, but have a smaller amplitude. Below that level the wave moves faster than the air, so that the streamlines are 180° out of phase with the relative streamlines. At the level of zero relative zonal wind ($\bar{v} = c$), the streamline amplitude is zero (according to (11.3)) and the streamlines are straight lines.

This three-dimensional structure of the wave can also be anticipated from the hydrostatic relationships. The warm trough at low levels vanishes with height and is surmounted by a warm wedge whose amplitude increases with height. Correspondingly, the cold wedge at low levels vanishes with height, and is surmounted by a cold trough whose amplitude increases with height.

Having further determined the level where the relative zonal wind has the critical value (from Table 1), we can now estimate the transversal, longitudinal, and total mass divergence in the various parts of the wave. This estimate as applied to the column A is shown qualitatively in the three graphs to the right in Figure 14.

The transversal divergence is zero at the level where the relative zonal wind is zero ($\bar{v} = c$). Above that level the relative zonal wind is positive so that there is transversal divergence to the east of the trough and transversal convergence to the west of it. Below the level where $\bar{v} = c$, the relative zonal wind is negative so there is transversal convergence to the east of the trough and transversal divergence to the west of it. But, since the low-level trough is 180° out of phase with the upper trough, the column A has transversal divergence both above and below the level where $\bar{v} = c$.

The longitudinal divergence is zero at the level where the relative zonal wind is critical ($\bar{v} - c = v_c$). Above that level the relative zonal wind is supercritical, and

the bottleneck is at the cyclonic bend. This involves longitudinal divergence to the east of the trough and longitudinal convergence to the west of it. Below the level where $\bar{v} - c = v_c$ the relative zonal wind is subcritical, and the bottleneck is at the anticyclonic bend. That means longitudinal convergence to the east of the trough and longitudinal divergence to the west of it. Therefore the column A has longitudinal divergence above the level where $\bar{v} - c = v_c$ and also below the level where $\bar{v} = c$. Between these levels the column A has longitudinal convergence, for the wind is there subcritical and the column is to the east of the trough.

The total horizontal divergence is obtained by addition of the transversal and longitudinal parts. For the column A it is positive at all levels except in a layer whose base is the level where $\bar{v} - c = 0$, and whose top is below the level where $\bar{v} - c = v_c$. For the column B the horizontal divergence is negative at all levels except in the same intermediate layer.

In Figure 15 the distribution of horizontal divergence in the columns A and B is indicated by arrows of outflow and inflow. If the level O is interpreted as a flat part of the surface of the earth, it seems rather likely that divergence may dominate at A and convergence at B . This would mean pressure fall at the base of column A and pressure rise at the base of B . This again would make the pressure wave at the ground move westward, contrary to our initial assumption.

The only place and time that a wave of this kind could move eastward would then be while the low-level trough is over a mountain range, as shown in Figure 16. The downward velocity on the lee side of the mountain could then be strong enough to overcompensate the effect of convergence in the column, and produce a pressure fall at the surface. Likewise the upslope wind on the west side of the mountain can overcompensate the effect of divergence, and produce a pressure rise behind the trough.

If now the wave moves eastward, columns like A with a preponderance of divergence will arrive over the lee side of the mountain. The influence of the down-

FIG. 15. Divergence distribution in stable baroclinic wave.

FIG. 16. Baroclinic wave with low-level trough over mountain range.

ward motion and the divergence will add up, and a strong pressure fall will result, which will continue until columns with preponderance of convergence arrive. This is equivalent to a deepening of the once stable wave and a temporary slowing down of the surface trough while it moves down the east slope of the mountain.

The above discussion admittedly is somewhat vague, but may perhaps be made to apply as an explanation of the very frequent deepening of warm troughs on the lee slope of the Rocky Mountains, mainly in the area of Alberta and Montana.

13. The first formation of the baroclinic wave. We shall now take up the fundamental problem announced in the introduction, namely, to describe a mechanism for cyclogenesis. In what follows we shall discuss the incipient wave in the baroclinic westerly current, characterized by the usual decrease of temperature along the isobaric layers toward the pole. To simplify the discussion we shall assume that the wave disturbance has infinite lateral extent with zero transversal divergence. We shall see later that the results are in principle the same for waves with finite width.

The fields of longitudinal and transversal divergence have the zero lines along the troughs and along the wedges. Consequently, horizontal divergence does not explain the first formation and intensification of the wave disturbance. The only alternative is vertical motion; downward where the trough is formed and upward where the wedge is formed. This would explain why the wave pattern of upper isobars at the outset is located with a definite phase lag relative to the

frontal wave disturbances in the lower atmosphere. The upward motion over the warm-front surface creates the incipient upper wedge, and the downward motion behind the center at the ground creates the incipient upper trough.

The complete description of that process in terms of the equations of dynamics and thermodynamics is not yet within reach, so we must confine our treatment to qualitative reasoning. Let us consider two limiting cases:

a. The influence of the vertical displacement on the pressure change at the level ϕ is completely compensated by divergence or convergence in the air column above, so that everywhere

$$\left(\frac{\partial p}{\partial t}\right)_{\phi} = (g\rho v_z)_{\phi} - \int_{\phi}^{\infty} \text{div}_H(\rho v) \delta\phi = 0. \quad (13.1)$$

This is the usual assumption when examining the vertical stability of small elements of air in an environment at rest.

b. The influence of the vertical displacement on the pressure change at the level ϕ is not at all compensated by divergence or convergence in the air column above, so that everywhere

$$\left(\frac{\partial p}{\partial t}\right)_{\phi} = (g\rho v_z)_{\phi}. \quad (13.2)$$

The real case is likely to lie somewhere between (*a*) and (*b*) and will depend on the character of the initial undisturbed flow and the intensity and spatial extent of the vertical disturbance.

Figure 17a illustrates case (*a*). The upper part represents a vertical zonal cross section through the center of the young frontal wave. The intersection with the frontal surface then appears as an approximate sine curve tangent to the ground. In the reference level the isobars were initially straight and parallel to the path of the incipient wave, so that the line $\phi\phi$ on the vertical cross section was an isobar in the initial stage. According to (13.1) the same isobar $p = p_0$ stays in the position $\phi\phi$ also after the vertical motion has started. The isotherms in the reference level were straight and parallel to the isobars, so that $\phi\phi$ in the vertical cross section also was an initial isotherm. The vertical displacement $\Delta\phi$ changes the temperature at the constant level $\phi\phi$ by

$$\Delta T = -\Delta\phi(\gamma_d - \gamma), \quad (13.3)$$

where γ_d is the dry adiabatic lapse rate, and γ the actual lapse rate. Assuming $\gamma < \gamma_d$, the new isotherm position is raised in the region of descent and lowered in the region of ascent.

In the initial stage the isobaric surfaces slope gently down towards the pole and the isothermal surfaces slope the same way at a steeper angle. The isobaric

FIG. 17a.

FIG. 17b.

FIG. 17a, 17b. The two limiting cases of formation of the upper wave by vertical impulse.

surfaces stay fixed also in the perturbed state, whereas the isothermal surfaces move up in the region of descent and down in the region of ascent, thereby maintaining their meridional slope. Consequently, the horizontal map in the reference level ϕ (lower half of Fig. 17a) retains its straight west-east isobars also in the perturbed state. The isotherm $T = T_0$, at first straight west-east coinciding with the isobar $p = p_0$, becomes sinusoidal in the new state. Isotherms to the north and south of $T = T_0$ behave similarly. They define a warm tongue extending northwards in the region of descent and a cold tongue extending southwards in the region of ascent.

The case (b) is illustrated by Figure 17b. Since no horizontal divergence is present, no internal changes occur in the vertical air columns. They move up and down as solid columns. The vertical cross section (upper part of the diagram) shows how isobars and isotherms alike are lifted and lowered just as much as the air particles themselves. The lower part shows how sinusoidal isobars and isotherms result on the horizontal map in the reference level. This time the cold tongue coincides with the pressure trough and the warm tongue with the pressure wedge. The north-south amplitude of the isobars is considerably greater than that of the isotherms, the reason being that the isobaric surfaces have much smaller meridional slope than the isothermal surfaces.

Figures 17a and 17b show only the initial effect of the lowering and raising of two adjacent portions of the baroclinic upper westerly current. After the upper wave has formed, and the isobars and isotherms intersect on the upper-level map, the horizontal advection starts to move the isotherms. This process is shown in Figure 18. The initial state has been selected as a compromise between the two limiting cases represented by Figures 17a and 17b. The initial isotherm in Figure 18

has been made straight west-east as a compromise between the sinusoidal isotherms with opposite phase in Figures 17a and 17b. The initial straight isotherm in Figure 18 is numbered 0, and the subsequent positions reached by the same isotherm under the influence of horizontal advection relative to the wave are numbered 1 and 2, . . . 6. Since we are dealing with a moving wave it is the wind relative to that moving wave which decides the resulting shape of the isotherms. For the case under consideration we shall stipulate that the air moves eastwards faster than the wave (the usual case in the upper air). The relative streamlines are then in phase with the isobars but have greater north-south amplitude (see Fig. 13). The result of the advection is obviously the formation of a warm tongue in the region of southerly wind components and a cold tongue in the region of northerly wind components. If the advection is allowed to continue undisturbed while the particles move through one quarter of a wave length, the warm tongue would arrive at coincidence with the pressure wedge and would have a north-south amplitude twice that of the

FIG. 18. Progressive formation of thermal wave by pre-existing pressure wave.

FIG. 19. Tilt of troughs and crests in young intensifying upper wave.

relative streamlines. However, extrapolation of the isotherm advection so far ahead has but limited practical value, because the isobars also change during the process.

Let us consider an early stage in the isotherm advection in Figure 18 and look for the change to be expected in the isobar pattern. The lower part of Figure 19 shows the horizontal map at the level ϕ . The temperature and pressure waves are out of phase with each other, so the crests and troughs of the pressure wave must tilt as shown in the upper part of the figure. This upper diagram, representing the vertical cross section along the path of the wave, contains the three successive isobars $p_0 + \Delta p$, p_0 and $p_0 - \Delta p$. The vertical distance between them is greatest in the warm tongue. The pressure crest therefore tilts westward (toward warm) with increasing height. And the pressure trough also tilts westward (toward cold) with height.

The variation of the zonal wind with height in the warm air immediately south of the polar front is shown in the upper right diagram in Figure 19. The zonal wind increases with height in the troposphere to a maximum value at the tropopause and decreases from there up. The wave is assumed to move with a slightly slower speed than the warm-sector air at the surface. This choice is plausible since the surface air always overtakes the wave and ascends over the warm front. The relative zonal wind then increases from a small positive value at the ground to a maximum value at the tropopause. The flow is, according to (5.14), non-diverging at the level where the relative zonal wind is critical. The horizontal divergence ahead of the trough is positive above the level of nondivergence and negative below that level. In a westerly current of given strength the height of the level of nondivergence depends on the wave length. For short waves the critical speed is small and hence the level of non-divergence is low. As the wave length increases the level of nondivergence is raised to greater heights

until the tropopause is reached. For still longer waves no level of nondivergence exists.

Let us first examine a short wave whose level of non-divergence is below the reference level ϕ , as indicated in the figure. Consider a vertical column of air extending upward from the reference level at the trough-line position in that level. The pressure change at the base of this column indicates whether the trough is deepening or filling on the reference-level map. First, the trough would deepen if the air continues to descend at the base of the column after the initial wave has developed. But, even if that downward motion has ceased, deepening would still continue if the integral of horizontal divergence in the column above the reference level shows depletion of air. Because of the westward tilt of the trough, the vertical column is east of the trough at all levels above the reference level. So the entire column has horizontal divergence, and the pressure tendency at its base is negative. Similar reasoning for the crest of the pressure profile shows that the pressure wedge on the map in the reference level must be building up, even though the upward motion, which once started the wedge, may have ceased. Thus, *a baroclinic westerly current is dynamically unstable for waves whose level of nondivergence is sufficiently low and tends to make these waves grow strong, however weak they may be at the start.*

Let us next consider long waves where no level of nondivergence exists. The relative zonal wind at all levels is then subcritical, so the flow has horizontal convergence at all levels ahead of the trough. The long waves therefore would move westwards. However, such waves would have no opportunity to develop. Since the trough and the wedge would tilt westwards, the pressure would rise in the trough and fall on the wedge at all levels, and the initial disturbance would be damped out. Therefore the baroclinic westerly current is stable for these long waves.

The waves which develop in the westerlies above the polar front always have finite lateral width and consequently have transversal divergence. The only way in which transversal divergence will influence the above result is to make the total horizontal divergence zero for a somewhat smaller value of the relative wind than the critical speed. Other things being equal, the transversal divergence therefore lowers the level of non-divergence, and makes the waves of finite width more unstable than the infinitely wide waves.

Both the short and the long waves considered above are evidently inertial waves, that is, the stabilizing influence is entirely due to the earth's rotation. These inertial waves show the same dynamical behavior as the inertial waves in the polar front, which was mentioned in the first section (see Fig. 1). When the waves are long the rotational stability will dominate, but for shorter waves the instability due to the vertical shear,

which always exists in the baroclinic current, becomes stronger than the rotational stability and the waves are unstable.

The same kind of dynamic instability exists wherever the temperature in a given isobaric surface decreases towards the left (right in S. hemisphere) across the current, or, as it also may be expressed, wherever the speed of the current increases with height. The principal region of dynamical instability is thus the temperate region of westerlies. These westerlies are dynamically more unstable in winter than in summer. This explains in a general way the storminess in middle latitudes and its seasonal cycle.

If the direction of the horizontal temperature gradient is reversed in the preceding discussion, we obtain a system where the troughs and wedges tilt eastward with height. Any disturbance in this system will therefore be damped out. In the stratosphere of the temperate zone the westerlies decrease with height. So independently intensifying waves seem excluded in this region. In winter at latitudes greater than 60° the stratosphere temperature decreases towards the pole, so the westerlies must increase with height. In this region and season active wave formations may have their birth in the stratosphere and grow due to dynamic instability.

Easterly currents are dynamically unstable only when they flow between a warm high and a cold low. This involves a reversal of the normal meridional temperature gradient and would be very rare in high latitudes. However, such a reversal is normal in summer in low latitudes between the thermal and the geographical equators. So the equatorial easterlies in summer (especially late summer) exhibit some dynamical instability. That is probably significant for the occurrence of the tropical cyclones.

14. Horizontal divergence in closed cyclonic isobar patterns. The layers from the ground up to roughly 3 km elevation show more complicated pressure patterns than the higher layers. In the lower layers closed isobars around centers of high and low pressure frequently occur. Occasionally such closed pressure patterns also extend to layers well above 3 km.

The simplest closed low (or high) for theoretical treatment is the one with the center at the pole surrounded by circular zonal isobars coinciding with the circles of latitude. Any ring-shaped channel between two neighboring isobars will then follow along a constant latitude and will therefore have the same transport capacity throughout. So there is no region of accumulation or depletion of air, and no pressure changes will occur. The circular vortex centered at the pole is thus a possible steady state. In fact it also exists in the atmosphere permanently. The upper westerlies of the temperate latitudes are part of the huge cyclonic vortex centered at the pole. (At the ground the

FIG. 20. Low with circular concentric isobars not centered at the pole.

polar low is concealed since the cold polar air contributes enough surplus weight in the lower layers to make the minimum of pressure disappear).

If a low with concentric circular isobars is centered at some latitude away from the pole, the isobaric channels surrounding the center will not have equal transport capacity all around. Let us compare the transport capacity at the northernmost and southernmost points of the ring-shaped isobaric channel (Fig. 20). On account of the symmetry these points are places of maximum or minimum speed where the wind is gradient. Assume for simplicity that the density is the same, and hence that the horizontal pressure force has the same value at the two points. When the gradient wind equation (5.1) is differentiated with respect to latitude, keeping the curvature and the pressure force constant we find

$$\frac{dv}{d\varphi} = - \frac{v\Omega \cos \varphi}{vK + \Omega \sin \varphi}. \quad (14.1)$$

This shows that the gradient wind increases with decreasing latitude. The southernmost point of the ring-shaped isobaric channel has the maximum speed and the northernmost point has the minimum speed. Since the isobaric channel has the same width at the two bends the transport capacity is directly proportional to the wind speed. So the bottleneck of the channel is at the northernmost point.

The whole western half of the vortex then exports more air to the eastern half than what it receives in return. Air will be depleted from the western half of the vortex and accumulated in the eastern half. If the vortex extends with a vertical axis all the way up through the atmosphere the pressure would fall in the western part and rise in the eastern part. The low and the associated vortex would *move westward*. The stationary concentric circular vortex reaching to the top of the atmosphere is thus an impossibility unless it is centered over the pole.

FIG. 21. Low with circular eccentric isobars.

The tendency for westward displacement of closed lows is counteracted if the pressure pattern is made eccentric as shown in Figure 21. The isobaric channel is narrow in the south and wide in the north. According to equation (5.2) the transport capacity, $\delta F = \rho v \delta n \delta z$, will be equal at the southernmost and northernmost points if

$$Kv + 2\Omega \sin \varphi = K'v' + 2\Omega \sin \varphi'. \quad (14.2)$$

It should be remembered that v' , the east wind at the northernmost point, is counted as positive in this equation and in those to be derived from it. If the central latitude, $\bar{\varphi}$, and the angular amplitude σ_p , are introduced from (5.6), (14.2) becomes

$$Kv - K'v' = 4\Omega \cos \bar{\varphi} \sin \sigma_p. \quad (14.3)$$

If the circular pressure pattern moves along the west-east direction, the relation between the curvature of the path and the curvature of the streamline at the southernmost and northernmost points of the isobaric channel is given by

$$\begin{aligned} Kv &= K_S(v - c), \\ K'v' &= K'_S(v' + c). \end{aligned} \quad (5.10)$$

The circular isobar has the angular radius of curvature σ_p and hence its horizontal curvature is $K_p = 1/(a \tan \sigma_p)$. (See footnote p. 4). Since, now, the streamlines coincide very nearly with the isobars, we have approximately that $K_S = K'_S = K_p = 1/(a \tan \sigma_p)$, and consequently

$$Kv - K'v' = K_S(v - v' - 2c) = \frac{v - v' - 2c}{a \tan \sigma_p}.$$

When this expression is substituted in (14.3) we get the condition for equal transport capacity at the southernmost and the northernmost points in a circular isobaric channel:

$$\begin{aligned} v - v' - 2c &= 4\Omega a \cos \bar{\varphi} \sin \sigma_p \tan \sigma_p \approx \\ &\approx 4\Omega a \sigma_p^2 \cos \bar{\varphi}. \end{aligned} \quad (14.4)$$

A circular isobaric pattern which in all isobaric channels satisfies equation (14.4) will be said to have the *critical eccentricity*, which makes the eastern and the western half of the vortex import equal amounts of air from each other during any given time.

TABLE 3.

($v - v' - 2c$) in ms^{-1} in a pressure pattern of critical eccentricity

$\bar{\varphi}$	σ_p			
	1°	5°	10°	20°
90°	0	0	0	0
80°	0.1	2.5	9.9	—
70°	0.2	4.9	19.4	78
60°	0.3	7.1	28.4	114
50°	0.4	9.1	36.6	146
40°	0.4	10.9	43.6	175
30°	0.5	12.3	49.4	198
20°	0.5	13.4	53.5	215
10°	0.6	13.9	55.6	—

The values of the right hand side in (14.4) are given in Table 3 for values of σ_p up to 20°. The table shows what velocity difference there must be between the southernmost and the northernmost point of a *stationary* ring-shaped isobaric channel, in order that the channel shall have the same transport capacities at the two points. The value of $v - v' - 2c$ for the critical eccentricity increases with increasing distance from the pole and increasing radius of the channel.

A stationary low ($c = 0$) with exactly the critical eccentricity for all isobaric channels would have almost concentric isobars near the center and increasingly eccentric isobars toward the outskirts. Isobaric channels of 20° radius and with critical eccentricity would require west winds of impossible velocities at the southernmost point.

If the low moves toward the east, the values in the table must be augmented by twice the speed of the center to give the velocity difference $v - v'$ needed for critical eccentricity. If the low moves westward the tabulated values must be diminished by the same amount. So the critical eccentricity is accentuated if the low moves toward the east and less pronounced (or inverted) if the low moves toward the west.

The actual eccentricity of closed flow patterns on the map, as expressed by the difference in wind velocity at the southernmost and northernmost points of individual isobaric channels, may be compared with the tabulated values for the critical eccentricity. It will generally be found that closed circulations have subcritical eccentricity. That is, they do not have enough eccentricity to make the eastern half of the vortex export as much air to the western half as it receives in return.

The total accumulation or depletion of air in a half-vortex depends not only on the exchange of air with the other half of the vortex but also on the inflow or

outflow across the limiting outer isobar. In an eccentric vortex, like that in Figure 21 the flow of air from the velocity maximum in the south to the velocity minimum in the north would be associated with a slight component of motion across the isobar toward high pressure. This represents a depletion of air from the eastern half of the vortex. Corresponding reasoning for the western half shows an inflow across the outer isobar and accumulation of air in that part.

Thus eccentricity tends in two ways to deplete air from the eastern half of the vortex, and accumulate air in the western half and thereby counteracts the latitude effect which makes the eastern half gain and the western half lose air. The flow is therefore actually nondiverging for slightly subcritical eccentricity.

15. Closed cyclonic isobar patterns surmounted by wave-shaped patterns. The above treatment of the cyclonic vortices in temperate latitude tends to show that only those low pressure patterns can move eastward which have packed isobars to the south and open isobars to the north of the center. The lower the latitude the greater the eccentricity required. However, that result is valid only for cyclones which reach with essentially the same closed isobaric patterns to the top of the atmosphere. The usual cyclone of the temperate latitudes has closed isobars only in the lowest 2 or 3 km of the atmosphere. Higher up it appears only as a trough in the upper west-east trend of the isobars. The treatment of such cyclones will show that although a pattern of packed isobars to the south and open isobars to the north of the center is favorable for

a rapid displacement eastward, such a pressure pattern is by no means necessary for the eastward displacement of the ordinary cyclone in temperate latitudes.

Figure 22 represents a schematic picture of the processes which produce the pressure changes in the ordinary eastward moving cyclone in temperate latitudes. It has been assumed that $(v - v') < 4\Omega a \sigma_p^2 \cos \bar{\varphi} + 2c$. According to (14.4) this implies that the pressure distribution around the closed center in the lower atmosphere is not sufficiently eccentric to cause a depletion of air in front of the center and an accumulation of air behind. In the layers with closed isobars we have horizontal convergence in front of the cyclone and horizontal divergence in the rear. Higher up where the cyclone is represented by a trough in the west-east isobars there will be, according to our earlier results, horizontal divergence in front of the trough and horizontal convergence behind.

A column fixed in space at *A* (while the air is flowing through it) will gain weight by the convergence in the lower layers and will lose weight by the divergence in the upper layers. In order that there shall be a net loss of weight of the whole column, more air must be removed in the upper layers than accumulates through convergence in the lower layers. The fall of pressure ahead of the cyclone depends on that.

A column fixed in space at *B* will gain some weight by the convergence in the friction layer, but will normally lose more than that by the divergence in the superjacent layers of closed isobars. Above the level where the closed center disappears column *B* will gain weight by the convergence prevailing behind the upper air trough. The net effect of the changes in weight of column *B* should be positive, so as to give a rise in pressure behind the moving cyclone. Again this requires that the influence from the upper layers shall overcompensate the influence from the lower layers.

Figure 22 explains in a qualitative way how low centers can have falling pressure in front and rising pressure in the rear, although the analysis of the horizontal flow in the low layers shows accumulation of air in front of the center and depletion of air in the rear. The general west-east drift of low centers is thereby explained very much in accordance with the old rule of thumb: The centers move along with the upper current. We now have obtained the physical explanation for this rule: The upper current provides for depletion of air in front of the center and accumulation of air behind the center.

Actually there is a phase difference between the upper trough and the cyclone center at the ground, the upper trough lagging a little. Because of that phase lag upper-air divergence takes place vertically above the central area of the cyclone. If the upper-air divergence at that place is strong enough to overcompensate the convergence of air into the cyclone center in

FIG. 22. Closed eccentric isobars in low levels coexistent with sinusoidal isobars in the upper levels.

the lower layers, there will be falling pressure at and around the center. In that case the cyclone is deepening. This process of intensification evidently depends on the phase lag and the backward tilt of the upper trough, which again (according to Section 13) reverts to the vertical shear as the primary cause for wave instability.

It is furthermore evident from Figure 22 that deepening is most likely to occur if the pressure pattern changes from closed isobars to wave-shaped isobars at a relatively low level. This is characteristic of young cyclones. The first level to have closed isobars around a new cyclone is the surface level. As the cyclone grows it develops closed isobars at successively higher levels. The older the cyclone, the higher up will be the level of transition from closed to wave-shaped isobars, and the more the pressure changes will be influenced by the lower pattern of closed isobars. Finally the influence of the lower pattern will cancel or overcompensate that of the upper pattern, and the cyclone will stop or turn slightly retrograde. This, too, is very well corroborated by observations. Cyclones move fast eastward while they are young, but slow down when they get old and deep, sometimes even retrograding a little toward the west in their last phase.

It is interesting to note that the vortex in the surface layers, while being forced along by the influence of the "upper current," counteracts that displacement by piling up air in front. Thus if a vortex of given eccentricity is forced to move to the east, that eccentricity becomes the more "subcritical," the faster its eastward speed. In an eastward moving low the air will accumulate in the eastern half at the expense of the air in the western half, even though the same pressure pattern, when stationary, maintained nondivergence. This again means that the pressure fall in the front half and the rise in the rear half, both originating in the upper current, will be reduced respectively by the convergence and divergence in the lower layers. This effect increases with increasing eastward speed of the system and thus acts as a regulator of the speed.

The next step in the treatment of the pressure changes in moving cyclones should be to modify the circular isobars at the surface so as to accommodate fronts. Obviously there are a great many problems left for future work concerning the distribution of divergence in pressure fields with noncircular isobars delimiting more or less sharply defined troughs, etc. However, it is expected that the fundamental difference between the front and the rear of the moving cyclone, as represented in Figure 22, will remain as a background pattern upon which frontal effects are superimposed.

16. Closed anticyclonic isobar patterns. The analysis of pressure changes associated with moving highs can be carried out in analogy with the above analysis of the pressure changes in moving lows.

The anticyclone with concentric circular isobars is impossible as a steady state except when centered at the pole. At other latitudes steady-state anticyclones extending to high levels must have eccentric isobars with the maximum pressure gradient on the polar side of the center. The usual moving anticyclone does not extend far from the ground as a system of closed isobars, but it is surmounted by a wedge. The upper wave-shaped flow pattern produces accumulation of air over the front half of the high and depletion of air over the rear half, and thereby makes it move eastward. The lower closed flow pattern produces the opposite effect, but is overcompensated by the contribution of pressure change superimposed from the high layers. Since the upper wedge usually lags behind the position of the center of high at the ground, the upper flow pattern produces an accumulation of air over the central region of the high. If that accumulation of air is sufficient to overcompensate the outflow of air in the frictional layer near the ground, the high will intensify.

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