A MECHANISM FOR CONVECTION OVER THE OCEAN

By Patricia A. Langwell

New York University

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ABSTRACT

The cumuli which form over the ocean near Puerto Rico cannot be the result of "thermals." A theory is evolved, based on the oscillation of the boundary between the well-mixed layer near the ocean and the cloud layer. The physical plausibility of this theory is investigated by a development of stability criteria from the frequency equation.

1. Introduction

The type of cloud most frequently observed over the water north of Puerto Rico is the cumulus. The clouds vary from cumulus humilis to cumulonimbus, and the cloud cover varies from almost clear to almost overcast.

Since the ocean may be considered an isothermal surface (relative to the area considered), the usual assumptions of uneven heating and of "thermals" seem unwarranted. Moreover, measurements of the vertical velocity during horizontal plane-traverses show no significant difference between the dimensions and amplitudes of eddies below clouds and those at an equal height below clear air. No "roots" of cumulus clouds can be found. The purpose of this paper is to describe a mechanism for convection over the ocean, and to demonstrate that this mechanism is physically plausible.

The first aspect of the problem may best be stated by a description of the observations (Bunker et al., 1950). The lowest 600 m of the atmosphere are well mixed, and dominated by small-scale turbulence. The lapse rate is almost exactly dry adiabatic, and the mixing ratio nearly constant. Above is a shallow layer in which the mixing ratio decreases rapidly, and the lapse rate is either zero or slightly positive. The base of the clouds appears to be at this inversion.

The average height of the inversion is 600 m, but it has been observed to be as high as 730 m and as low as 300 m (Bunker et al., 1950, table 33). Its average height is about 85 per cent of the height of the lifting condensation level, but some of the observations of the inversion are above the lifting condensation level. If the height of the inversion oscillates about its mean position sufficiently for it to be located at or above the lifting condensation level for substantial areas, the latent heat of condensation would provide sufficient

energy for a parcel of air to break through the inversion. In this way, the lowest layer acts as a source of moisture for the clouds, and, rather than sensible heat, it is latent heat which is transferred upward from the lowest layer of the atmosphere. Then, if this is the way in which cumulus clouds are formed over the ocean, the amount of cloud cover depends on the wave length of the oscillations and on their instability. There remains the problem of showing that such unstable oscillations are physically plausible.

2. The frequency equation

The physical model is a compressible atmosphere on a flat, non-rotating earth. If the fluid is initially at rest, and in hydrostatic equilibrium,

\[ g = -Q \frac{\partial P}{\partial z}, \]  
\[ \frac{\partial u}{\partial t} = -Q \frac{\partial p}{\partial x}, \]  
\[ \frac{\partial w}{\partial t} = -Q \frac{\partial p}{\partial z} - (g/Q), \]  

where \( P \) and \( p \) are the undisturbed and perturbation pressures, \( Q \) and \( q \) the undisturbed and perturbation densities, \( g \) is gravity, \( x \) and \( z \) are the horizontal and vertical directions, and \( u \) and \( w \) are the perturbation velocities in these respective directions. If the temperature lapse rate is constant and equal to \( \epsilon \), the coefficient of barotropy is

\[ \Gamma = \frac{dQ/dP}{RT}, \]  

where \( T = T_0 - \epsilon z = P/QR \), and \( R \) is the gas constant. The other equations used are the equation of continuity and the adiabatic equation:

\[ \frac{\partial q}{\partial t} + Q(\partial u/\partial x + \partial w/\partial z) + w \partial Q/\partial z = 0, \]  

and

\[ \frac{\partial q}{\partial t} + w \partial Q/\partial t - \gamma (\partial p/\partial t + w \partial P/\partial z) = 0, \]  

where \( \gamma \) is the coefficient of piezotropy. Since, with adiabatic changes,

\[ (p + P)(q + Q)^{-\gamma} = \text{constant}, \]
where \( \lambda = \frac{c_p}{c_s} \),
\[
\gamma = \frac{\partial (q + Q)}{\partial (\rho + P)} = \frac{1}{\lambda R T},
\]
and
\[
dQ/dz = (dQ/dP) dP/dz = -gQ\Gamma.
\]

It is assumed that
\( u, w, \rho, q \sim A, C, D, E \exp [i\mu (x - kt)] \),
where the amplitudes are functions of height. There are two layers. In the lower, denoted by a superscript I, \( \epsilon = (\lambda - 1)(g/\lambda R) \), the adiabatic lapse-rate, and in the upper, denoted by a superscript II, \( \epsilon = 0 \). There is no reason that arbitrary lapse rates may not be used. Other treatments of this problem (Haurwitz, 1932, for example) have done so. However, the adiabatic or isothermal conditions shorten the computations considerably, in addition to conforming most closely to the model described above. With the foregoing assumptions about the wave character of the disturbance, the variations with time and with horizontal distance are eliminated. The height variations of \( u, \rho \) and \( q \) are eliminated from (2), (3), (5) and (6). Then, if a prime indicates a derivative with respect to \( z \),
\[
C'''' - g\gamma \left[ 1 - \frac{(\lambda - 1)\gamma^2 k} {1 - \gamma^2 k^2} \right] C'' - \left[ \mu^2 (1 - \gamma^2 k^2) + \frac{g^2 \gamma^2 (\lambda - 1)} {1 - \gamma^2 k^2} \right] C' = 0, \tag{7}
\]
and
\[
C'''' - g\lambda \gamma^2 C'''' - \left[ \mu^2 (1 - \gamma^2 k^2) - \frac{g^2 \gamma^2 (\lambda - 1)} {k^2} \right] C'' = 0. \tag{8}
\]

It is assumed that \( \gamma^I \) and \( \gamma^II \) are constant, near the interface, and that \( \gamma k^2 \ll 1 \), for either \( \gamma^I \) or \( \gamma^II \), because the velocity of the oscillations to be studied is much smaller than the velocity of sound. Equations (7) and (8) are then simply
\[
C'''' - g\gamma C'' - \left[ \mu^2 + g^2 \gamma^2 (\lambda - 1) \right] C' = 0, \tag{9}
\]
and
\[
C'''' - g\lambda \gamma^2 C'''' - \left[ \mu^2 + g^2 \gamma^2 (\lambda - 1)/k^2 \right] C'' = 0. \tag{10}
\]

Since the coefficients are not functions of height, the solutions of (9) and (10) are exponential.

Four boundary conditions are necessary. It is assumed that \( z = 0 \) at the interface, and two conditions are applied there:
\[
\frac{d}{dt} (p^I + P^I) - \frac{d}{dt} (p^II + P^II) = 0, \tag{11}
\]
and
\[
w^I = w^II. \tag{12}
\]

It is necessary that the oscillations be bounded as the height becomes great, so
\[
C''\to 0 \quad \text{as} \quad z \to \infty. \tag{13}
\]

The fourth condition is applied at the bottom of the atmosphere. If the amplitude of even unstable oscillations is small compared to the distance of the inversion from the ground,
\[
C''' \to 0 \quad \text{as} \quad z \to -\infty \tag{14}
\]
may be used as the fourth boundary condition. If the inversion is sufficiently close to the surface of the earth, this assumption becomes untenable, and the boundary condition is
\[
C'' = 0 \quad \text{when} \quad z = -h. \tag{15}
\]

Both cases are discussed below.

If condition (13) is applied to the solution of (10),
\[
C'' = C_2 \exp \left[ \frac{g\lambda^2 \gamma^II}{2} - \sqrt{\left( \frac{g\lambda^2 \gamma^II}{2} \right)^2 + \mu^2 - \frac{g^2}{k^2} (\lambda - 1) \gamma^II} \right] z. \tag{16}
\]

If condition (14) is applied to the solution of (9),
\[
C''' = C_1 \exp \left[ \frac{g\gamma^I}{2} + \sqrt{\left( \frac{g\gamma^I}{2} \right)^2 + \mu^2 + g^2 \gamma^2 (\lambda - 1)} \right] z. \tag{17}
\]

From condition (12), \( C_1 = C_2 \). If (11) is reduced to the perturbation form,
\[
\frac{d}{dt} (p^I - p^II) - g\omega (Q^I - Q^II) = 0. \tag{18}
\]

Since \( p = D(z) \exp [i\mu(x - kt)] \), and
\( D = (iQk/\mu)(C' - g\gamma C) \),
(18) becomes
\[
k^2 Q^I \left[ \frac{g\gamma^I}{2} + \sqrt{\mu^2 + g^2 \gamma^2 (\lambda - 1)} - g\gamma^I \right] - k^2 Q^II \left[ \frac{g\lambda^2 \gamma^II}{2} - \sqrt{\left( \frac{g\lambda^2 \gamma^II}{2} \right)^2 + \mu^2 - \frac{g^2}{k^2} (\lambda - 1) \gamma^II} - \gamma^II \right] - (Q^I - Q^II) g = 0. \tag{19}
\]

If the condition that \( \gamma k^2 \ll 1 \) is used, (19) reduces to
\[
k^2 Q^I \left[ \mu^2 + g^2 \gamma^2 (\lambda - 1) \right] + k^2 Q^II \sqrt{\left( \frac{g\lambda^2 \gamma^II}{2} \right)^2 + \mu^2 + g^2 \gamma^2 (\lambda - 1) \gamma^II} - (Q^I - Q^II) g = 0. \tag{20}
\]
This is the frequency equation. To write it explicitly in powers of \( k \), (20) is squared, and

\[
k^4 \left( Q^3 \left( \frac{\mu^2}{g^2} + (\lambda - \frac{3}{2}) \gamma^{III} \right) - Q^II \left( \frac{\mu^2}{g^2} + \frac{\lambda \gamma^{III}}{4} \right) \right) + k^2 \left( (\lambda - 1) \gamma^{II} Q^{III} \right) - 2(Q^I - Q^{II}) Q^I \sqrt{\frac{\mu^2}{g^2} + (\lambda - \frac{3}{2}) \gamma^{II}} \right) + (Q^I - Q^{II})^2 = 0. \tag{21}
\]

3. Stability criteria

The problem is to find the conditions under which the oscillations are unstable. The oscillations are unstable if the wave velocity \( k \) is complex. Therefore, the problem may be restated as a search for the conditions under which \( k \) is complex. According to Descartes’ rule of signs, there are no more real positive roots than there are changes of sign of the coefficients of \( f(\pm k) \), and no more real negative roots than there are changes of sign of \( f(\pm k) \). Since (21) contains only even powers of \( k \), there are as many changes of sign in \( f(\pm k) \) as in \( f(\pm k) \). A sufficient condition that all the roots of (21) be complex is that the coefficients of the powers of \( k \) all have the same sign. Even if roots have been introduced by squaring (20), if all the roots of (21) are complex, the same is true of (20). Since the last term of (21) is positive, it is necessary that the coefficients of \( k^4 \) and \( k^2 \) be positive. The coefficient of \( k^8 \) is positive when

\[
Q^I - Q^{II} < \frac{(\lambda - 1) \gamma^{II} Q^{III}}{2Q^I \sqrt{\mu^2/g^2 + (\lambda - \frac{3}{2}) \gamma^{II}}}. \tag{22}
\]

The right-hand member is a positive quantity. The order of magnitude of every term but \( \mu^2/g^2 \) can be estimated. However, since \( \mu = 2\pi/L \), the limiting values of this expression can be calculated. As the wavelength approaches infinity,

\[
Q^I - Q^{II} < \frac{\gamma^{II} Q^{III}}{4 \gamma^{II}} \approx \frac{Q^{II}}{4}. \tag{23}
\]

As the wavelength approaches zero, \( Q^I - Q^{II} < 0 \).

If condition (22) is not satisfied, it is impossible to be sure that unstable oscillations exist, since (21) may have as many as four real roots. When \( Q^{II} > Q^I \), condition (22) is satisfied for all wavelengths, but for each wavelength there is a small positive quantity \( \delta_L \), such that condition (22) is satisfied when \( Q^{II} > Q^I + (\lambda - 2\pi/L) \). In the problem investigated here, since the virtual temperature of the upper layer is less than that of the lower, \( Q^{II} > Q^I \). The requirement that \( Q^{II} > Q^I \) is the same as the condition that oscillations between two similar layers of an incompressible atmosphere be unstable.

The coefficient of \( k^4 \) is positive when

\[
\mu^2 < \frac{g^2 \gamma^{III} (\lambda - \frac{3}{2}) - \frac{\lambda \gamma^{III} Q^{III}}{4}}{Q^{II} - Q^I}. \tag{24}
\]

If condition (23) is not satisfied, (21) may have as many as two real roots. A mathematical evaluation of this expression is given near the end of this paper.

This same problem was considered without using the requirement that \( \gamma k^2 \ll 1 \). Inasmuch as \( \gamma k^2 \) may be as large as 0.01, for values of \( k \) which are large but not beyond possibility, and some problems do not allow for approximations of one order of magnitude, this computation was considered to have some value. Without the approximation that \( \gamma k^2 \ll 1 \), the frequency equation is of the sixteenth order.

One of the coefficients in the frequency equation is of a form which must be negative. The frequency equation may always have as many as four real roots. As a result, there are no conditions which require the oscillations to be unstable if they move very rapidly. However, all the other coefficients of the equation are positive, if \( Q^{II} > Q^I \) and if \( \mu \) is less than a minimal value. These conditions are similar to those which produce instability in the slower-moving waves.

To test the effect on the ground of the instability of the oscillations, condition (15) is used instead of condition (14). Equation (16) remains the same, but (17) is replaced by

\[
C^I = C_e \left( e^{y_q^2} \left[ e^{ax} - e^{-a(2\pi + a)} \right] \right), \tag{24}
\]

where

\[
a = \sqrt{\mu^2 + g\gamma^{II} (\lambda - \frac{3}{2})}. \tag{25}
\]

If condition (12) is applied,

\[
C_I(1 - e^{-2\pi a}) = C_e, \tag{25}
\]

and (19) becomes

\[
Q^I k^2 (1 + e^{-2\pi a})
\]

\[
+ \left[ Q^{III} k^2 \sqrt{\frac{g\lambda \gamma^{II}}{2}} \right] + \mu^2 - \frac{g^2}{k^2} (\lambda - 1) \gamma^{II}
\]

\[
- g(Q^I - Q^{II}) (1 - e^{-2\pi a}) = 0. \tag{26}
\]

Once again, it has been assumed that \( \gamma k^2 \ll 1 \). By the same reasoning as before, the oscillations are unstable if the coefficients of the equation are all positive. The coefficient of \( k^8 \) is positive when

\[
Q^I - Q^{II} < \frac{(\lambda - 1) \gamma^{II} Q^{III}}{2Q^I \sqrt{\mu^2/g^2 + \gamma^{II} (\lambda - \frac{3}{2}) \tanh a h}}. \tag{27}
\]

Since \( \tanh a h \) varies from zero to one as its argument varies from zero to infinity, condition (27) reduces to (22) as \( h \) approaches infinity. For all finite \( h \), the right-hand member is positive and larger than the
similar term in (22). Condition (27) is certainly fulfilled if \( Q^n > Q^l \). The coefficient of \( k^4 \) is positive when
\[
\mu^2 < \frac{g^2 [Q^n \gamma^2 (\lambda - \frac{3}{4}) - \frac{1}{4} \lambda^2 \gamma^4 Q^n \tanh^2 ah]}{(Q^n \tanh^2 ah) - Q^n}. \tag{28}
\]
Because of the limits of \( \tanh ah \), (28) reduces to (23) as \( h \) approaches infinity, but it seems at first that the numerator of (28) is larger and the denominator smaller than the similar terms in (23), and, as a result, that the quotient of (28) is always larger than (23). This would imply that the condition for instability is less stringent for a finite depth than for an infinite depth. However, implicit in this work is the assumption that \( \mu^2 > 0 \). This restriction imposes no difficulty in the first case, the denominator of (23) being positive from condition (22) and the numerator positive if
\[
\frac{\gamma^2 Q^l}{\gamma^2 Q^n} > \sqrt{\frac{\lambda^4}{4(\lambda - \frac{3}{4})}} \approx \frac{7}{8}. \tag{29}
\]
Since, in all practical examples, the density difference is never greater than about 5 per cent, this condition is automatically fulfilled. However, when the proximity of the ground to the inversion is considered, the condition that \( \mu^2 > 0 \) is not trivial. This condition is equivalent to the condition that numerator and denominator of (28) have the same sign.

If the numerator is negative,
\[
\tanh ah > \frac{\gamma^2 Q^n}{\gamma^4 Q^n} \sqrt{\frac{(\lambda - \frac{3}{4}) \lambda^2}{\lambda^2}} \approx 1.12 \frac{\gamma^2 Q^l}{\gamma^4 Q^n}.
\]

If the density difference is less than 5 per cent, this requires \( \tanh ah \) to be greater than one. Since this is impossible, the numerator and denominator must both be positive. This is equivalent to the statement that
\[
\tanh ah > Q^l/Q^n. \tag{30}
\]
If \( Q^n/Q^l \) is about 0.95, condition (30) is equivalent to the statement that
\[
\sqrt{\mu^2 + g^2 \gamma^2 (\lambda - \frac{3}{4}) h} > 1.85,
\]
or
\[
\mu^2 > \left( \frac{1.85}{h} \right)^2 - g^2 \gamma^2 (\lambda - \frac{3}{4}). \tag{31}
\]
For very large \( h \), condition (31) is automatically fulfilled, since the second term is larger than the first. When
\[
h < \frac{1.85}{g^2 \gamma^2 (\lambda - \frac{3}{4})}, \tag{32}
\]
the unstable \( \mu \) has a lower bound different from zero.

In terms of wavelength, since \( \mu = 2\pi/L \), when the ground is an "infinite" distance from the inversion, all wavelengths greater than \( L_\infty \) defined by (23) are unstable. As the depth of the lower layer decreases, the critical wavelength \( L_\kappa \), beyond which all wavelengths are unstable, decreases. In other words, the finite thickness is at first a destabilizing effect, since more wavelengths are unstable. However, when the lower layer becomes so shallow that condition (32) is fulfilled, the unstable wavelengths have a finite upper bound.

If \( g = 10^3 \), \( \gamma = 10^{-9} \), \( \lambda = 1.4 \) and \( Q^l = 0.95 Q^n \), in c.g.s. units, the following approximate values result:

- If the lower layer is infinitely deep, \( L_\infty \) is 62 km. The depth below which the unstable wavelengths are bounded is 23 km. When the depth of the lower layer is 600 m, the upper limit of unstable wavelengths is 2 km.

This last value seems rather low but, in view of the various approximations involved, it is probably a reasonable estimate of a cloud diameter.

In conclusion, it appears that it is possible for lengths of cloud dimensions to oscillate unstably at an interface between an adiabatic and an isothermal layer. These oscillations permit the air in the lower layer to reach saturation in limited areas. Through the release of latent heat, the lower layer acts as a source of moisture, whereby cumulus clouds form over the ocean.

REFERENCES
