ESTIMATES OF THE VERTICAL VELOCITY BASED ON THE VORTICITY EQUATION

By Leon Sherman

Florida State University

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The vertical component of the vorticity equation may be written (see, for example, Sherman, 1952),

$$D\zeta_a/Dt = D\zeta/Dt + \beta v$$
$$= -\zeta_a \nabla_H \cdot V + \left(\frac{\delta c}{\delta z} + \frac{c}{r}\right)\frac{\delta w}{\delta n}$$
$$+ 2\Omega \cos\varphi \frac{\delta w}{\delta y} + c\frac{\delta \psi}{\delta z}\frac{\delta w}{\delta s} + N + F, \quad (1)$$

where c, ψ and w are, respectively, the horizontal speed and direction and the vertical speed of the wind, and $\delta/\delta s$, $\delta/\delta n$ and $\delta/\delta z$ are rates of change with respect to distance in the tangential, normal and vertical directions. For short term considerations we may ignore the solenoidal and frictional terms, N and F. Doing so, and dropping some terms of (1) demonstrably small (compared, say, to the βv term) we have, for systems in which the wind direction changes little with elevation:

$$D\zeta_a/Dt \approx -\zeta_a \nabla_H \cdot V_H + (\delta c/\delta z)\, \delta w/\delta n. \quad (2)$$

The last term on the right drops out for the case of horizontal flow. For that case, the sign of the horizontal divergence is determined by the sign of the individual rate of change of the absolute vorticity. Since this individual rate of change is zero along a constant absolute vorticity trajectory (CAVT), it has been useful to compare actual trajectories to CAVT's to determine the sign of $D\zeta_a/Dt$, and hence, for "horizontal" flow, to determine the sign of $\nabla_H \cdot V_H$. In practice, it has often been assumed that in the real atmosphere the terms of (2) involving the vertical velocity can be neglected and that the equation

$$D\zeta_a/Dt \approx -\zeta_a \nabla_H \cdot V_H \quad (3)$$

can be used for estimates of the vertical velocity. An example of this sort of reasoning is the explanation sometimes given for the rain area to the east-southeast of a cold low aloft (*e.g.*, at 500 mb). Such a rain area forms part of the cold low model which has been described by Palmen (1951).

The explanation usually advanced (*ibid.*) is essentially the following: Such cold lows are slowly moving systems whose shapes are relatively constant in the middle troposphere. Hence (2) ought to apply for short term considerations. Further, if the solid line in fig. 1 is a trajectory (in practice approximated by a nearly steady state pressure-contour line in the warm air), then it will typically be of shorter wavelength than a CAVT for a corresponding initial wind velocity and latitude. Hence if the curvature term of the vorticity dominates, the individual rate of change of absolute vorticity at point A of fig. 1 will be negative. If now (3) is assumed, then at A one would have horizontal divergence. If the map is above the level of nondivergence, this upper divergence will correspond to rising motion and hence to cloud and rain at A and any other point along the trajectory where the decrease in curvature is greater than that along a CAVT.[1] Similar considerations apply along other contours. This reasoning then leads to the expectation of a rain area where, indeed, one generally is found in practice. But I believe the explanation to be in error.

It has already been noted that (2) ought to apply, and the reasoning used to determine the sign of $D\zeta_a/Dt$ is not in question here. However the terms $-\zeta_a \nabla_H \cdot V_H$ and $(\delta c/\delta z)\, \delta w/\delta n$ are, in general, of the same order of magnitude (Sherman, 1952); hence one ought not discard the second term out of hand. In fact, for the estimate of the mid-tropospheric vertical velocity pattern, it seems reasonable to operate near the level of non-divergence, where the w pattern (and so, presumably, the $\delta w/\delta n$ pattern) is most intense. At this level, $\nabla_H \cdot V_H$ vanishes. So instead of reducing (2) to the form (3), we ought to reduce it to

$$D\zeta_a/Dt \approx (\delta c/\delta z)\, \delta w/\delta n. \quad (4)$$

The vertical shear of the wind, $\delta c/\delta z$, is typically positive in the part of the atmosphere under consideration. Hence at point A of fig. 1 we would expect $\delta w/\delta n$, the

[1] But note that if it is mid-tropospheric, the map may be below the level of non-divergence, and we arrive at the opposite conclusion—an uncertain situation for *mid*-tropospheric estimates!

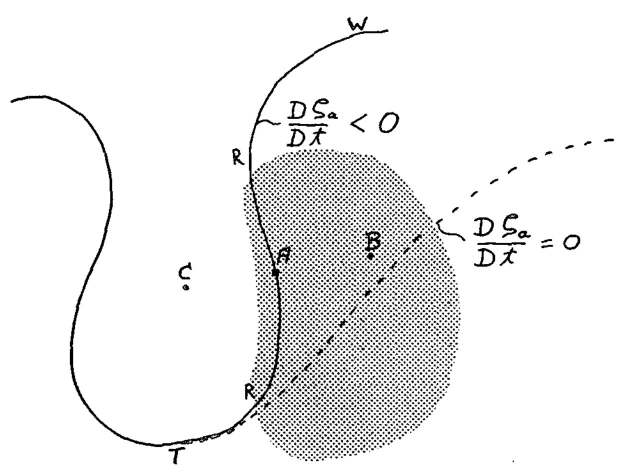

FIG. 1. Trajectories at upper level. Area of precipitation is stipled.

rate of change of w along the leftward pointing normal, to be negative. Hence we expect greater rising motion at point B than at C. If this difference is sufficiently large, then from the empirical upper bounds to the magnitude of the vertical velocity we might deduce actual subsidence in the cold dome, at C, as well as rising motion in the warm air, at B.

We can in fact draw this conclusion. To estimate $D\zeta_a/Dt \approx c[(\delta\zeta/\delta s) + (\delta f/\delta s)]$, we treat ζ as c/R, where R is the radius of curvature. The change in ζ from the trough to the wedge is $(c/-R) - (c/R) = -2c/R$, if we assume the same speed and absolute value of the radius of curvature for both trough and wedge. If we assume the length of the contour segment from T to W to be 20 deg lat, and the change in latitude from T to W to be 15 deg lat, taking $\delta c/\delta z \approx 10$ mps/3 km, $c \approx 25$ mps and R, the radius of curvature at the trough, to be 6 deg lat, we have:

$$\delta w/\delta n \approx c[-(2c/R) + \delta f/\delta s]/\delta w/\delta z \approx 2\tfrac{1}{4}\ \text{(cm/sec)/deg lat.}$$

Over a distance of 200–300 miles, we then expect a change in vertical velocity of the order of magnitude of $6\frac{1}{2}$–10 cm/sec. This is sufficient to include almost the entire range from greatest descending to greatest ascending motions usually measured (Panofsky, 1946). One could also make comparison to the expected vertical velocity in the upglide motion at a warm front. For a wind *component* of 5 mps, normal to the front, and a frontal slope of 1:250, one gets a vertical velocity of 2 cm/sec. Clearly, the term $(\delta c/\delta z)\ \delta w/\delta n$ is sufficient to explain the transition from subsidence in the cold dome to rain in the region to the east.

Furthermore, our argument, because it involves the derivative $\delta w/\delta n$ rather than w itself, can be applied to that contour corresponding approximately to the axis of the jet. This makes more valid the identification of vorticity with curvature involved in the argument which led to the sign of $D\zeta_a/Dt$. Having gotten $D\zeta_a/Dt$ in this more valid manner at one point, we need only assume it to be reasonably continuous in order to reach our conclusion concerning vertical velocities at points on adjacent contours.

Finally it may be noted that our argument (given a large estimate for the term $D\zeta_a/Dt$) implies a fairly definite edge to the rain area along RR (that is, along the edge of the cold air). It also leads to the subsidence within the cold dome.

In summary, estimates of the occurrence of such rain areas, based upon comparison of contours with CAVT's, should be expected to be valid—as they have been in practice. However the proper basis for these estimates made at mid-tropospheric levels is not the one that was previously assumed.

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