PLANETARY WAVES IN THE ATMOSPHERE

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ABSTRACT

A numerical method is presented for solving the non-linear barotropic vorticity equation in spherical coordinates. It is shown that, when the stream function is expressed as a sum of surface spherical harmonics, the barotropic vorticity equation gives rise to harmonic tendency equations which express the time rate of change of the harmonic coefficients as functions of all the harmonic coefficients. The harmonic tendency equations may be used in an iterative process to find the flow pattern at a future time from a knowledge of the harmonic coefficients at an initial time.

The amount of computation is reduced greatly if the planetary waves are regarded as composed of perturbations superimposed on a steady zonal flow in which the angular velocity varies with colatitude. An example based on an actual synoptic situation is given.

The effect of large scale lateral mixing is to curb the development of harmonics of large degree.

1. Introduction

It is well known from the fundamental work of Rossby (1939) and Haurwitz (1940a; 1940b) that the movement of planetary waves may be predicted, at least to a first approximation, with the aid of the barotropic vorticity equation,

$$\frac{\partial \zeta}{\partial t} = -\,\boldsymbol{v}\cdot\nabla(\zeta + f), \tag{1}$$

where ζ is the vertical component of the relative vorticity, $\boldsymbol{v}$ the velocity, and f is the Coriolis parameter.

Although a number of investigators have extended the theory of planetary waves by studying more general solutions of (1) than those obtained by Rossby and by Haurwitz, it was not until the recent work of Charney *et al* (1950) that the integration of the non-linear equation (1) under general conditions was carried out. In their method, the right-hand side of (1) is evaluated at the initial time by finite differences, and (1) is then regarded as a Poisson equation in the height tendency of a constant-pressure surface. The Poisson equation is solved, the height extrapolated for a short time-interval, and the procedure repeated as many times as necessary to produce a forecast for some desired time.

Another method, to be presented here, differs principally from that of Charney *et al* in that it is unnecessary to evaluate the right-hand side of (1) by finite differences either at the initial time or at any stage of the iteration. This advantage is gained at the cost of evaluating a large number of constants. However, once these constants have been calculated, they may be used in future forecasts and in theoretical studies such as that of barotropic instability.

2. The harmonic tendency equation

For a spherical earth, (1) is

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{a}\left(v_\theta \frac{\partial}{\partial \theta} + \frac{v_\lambda}{\sin\theta}\frac{\partial}{\partial \lambda}\right)(\zeta + 2\omega\cos\theta), \tag{2}$$

and

$$\zeta = \frac{1}{a\sin\theta}\left[\frac{\partial}{\partial\theta}(v_\lambda \sin\theta) - \frac{\partial v_\theta}{\partial\lambda}\right],$$

where θ is the colatitude, λ the longitude, a the radius of the earth, ω the angular velocity of the earth's rotation, and v_θ and v_λ are the components of the velocity in the directions of increasing θ and λ, respectively. If a stream function ψ is introduced by

$$v_\lambda = \frac{1}{a}\frac{\partial\psi}{\partial\theta} \quad\text{and}\quad v_\theta = -\frac{1}{a\sin\theta}\frac{\partial\psi}{\partial\lambda},$$

equation (2) becomes

$$\nabla_h^2 \frac{\partial\psi}{\partial t} = \frac{1}{a^2\sin\theta}\left(\frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial\lambda}\right) \times (\nabla_h^2\psi + 2\omega\cos\theta), \tag{3}$$

where

$$\nabla_h^2 = \frac{1}{a^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\lambda^2}\right]$$

is the surface spherical Laplacian operator.

Let

$$Y_n^m = e^{im\lambda} P_n^m,$$

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where P_n^m is the associated Legendre function of the first kind of order m and degree n which has been normalized to unity, *i.e.*,

$$\int_0^\pi P_n^m P_s^m \sin\theta\, d\theta = \delta_{n,s},$$

where

$$\delta_{n,s} = 1 \quad \text{if} \quad n = s$$

$$\delta_{n,s} = 0 \quad \text{if} \quad n \neq s.$$

The order may be negative, and

$$P_n^{-m} = (-1)^m P_n^m.$$

Then Y_n^m is a surface spherical harmonic of order m and degree n, and satisfies the equation

$$a^2\nabla_h^2 Y_n^m = -n(n+1)Y_n^m.$$

At a particular time t', the stream function of the planetary flow may be given by the finite sum

$$\psi_{t=t'} = a^2\omega \sum_{n=|m|}^{n'} \sum_{m=-m'}^{m'} K_n^m(t')\, Y_n^m, \qquad (4)$$

where m' and n' are positive integers which are chosen to eliminate disturbances of smaller scale than planetary waves. The harmonic coefficients K_n^m are complex, and the condition that ψ be real is

$$K_n^{-m} = (-1)^m \overline{K_n^m},$$

where $\overline{K_n^m}$ is the conjugate of K_n^m. At the time t', the stream-function tendency is

$$\left.\frac{\partial\psi}{\partial t}\right]_{t=t'} = a^2\omega \sum_{n=|m|}^{n''} \sum_{m=-m''}^{m''} \left.\frac{dK_n^m}{dt}\right]_{t=t'} Y_n^m, \qquad (5)$$

where m'' and n'' are positive integers, and where it is not necessarily true that $m' = m''$ and $n' = n''$. The quantity dK_n^m/dt is the harmonic tendency.

If the expressions for $\psi_{t=t'}$ and $\partial\psi/\partial t]_{t=t'}$ are substituted into (3), it is seen that

$$\sum_{n=|m|}^{n''} \sum_{m=-m''}^{m''} n(n+1) \left.\frac{dK_n^m}{dt}\right]_{t=t'} Y_n^m$$
$$= 2i\omega \sum_{n=|m|}^{n'} \sum_{m=-m'}^{m'} mK_n^m(t')\, Y_n^m$$
$$+ \frac{i\omega}{2\sin\theta} \sum_{s=|r|}^{n'} \sum_{r=-m'}^{m'} \sum_{k=|j|}^{n'} \sum_{j=-m'}^{m'}$$
$$\times \Big\{ [s(s+1) - k(k+1)] K_k^j(t')\, K_s^r(t')$$
$$\times e^{i(j+r)\lambda} \left[jP_k^j \frac{dP_s^r}{d\theta} - r \frac{dP_k^j}{d\theta} P_s^r \right] \Big\},$$

where j and r are dummy indices for m, and k and s are dummy indices for n, which must be introduced because of the non-linearity of (3). If both sides of the above equation are multiplied by $Y_n^{-m}\sin\theta$ and integrated with respect to λ from 0 to 2π, and with respect to θ from 0 to π, it follows from the resulting equation that

$$\left.\frac{dK_n^m}{dt}\right]_{t=t'} = \frac{2i\omega m K_n^m(t')}{n(n+1)}$$
$$+ \frac{i\omega}{2} \sum_{s=|r|}^{n'} \sum_{r=-m'}^{m'} \sum_{k=|j|}^{n'} \sum_{j=-m'}^{m'} K_k^j(t')\, K_s^r(t')\, H_{k,n,s}^{j,m,r}, \qquad (6)$$

where $H_{k,n,s}^{j,m,r}$ is zero unless

$$j + r = m. \qquad (7)$$

When the above condition is satisfied,

$$H_{k,n,s}^{j,m,r} = \frac{s(s+1) - k(k+1)}{n(n+1)} \int_0^\pi P_n^m$$
$$\times \left(jP_k^j \frac{dP_s^r}{d\theta} - r\frac{dP_k^j}{d\theta} P_s^r \right) d\theta. \qquad (8)$$

It will be shown in the next section that $H_{k,n,s}^{j,m,r}$ is zero unless

$$k + n + s = \text{odd integer}, \qquad (9)$$

and

$$|k - s| < n < k + s. \qquad (10)$$

Therefore, it is evident that

$$m'' \le 2m', \quad n'' \le 2n' - 1. \qquad (11)$$

The quantities $H_{k,n,s}^{j,m,r}$ will be called the interaction coefficients, and equation (6) the harmonic tendency equation.

The values of K_n^m may be found for any time $t \neq t'$ by iterating with the equation

$$K_n^m(t + \Delta t) = K_n^m(t - \Delta t) + 2\,\Delta t \frac{dK_n^m(t)}{dt}, \qquad (12)$$

except at the first step, where uncentered time differences must be used. At each stage of the iteration the number of harmonics would be increased according to the inequalities (11), and this would have the effect of introducing disturbances whose dimensions are smaller than the wavelengths of planetary waves. However, it has been shown by Charney *et al* (1950) that these disturbances may be neglected, provided that the ratio of the diameter of the largest neglected disturbance and Δt is sufficiently large.

Equations (6) and (12) permit the calculation of planetary flow patterns without employing the concept of phase velocity. However, since this concept has a central position in the literature of planetary waves, it is of interest to show that the phase velocity may be calculated when K_n^m and dK_n^m/dt are known.

If α_n^m is the instantaneous complex angular phase velocity,

$$K_n^m(t) = K_n^m(0) \exp\left(-im\int_0^t \alpha_n^m\, dt\right).$$

If the above equation is differentiated logarithmically with respect to time, it is seen that

$$\alpha_n^m = \frac{i}{mK_n^m}\frac{dK_n^m}{dt}. \qquad (13)$$

In general, α_n^m will be complex and will vary with time. Cases for which α_n^m is a real constant have been given by Ertel (1943), Craig (1945) and Neamtan (1946). The work of Neamtan includes that of Ertel and Craig as special cases.

For the type of flow studied by Neamtan,

$$\psi/(a^2\omega) = K_1^0 P_1^0 + \sum_{m=-n}^{n} K_n^m Y_n^m. \qquad (14)$$

From (6) and (7) it follows that

$$\frac{dK_n^m}{dt} = \frac{2i\omega m K_n^m}{n(n+1)} + i\omega K_1^0 K_n^m H_{1,n,n}^{0,m,m} + \frac{i\omega}{2}\sum_{r=-n}^{n}\sum_{j=-n}^{n} K_n^j K_n^r H_{n,n,n}^{j,m,r}.$$

From (8) it is seen that $H_{n,n,n}^{j,m,r} = 0$, and

$$H_{1,n,n}^{0,m,m} = \frac{n(n+1)-2}{n(n+1)} m(3/2)^{\frac{1}{2}}.$$

Therefore,

$$\alpha_n^m/\omega = \frac{2(3/2)^{\frac{1}{2}}K_1^0 - 2}{n(n+1)} - (3/2)^{\frac{1}{2}}K_1^0. \qquad (15)$$

Since the angular velocity of the zonal flow resulting from the first term on the right-hand side of (14) is $-\omega(3/2)^{\frac{1}{2}}K_1^0$, equation (15) is identical to that derived by Neamtan. Equation (15) was first given by Haurwitz (1940b) for linearized flow where the unperturbed motion is a zonal flow of constant angular velocity. The formal equivalence of the results of Haurwitz and Neamtan results from the vanishing of $H_{n,n,n}^{j,m,r}$.

3. Calculation of the interaction coefficients [4]

Since the interaction coefficients are zero when (7) is not satisfied, it will be assumed in this section that (7) holds.

Let

$$L_{k,n,s}^{j,m,r} = \int_0^\pi P_n^m \left(jP_k^j \frac{dP_s^r}{d\theta} - r\frac{dP_k^j}{d\theta}P_s^r\right) d\theta. \qquad (16)$$

[4] The reader who wishes to omit this section may pass directly to section four.

Then

$$H_{k,n,s}^{j,m,r} = \frac{s(s+1) - k(k+1)}{n(n+1)} L_{k,n,s}^{j,m,r}.$$

The integral (16) occurs in papers on terrestrial magnetism by Elsasser (1946a; 1946b; 1947) and Bird (1949). Bird gives a table from which the values of $L_{k,n,s}^{j,m,r}$ (for k, n and s less than or equal to four) may be obtained. The table was constructed by expressing the integrand of integral (16) as the product of $\sin\theta$ and a polynomial in $\cos\theta$. For higher degrees, it is desirable to have a formula for the value of integral (16). A method for obtaining such a formula was given by Infeld and Hull (1951). Their method contains an error, and another method will be given here.

It follows directly from (16) that

$$L_{k,n,s}^{j,m,r} = -L_{k,n,s}^{-j,-m,-r} = -L_{s,n,k}^{r,m,j}.$$

It may be shown, by integration by parts, that

$$L_{k,n,s}^{j,m,r} = (-1)^j L_{k,s,n}^{-j,r,m} = (-1)^r L_{n,k,s}^{m,j,-r}.$$

Therefore in the following evaluation of the integral (16) it may be assumed, without any loss of generality, that j, m and r are non-negative.

By definition,

$$P_k^j = \left[\frac{(k-j)!}{(k+j)!}\right]^{\frac{1}{2}} \sin^j\theta \frac{d^jP_k^0}{d(\cos\theta)^j}.$$

If the above expression for P_k^j, and the similar expression for P_s^r, are substituted into (16), it follows that

$$L_{k,n,s}^{j,m,r} = E_{k,n,s}^{j,m,r} - E_{s,n,k}^{r,m,j},$$

where

$$E_{k,n,s}^{j,m,r} = r[(k+j+1)(k-j)]^{\frac{1}{2}}\int_0^\pi P_k^{j+1} P_n^m P_s^r\, d\theta.$$

With the aid of the identity (MacRobert, 1948, p. 97)

$$\frac{dP_k^0}{d\cos\theta} = (2k+1)^{\frac{1}{2}}\sum_q (2q+1)^{\frac{1}{2}} P_q^0,$$

where

$$q = k-1,\ k-3,\ k-5,\ \cdots,\ 1 \text{ or } 0,$$

it may be shown that

$$E_{k,n,s}^{j,m,r} = r\left[\frac{(k-j)!\,(2k+1)}{(k+j)!}\right]^{\frac{1}{2}} \times \sum_q \left[\frac{(2q+1)(q+j)!}{(q-j)!}\right]^{\frac{1}{2}} \int_0^\pi P_q^j P_n^m P_s^r \sin\theta\, d\theta,$$

where

$$q = k-1,\ k-3,\ k-5,\ \cdots,\ j+1 \text{ or } j.$$

It has been shown by Gaunt (1929) and Infeld and

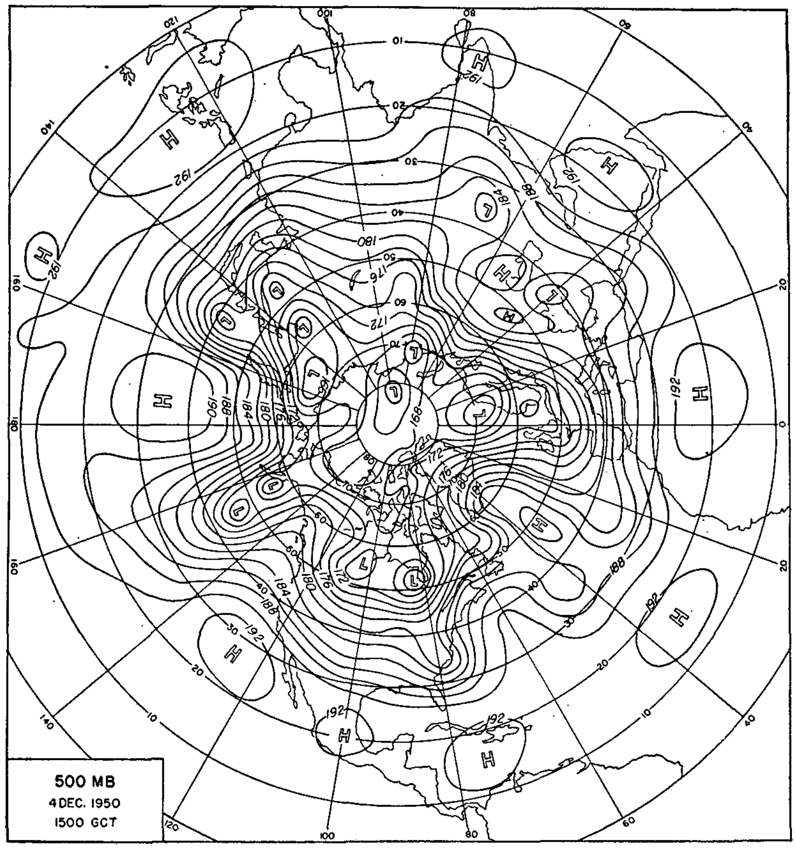

FIG. 1. 500-mb chart for 1500 GCT 4 December 1950. Height contours in hundreds of feet.

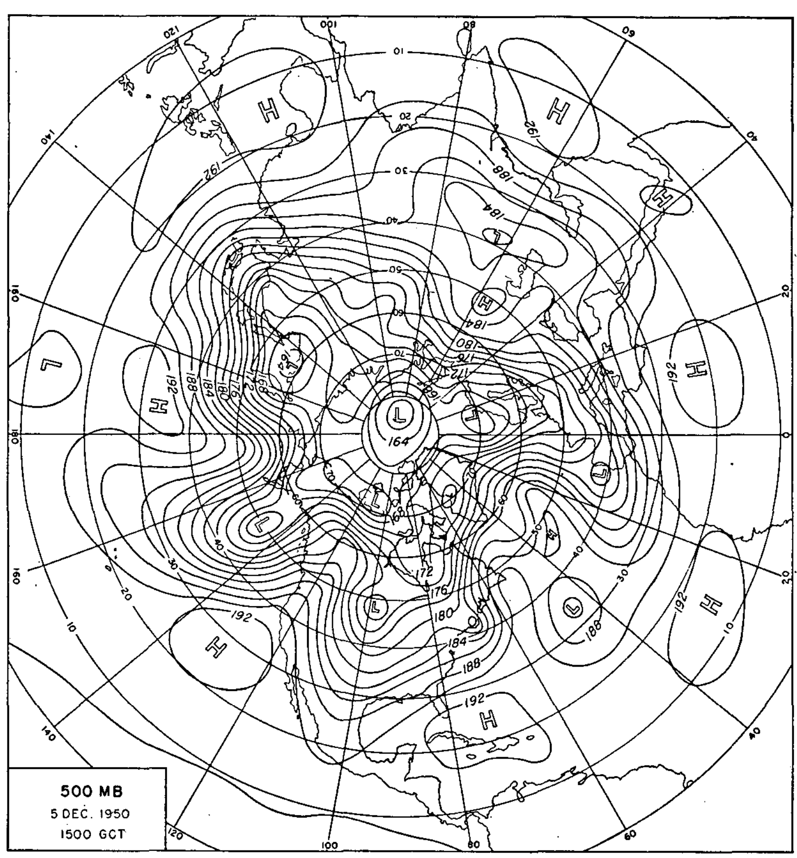

FIG. 2. 500-mb chart for 1500 GCT 5 December 1950. Height contours in hundreds of feet.

Hull (1951) that

$$\int_0^\pi P_q^j P_n^m P_s^r \sin\theta\, d\theta$$

$$= \frac{(s+q-n-1)!!\,[(2n+1)(2q+1)(2s+1)]^{\frac{1}{2}}}{(s+n-q)!!\,(q+n-s)!!\,(n+q+s+1)!!}$$

$$\times \left[\frac{(n+m)!\,(n-m)!\,(q-j)!\,(s-r)!}{2(q+j)!\,(s+r)!}\right]^{\frac{1}{2}}$$

$$\times \sum_{h=0}^{n-m} \frac{(-1)^{\frac{1}{2}(s-q+n)+r+h}(s+r+h)!\,(q+n-r-h)!}{(n-m-h)!\,h!\,(s-r-h)!\,(q-n+r+h)!},$$

where

$$n!! = n(n-2)(n-4)\cdots 2 \text{ or } 1,$$

$$0!! = (-1)!! = 1.$$

The above integral vanishes unless $q + n + s$ = even integer and $|s - q| \leq n \leq s + q$.

From these two conditions it is evident that $E_{k,n,s}^{j,m,r}$ is zero unless

$$k + n + s = \text{odd integer}, \tag{9}$$

and

$$n < k + s.$$

In addition, when $s \geq k$, $E_{k,n,s}^{j,m,r}$ is zero unless $s - k < n$. However, since $E_{k,n,s}^{j,m,r}$ may also be expressed as a sum of integrals of the type

$$\int_0^\pi P_k^{j+1} P_n^m P_q^{r-1} \sin\theta\, d\theta,$$

where

$$q = s-1, s-3, s-5, \cdots, r \text{ or } r-1,$$

it follows that, when $s \leq k$, $E_{k,n,s}^{j,m,r}$ is zero unless $k - s < n$. Consequently $E_{k,n,s}^{j,m,r}$ is zero for $s \gtreqless k$, unless

$$|k - s| < n < k + s. \tag{10}$$

Therefore, $H_{k,n,s}^{j,m,r}$ is zero unless (9) and (10) are satisfied.

4. Linearized flow

Because of the large number of interaction coefficients that would be required, an electronic computor with a large memory capacity would be needed to apply the non-linear method of section two to the large-scale flow of a hemisphere. However, because of the zonal nature of the large-scale flow at the non-divergent level, it is possible to reduce greatly the required number of interaction coefficients by a linearization procedure.

It will be assumed that

$$\psi = \bar{\psi} + \psi',$$

where

$$\bar{\psi} = a^2\omega \sum_{n=0}^{n'} K_n^0 P_n^0$$

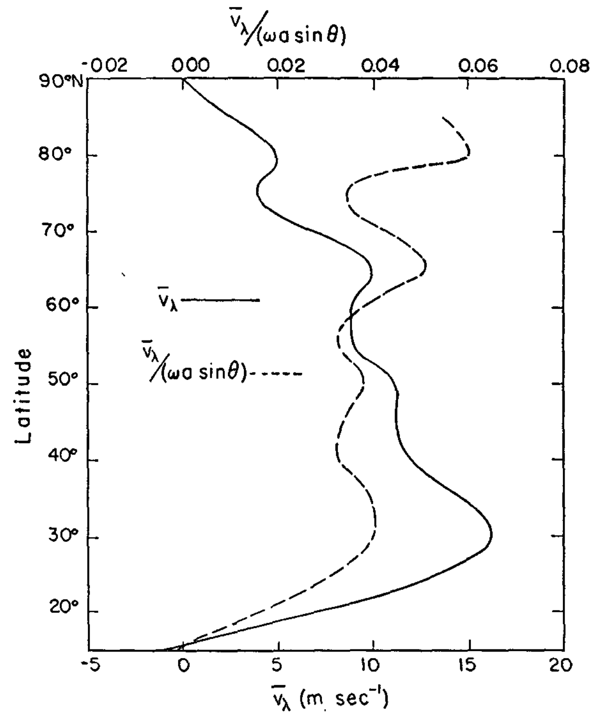

FIG. 3. Geostrophic zonal-velocity and angular-velocity profiles at 500 mb, 1500 GCT 4 December 1950.

is the unperturbed stream function, and

$$\psi' = a^2\omega \sum_{n=|m|}^{n'} \sum_{m=-m'}^{m'} K_n^m Y_n^m, \quad m \neq 0,$$

is the perturbation stream-function. The values of K_n^0 do not vary with time. The unperturbed flow is, then, a steady zonal flow in which the angular velocity varies with colatitude. The values of $|K_n^m|$, for $m \neq 0$, are very small in comparison with values of K_n^0. Therefore, every term in the summation on the right-hand side of (6) which does not contain the coefficient of a zonal harmonic as one of its factors may be neglected, and the harmonic tendency equation becomes

$$\frac{dK_n^m}{dt} = i\omega \sum_{s=|m|}^{n'} K_s^m G_{n,s}^m, \tag{17}$$

where

$$G_{n,s}^m = \frac{2m\delta_{n,s}}{n(n+1)} + \sum_{k=1}^{n'} K_k^0 H_{k,n,s}^{0,m,m}.$$

As an example, the synoptic situations in figs. 1 and 2, showing the 500-mb surface for the northern hemisphere for 1500 GCT 4 December 1950 and 1500 GCT 5 December 1950, respectively, will be considered. The geostrophic zonal-velocity and angular-velocity profiles for 4 December are shown in fig. 3.

The coefficients of the zonal harmonics of the stream function are given by

$$K_n^0 = -\frac{1}{2\pi a\omega[n(n+1)]^{\frac{1}{2}}} \times \int_0^\pi \int_0^{2\pi} v_\lambda P_n^1 \sin\theta\, d\lambda\, d\theta. \tag{18}$$

If it is assumed that v_λ is symmetrical with respect to the equator, and zero between 15°N and 15°S, it is seen that for 4 December

$$\bar{\psi}/(a^2\omega) = -0.0146\, P_1^0 - 0.0047\, P_3^0 + 0.0017\, P_5^0 + \cdots.$$

The coefficients of the tesseral and sectorial harmonics of the stream function are given by

$$K_n^m = \frac{i(-1)^m}{2\pi m a\omega}\int_0^{\pi}\int_0^{2\pi} v_\theta\, Y_n^{-m} \sin^2\theta\, d\lambda\, d\theta, \qquad m \neq 0. \quad (19)$$

The determination of the values of K_n^m for $m \neq 0$, thus, requires a surface spherical harmonic analysis[5] of $v_\theta \sin\theta$.

It is seen from the map of 4 December (fig. 1) that the shape of the planetary waves varies somewhat in different parts of the hemisphere. This is due to the perturbation stream-function being composed of a variety of harmonics of different orders and degrees, which may be determined by application of the above equation. However, there are properties that are common to all the planetary waves and allow perturbations, typical of the hemispherical perturbations causing the planetary waves, to be devised without carrying out the surface spherical harmonic analysis.

On 4 December the predominant wave number is five, and the troughs and ridges appear to be due to perturbations which extend from low latitudes to the pole with maximum intensity in middle latitudes. A perturbation of this type is shown in fig. 4. The zero of longitude is arbitrary in fig. 4, *i.e.*, the perturbation may be regarded as typical of that causing any particular wave if the zero longitude of the perturbation coincides with the ridge line of the wave. The analytical representation of this initial perturbation is

$$\psi' = 2a^2\omega\eta \cos 5\lambda\, (P_6^5 + 0.70\, P_8^5),$$

where η is a real constant which is small in comparison with values of K_n^0. Therefore, for the initial perturbation,

$$K_6^{\pm 5} = \pm\,\eta, \quad \text{and} \quad K_8^{\pm 5} = \pm\, 0.70\,\eta.$$

This particular combination of harmonics has been chosen so as to have single cells between equator and pole, and so that ψ' will have its maximum values in middle latitudes.[6]

If (12) and (17) are applied, with Δt equal to three hours and the values of Re K_n^5 and Im K_n^5 rounded off to two decimal places during the computation, it is

[5] See Haurwitz and Craig (1952) for a comprehensive discussion of methods of surface spherical harmonic analysis.

[6] The single harmonic cos 5λ P_6^5 has single cells between equator and pole, but its maximum values are in low latitudes.

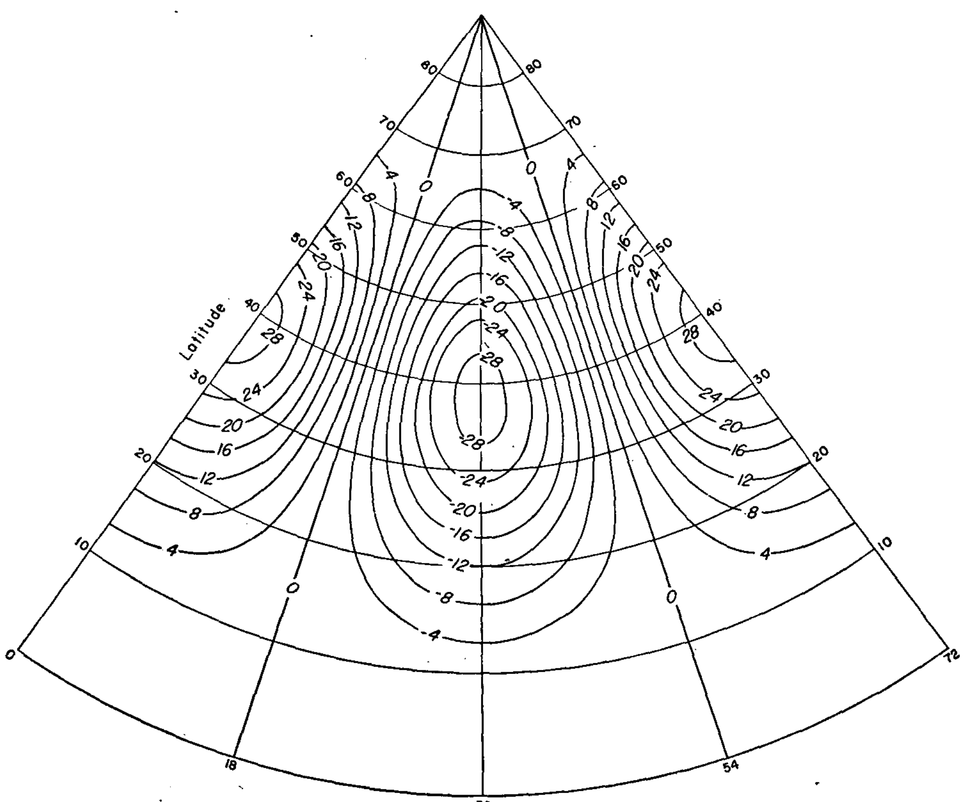

FIG. 4. Perturbation streamlines at initial time. Values in units of 0.1 $\eta a^2\omega$.

found that after twenty-four hours

$$\begin{aligned}\psi'/(2a^2\omega\eta) = \cos 5\lambda\ (0.75\, P_6^5 \\ + 0.70\, P_8^5 - 0.02\, P_{10}^5 + \cdots) \\ + \sin 5\lambda\ (-\,0.50\, P_6^5 + 0.07\, P_8^5 \\ + 0.12\, P_{10}^5 - 0.04\, P_{12}^5 + \cdots).\end{aligned}$$

The round-off procedure was sufficient to keep the growth of new harmonics from becoming excessive. However, if the iteration were continued past twenty-four hours, harmonics of degree greater than twelve could have been neglected without leading to computational instability.

At the initial time the stream function is antisymmetrical with respect to the equator, and it continues to be because of the condition given by (9).

From (7) it is seen that, if the linearization assumption had not been made, the harmonics $Y_n^{\pm 5}$ could have interacted to produce changes in the zonal harmonics and to produce new tesseral harmonics whose orders are multiples of five. The changes in the zonal harmonics, due to such interactions, are probably small in comparison with the changes in the zonal flow due to other causes, and the new tesseral harmonics of order ten or greater could be neglected since their wavelengths are smaller than those of planetary waves. It is evident, then, why the linearization assumption may be used in the computation of the planetary flow.

The perturbation streamlines after twenty-four hours are shown in fig. 5. It is seen that the portion of the perturbation in high latitudes has moved very little, but that the portion in low latitudes undergoes retrogression which increases with decreasing latitude. This may be compared to the movement of the planetary waves on the 500-mb surface between 1500 GCT 4 December 1950 and 1500 GCT 5 December 1950. It is seen, likewise, that the troughs and ridges do not move much in high latitudes, but in low latitudes retrogression occurs which increases southward.

5. Effect of friction

Although the computation requires the neglect of harmonics of large degree, their neglect may be justified by physical considerations.

The assumptions underlying (1) are that the atmosphere is barotropic and non-viscous, and that its motion is non-divergent and horizontal. If the non-viscous assumption is dropped, and if it is assumed that the velocity does not vary with height, it follows from the equations of motion that

$$\frac{d}{dt}(\zeta + 2\omega\cos\theta) = -\,\nu\,\mathrm{curl}_a\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{v}, \quad (20)$$

where ν is the kinematic coefficient of eddy viscosity

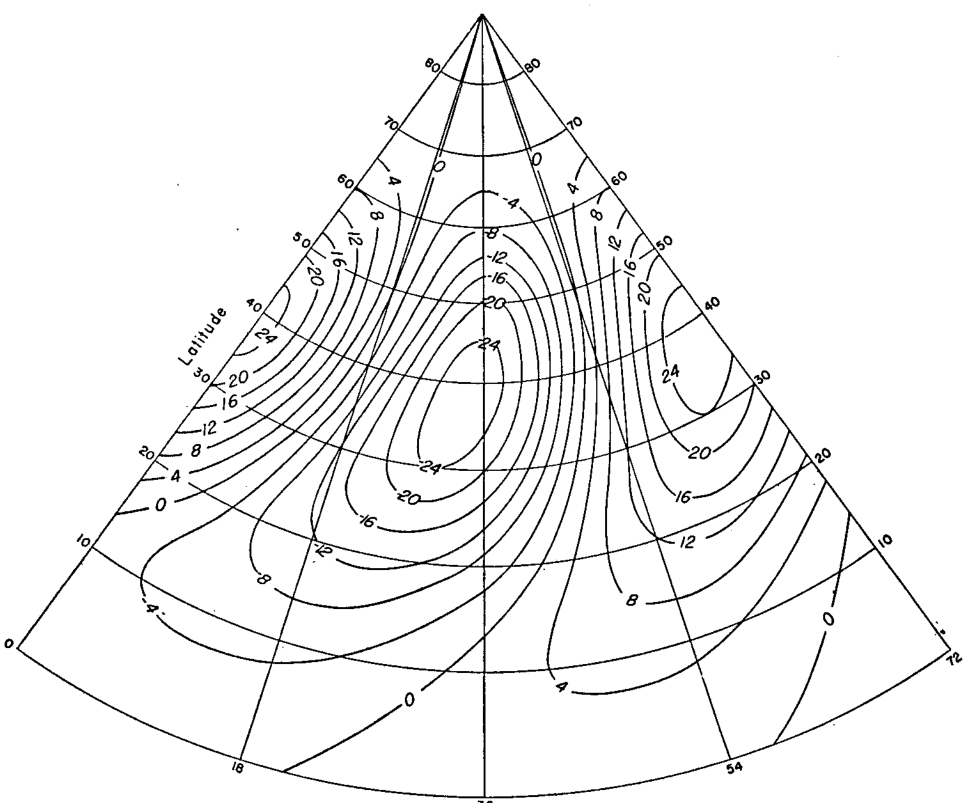

FIG. 5. Perturbation streamlines after twenty-four hours. Values in units of 0.1 $\eta a^2\omega$.

associated with large-scale lateral shear, and the suffix a indicates the component in the radial direction in a spherical coordinate system.

Jeffrey (1915) has given an equation from which the components of curl curl A, where A is any vector, may be determined in any orthogonal curvilinear coordinate system. When specialized to the radial component in a spherical coordinate system, this equation of Jeffrey is

$$\mathrm{curl}_a\,\mathrm{curl}\,\boldsymbol{A} = \partial(\mathrm{div}\,\boldsymbol{A})/\partial a - \nabla^2 A_a + (2A_\theta \cot\theta)/a^2 + (2/a^2)\,\partial A_\theta/\partial\theta + (2/a^2 \sin\theta)\,\partial A_\lambda/\partial\lambda.$$

If now A is taken as curl $\boldsymbol{v}$, with the above assumptions

$$A_a = \zeta, \quad A_\theta = -v_\lambda/a \quad \text{and} \quad A_\lambda = v_\theta/a,$$

and (20) becomes

$$d(\zeta + 2\omega\cos\theta)/dt = \nu\nabla_h^2\zeta + 2\nu\zeta/a^2. \qquad (21)$$

From the above equation it is seen that the term

$$-\nu[n(n+1) - 2]\,K_n^m/a^2$$

must be added to the right-hand side of the harmonic tendency equation, and that the damping effect of lateral mixing is much greater for harmonics of large degree than for harmonics of small degree.

The second term on the right-hand side of (21), which is not present when a plane earth is considered, is necessary so that the terms Y_1^0 and $Y_1^{\pm 1}$ in the stream function, which only contribute to the atmosphere's motion as rigid-body rotations relative to the earth, do not produce lateral stresses.

6. Concluding remarks

In the preceding sections the harmonic tendency equation has been derived, and it was shown how it may be applied to the calculation of planetary flow patterns. Although it is possible to carry out calculations for non-linear flow, it was shown in section four that the amount of computation is reduced greatly if the planetary waves are regarded as composed of perturbations superimposed on a steady zonal flow in which the angular velocity varies with colatitude.

In the example of section four, the coefficients of the zonal harmonics were computed from the actual zonal flow. In lieu of use of the actual coefficients of the tesseral harmonics, computations were carried out for the coefficients of a hypothetical perturbation, which may be regarded as typical of the actual perturbations. In the computation of actual hemispherical prognostic charts, the coefficients of the initial perturbations may be computed from (19) by using either actual or geostrophic values of v_θ and assuming that v_θ is antisymmetrical with respect to the equator.

In section five it was shown that the effect of large-scale lateral mixing is to dampen the harmonics of large degree. This justifies on a physical basis the neglect of the harmonics of large degree, which is required by computational stability considerations.

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