THEORY OF THE MEAN ATMOSPHERIC PERTURBATIONS PRODUCED BY DIFFERENTIAL SURFACE HEATING

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ABSTRACT

When air flows over terrain where the surface temperature varies with position, non-adiabatic heat will be added or subtracted. We consider the time-dependent disturbances induced in a uniform basic current as the result of differential heating on a flat rotating earth.

The problem of obtaining a plausible mathematical-physical description of the heating function is then discussed. A method which was proposed in a previous paper, Stern and Malkus (1953), is re-examined and restated, to clarify the underlying assumptions and the point of departure from the classical theory of the eddy conduction of heat.

With this description of the heating function, it is shown that the mean motions may be specified in terms of an "equivalent mountain." This depends, in general, on the Coriolis parameter as well as the surface temperature, undisturbed wind speed, and the eddy conductivity of the heated region. The theory is applied to the small-scale sea-breeze problem, and it is shown that, by retaining the linearized advective term in the momentum equation, it is possible to explain the frequently observed phase relation between the diurnal temperature wave and the sea breeze without introducing friction. Additional derived relations for the hodograph suggest tests by means of future observations.

1. Introduction

The present paper is part of a theoretical investigation of the mean perturbations induced in an atmosphere as it moves over terrain which has a prescribed temperature variation. It follows a series of observational and theoretical studies of the air flow over a small island: Malkus and Bunker (1952), Malkus and Stern (1953), and Stern and Malkus (1953). On this scale, such well known phenomena as cloud streets and sea-breeze effects are produced by the land-water thermal contrast. One of the reasons for the intensive studies of these phenomena was the belief that they may furnish touchstones for the investigation of other meteorological phenomena where the mechanism of heat transfer is similar, but where the scale of the mean motions is much larger.

Fig. 1 is a schematic diagram of the model to be considered. A horizontally uniform, undisturbed current of air moves over a region of differential heating. If the land is at a constant temperature above the surrounding water, heat will be added and transported upward by turbulence at the windward shore, and the isolines of heating are schematically indicated by the region B. Similarly, in passing from the land to the water, the air will be cooled. The heating produced by the advection of the air across the temperature gradient at the surface is confined to a relatively shallow, turbulent ground layer. Within this layer, the undisturbed state of the atmosphere will be characterized by a wind speed and stability which are constants in the vertical as well as in the horizontal. In section 2, below, the hydrodynamic perturbation equations for the lower atmosphere are derived in terms of an unspecified distribution of heat sources. Before these equations can be solved, it is necessary to specify the distribution of heating. In section 3, below, the turbulent heating process is considered, and a differential equation is derived. In this manner, it is possible to determine an analytic expression for the heating function in terms of the temperature distribution along the ground. When the results of this section are combined with that of section 2, it is seen that the vertical displacement of the streamlines from the undisturbed state may be broken down into two components. The first one is intimately connected with the turbulent heating process; indeed, it satisfies the same type of conduction equation as the heating function, and decays rapidly in the vertical. The second component satisfies the homogeneous equation (which is the same as for the adiabatic flow of air over a mountain), and is only related to the heating process by the fact that the two components taken together must give zero vertical velocity at the flat ground. With this single boundary condition, we are able to introduce the so-called "equivalent mountain," which depends only on the low-level structure of the atmosphere. Then the perturbations produced by the surface heating, at large elevations and at all points where the conduction component is zero, are the same as the corresponding motions that are produced in the adiabatic flow over the equivalent mountain.

2. The equations of motion

Fig. 1 is a schematic representation of the model, in which the undisturbed wind is shown as constant up to some height which is outside the turbulent ground layer. As long as this condition is fulfilled, it is unnecessary, for present purposes, to specify the location of the transition regions or the nature of the winds aloft. The generality of the upper boundary conditions, while of course preventing immediate solution of the equations, is deliberately strived for in this section. With these considerations, we now proceed to define the symbols used in the hydrodynamic equations, to state some of the less obvious simplifying assumptions, and then to solve the set of equations to obtain a linearized partial differential equation relating the mean vertical velocity to the distribution of heat sources.

Let ξ, η, ζ and t_1 be three rectangular coordinates and time (c.g.s. units), respectively, which will later be replaced by corresponding dimensionless coordinates x, y, z and t; let U and V be the low-level components of the undisturbed wind in the ξ and η directions, respectively, assumed constant; let u, v and w be the components of disturbed velocities in the ξ, η and ζ directions, respectively; let p_m, ρ_m and T_m be the undisturbed pressure, density and absolute temperature, respectively; let p, ρ and T be the disturbed pressure, density and temperature, respectively; and let Γ, α and $s = (\Gamma - \alpha)/T_m$ be the adiabatic lapse rate, undisturbed lapse rate and stability, respectively.

The ζ axis is taken as vertical, and the ξ axis is in the direction of the horizontal temperature gradient at the ground ($\partial T/\partial\eta = 0$). The derivation that follows pertains to air flowing over small-scale heat sources; hence we consider the Coriolis parameter (f) as constant, and furthermore all η-derivatives may be omitted in the hydrodynamic perturbation equations which are listed below:

$$D_1 u - fv = -\rho_m^{-1}\,\partial p/\partial\xi, \tag{1}$$

$$D_1 v + fu = 0, \tag{2}$$

$$D_1 w = -\rho_m^{-1}\,\partial p/\partial\zeta - (\rho/\rho_m)g, \tag{3}$$

$$\partial u/\partial\xi + \partial w/\partial\zeta = 0, \tag{4}$$

$$\rho/\rho_m = -T/T_m, \tag{5}$$

and

$$H = D_1 T + w(\Gamma - \alpha), \tag{6}$$

where

$$D_1 = \left(\frac{\partial}{\partial t_1} + U\frac{\partial}{\partial\xi}\right). \tag{7}$$

Equation (6) arises from the first law of thermodynamics, where $H(\xi, \eta, \zeta, t_1)$ is proportional to the rate at which non-adiabatic heat is being supplied at a given point, and the right-hand side of (6) is the linearized approximate value of the total derivative of potential temperature. Equation (5) is an approximate form of the equation of state (Charles' law). The more exact form of the gas law is obtained by adding the term p/p_m to the right-hand side of (5). It may be shown that inclusion of this would lead to a "damping" term in the final differential equation, which only becomes important at very large heights. The density variation in the continuity equation has been neglected, but its influence on the potential term in (3) is, of course, considered.

Eliminating v between (1) and (2), and then p by means of (3), one obtains

$$(D_1^2 + f^2)\frac{\partial u}{\partial\zeta} = D_1^2\frac{\partial w}{\partial\xi} + \frac{g}{\rho_m}D_1\frac{\partial\zeta}{\partial\xi}. \tag{8}$$

Now, using (5) and (6), we find

$$(D_1^2 + f^2)\frac{\partial u}{\partial\zeta} = D_1^2\frac{\partial w}{\partial\xi} - \frac{g}{T_m}\frac{\partial H}{\partial\xi} + gs\frac{\partial w}{\partial\xi}. \tag{9}$$

Finally, a stream function $U\psi$ may be introduced by means of (4), such that $w = U\,\partial\psi/\partial\xi$ and $u = -U\,\partial\psi/\partial\zeta$. Then, the final equation is

$$\left(\frac{\partial}{\partial t_1} + U\frac{\partial}{\partial\xi}\right)^2\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2}\right)\psi + f^2\frac{\partial^2\psi}{\partial\zeta^2} + gs\frac{\partial^2\psi}{\partial\xi^2} = \frac{g}{UT_m}\frac{\partial H}{\partial\xi}. \tag{10}$$

At this point, dimensionless coordinates (x, z, t) will be introduced to replace (ξ, ζ, t_1). Let

$$x = \xi(gs/U^2)^{\frac{1}{2}},\quad z = \zeta(gs/U^2)^{\frac{1}{2}},$$

and

$$t = t_1(gs)^{\frac{1}{2}}. \tag{11}$$

In addition, let

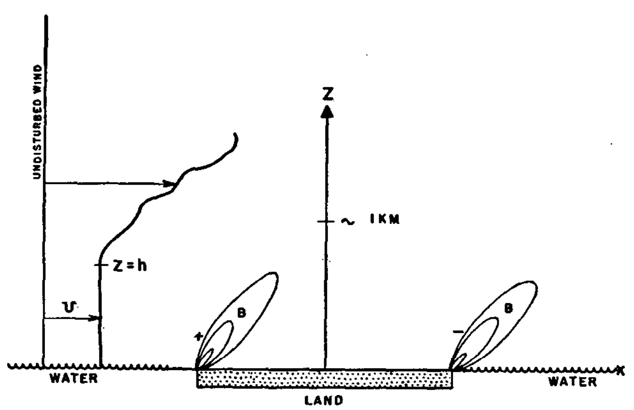

FIG. 1. Schematic diagram of model to be considered. Air approaches heat source with undusturbed wind profile as shown at left. If land is at constant temperature relative to water, heat is added to air as it crosses windward coast. Heat is transported upwards, and isolines for heating function are indicated by region B. Similarly, air is cooled in going from warm land to cool water. Problem is to predict mean vertical and horizontal displacements of air stream in terms of boundary value of surface temperature.

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right), \quad D = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right),$$

and $f_0^2 = f^2/gs$ = a non-dimensional Coriolis number.
Therefore, (10) becomes

$$D^2\,\nabla^2\psi + f_0^2\psi_{zz} + \psi_{xx} = (sT_m)^{-1}(gs)^{-\frac{1}{2}}\,\partial H/\partial x. \quad (12)$$

If the heat sources are removed, so that $H = 0$, and if the time derivatives are set equal to zero, (12) reduces to the same equation as that for the adiabatic flow of air over a mountain (Queney, 1948). However, the inhomogeneous term is the essential forcing function which produces the mean perturbations over flat heated terrain, and it must be specified by another relation before (12) can be solved.

3. Description of the heating function

To describe the turbulent heating, one might be tempted to use the classical equation of eddy conduction and set the total derivative of the potential temperature equal to a constant times the second derivative of the temperature with respect to height. It is well known that, although the eddy conductivity varies considerably in space and time, it is often possible to obtain a first orientation into many phenomena by assuming an effective constant value for this quantity. It is contended, however, that in the present type of problem, where the heating induces non-negligible mean vertical velocities, this approach is inapplicable, at least without considerable modification. To give a simple illustration of this inadequacy, consider the essentially adiabatic waves which occur outside the ground layer, *e.g.*, lee waves. These waves vary in magnitude in the vertical as well as the horizontal direction, and the second derivative of T with height is not zero. In fact, application of the simple eddy-conduction equation would lead to the untenable conclusion that the amount of heat added at a point far in the lee may, under certain circumstances, be comparable with the heat that is supplied at a point in the middle of the heated region B (of fig. 1). If one is to use the ideas of eddy conduction at all, it is necessary to distinguish between the temperature gradients that are maintained by turbulent transport and those that are due to adiabatic convective motions. This difficulty might, in principle, be surmounted by using the classical eddy equation with a conductivity that was much less outside B than inside; but because of the complicated shape of B, this formalism seems hopelessly complex to introduce into the hydrodynamic equations. On the other hand, an adequate formalism must insure that the heating function approaches zero at far distances from the source, even when mean vertical velocities due to the heating exist in these regions. (It is assumed, of course, that there are no other mechanisms, such as condensation, which are producing heat sources. These can be treated separately.) The following discussion of the heating function is an extension and elaboration of the method of Stern and Malkus (1953).

Because the temperature perturbation is due to a combination of two different mechanisms, namely turbulent transport and adiabatic convective motions, it is inconvenient to attempt to describe the heating in terms of this quantity as the dependent variable. Instead, the present formalism evolves about the heating function which is proportional to the divergence of the eddy flux, and we shall try to justify the major premise, that this can be determined independently of the mean vertical velocities which it produces. This is clearly the case at the lower boundary, as is seen from (6), and here the heating function can be determined from the temperature along and the shape of this boundary, and is not *explicitly* related to the vertical velocities that are produced aloft. What can be said about the variation of the heating function in the vertical?

In fig. 1, consider the effect of inserting a series of horizontal grids, or large rigid screens, into the field of motion. At any point the spacing of the gridwork is large compared with the mean eddy size, and small compared to the distance over which one averages to obtain the mean vertical velocity and temperature. Then the turbulent eddies will pass through the gridwork relatively unaffected, while the mean vertical velocities will be reduced towards zero. The screens are effectively a solid barrier to the mean motions. It is hypothesized that the eddy flux of heat, and in particular its divergence, is unaffected by variations of the mean perturbations due to the imposition of these constraints. In this way, it is possible to consider the extremely complex turbulent heating process independently; then, acting as a fixed driving force, the heating function produces streamline displacements as determined by (12) and the boundary conditions.

This hypothesis being accepted, a differential equation for H is now derived by applying the simple ideas of eddy conduction to the model in which the mean vertical velocities have been constrained to zero. Denote the temperature in this model by $T_e(\xi, \eta, \zeta, t_1)$, and note that for $\zeta \neq 0$ this is different from $T(\xi, \eta, \zeta, t_1)$, the temperature distribution in the model whose mean motions it is desired to investigate.

Applying the first law of thermodynamics and the eddy-conduction equation to the model in which the mean vertical velocities have been eliminated, we obtain the following relations:

$$H = \frac{\partial T_e}{\partial t_1} + U\frac{\partial T_e}{\partial \xi} + V\frac{\partial T_e}{\partial \eta}, \quad (13)$$

and

$$H = K\,\partial^2 T_e/\partial\zeta^2. \tag{14}$$

Upon elimination of T_e, there results

$$\frac{\partial H}{\partial t_1} + U\frac{\partial H}{\partial \xi} + V\frac{\partial H}{\partial \eta} = K\frac{\partial^2 H}{\partial \zeta^2}. \tag{15}$$

In the derivation of (15), it is assumed that the eddy conductivity (K) has a constant effective value for the heated region B. The fact that the measured K may be an order of magnitude less far outside B is of little moment in this formalism, because (15) insures that H decreases rapidly away from the ground. It is to be noted that, when the mean vertical velocities produced by the turbulent heating are actually negligible, (15) reduces to the familiar eddy-conduction equation, where temperature replaces H.

The value of the heating function along the ground follows directly from the first law of thermodynamics and is independent of any previous assumptions regarding the eddy transport:

$$H(\xi, \eta, 0, t_1) = \left(\frac{\partial}{\partial t_1} + U\frac{\partial}{\partial \xi} + V\frac{\partial}{\partial \eta}\right) \times T(\xi, \eta, 0, t_1). \tag{16}$$

This, of course, serves as one boundary condition for the solution of (15). As the other boundary condition, we require that H vanish as $z \to \infty$. It is seen, therefore, that (15) and (16) unite in one simple formalism the ideas of eddy transport of heat in the mixed region near the ground and adiabatic flow at large distances. A number of examples of the distribution of the heating function for various steady-state temperature distributions have been given by Stern and Malkus (1953).

Returning to the model in which the horizontal temperature gradient at the ground is in the direction of the x-axis, and writing (15) and (16) in terms of the dimensionless coordinates, we have

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) H = b^2 \frac{\partial^2 H}{\partial z^2},$$
$$H(x, 0, t) = (gs)^{\frac{1}{2}}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) T(x, 0, t), \tag{17}$$

where

$$b^2 = (K/U^2)(gs)^{\frac{1}{2}}.$$

4. Solution of the equation

Since H is specified in terms of the given thermal boundary conditions by (17), this may be solved first and then substituted in (12). If ψ and H are written as Fourier expansions whose typical terms are $\bar{\psi} e^{ikx} e^{i\lambda t}$ and $\bar{H} e^{ikx} e^{i\lambda t}$, where $\bar{H}$ is the amplitude of the harmonic with an x-wave number k, and a t-wave number λ, (17) becomes

$$d^2\bar{H}/dz^2 - i[(k+\lambda)/b^2]\bar{H} = 0,$$
$$\bar{H}(z=0) = (gs)^{\frac{1}{2}} i (k+\lambda)\bar{T}_0,$$
$$\bar{H} = (gs)^{\frac{1}{2}} i(k+\lambda)\bar{T}_0 \exp - [(\pm i)^{\frac{1}{2}} |k+\lambda|^{\frac{1}{2}} b^{-1} z]. \tag{18}$$

Here, $\bar{T}_0$ is the amplitude of the (k, λ) harmonic of the surface temperature distribution; $(\pm i)^{\frac{1}{2}} = e^{\pm\pi i/4}$, according as $(k + \lambda)$ is greater or less than zero; and $|k + \lambda|$ denotes the magnitude of the quantity. It is also to be noted that the boundary condition at infinity has been used, namely that H shall be finite and, in particular, zero.

If now the Fourier expansions are substituted in (12), together with (18), there results

$$[f_0^2 - (k+\lambda)^2]\, d^2\bar{\psi}/dz^2 - [k^2 - k^2(k+\lambda)^2]\bar{\psi} = -\{[k(k+\lambda)\bar{T}_0]/sT_m\} \times \{\exp - [(\pm i)^{\frac{1}{2}} |k+\lambda|^{\frac{1}{2}} b^{-1} z]\}. \tag{19}$$

Solving, one obtains

$$\bar{\psi} = \bar{M}(k, \lambda)[h(z, k, \lambda) - F(z, k, \lambda)], \tag{20}$$

where

$$\bar{M}(k, \lambda) = \frac{\bar{T}_0}{sT_m} \frac{k(k+\lambda)}{[(k+\lambda)^2 - f_0^2][\{i(k+\lambda)/b^2\} - r^2]},$$
$$r^2 = [k^2 - k^2(k+\lambda)^2]/[f_0^2 - (k+\lambda)^2], \tag{21}$$

$F(z, k, \lambda)$ = linear combination of e^{-rz} and e^{+rz},

and

$$h(z, k, \lambda) = \exp - [(\pm i)^{\frac{1}{2}} |k+\lambda|^{\frac{1}{2}} b^{-1} z].$$

If now the various harmonics are added together, the solution for the displacement ψ corresponding to an arbitrary temperature distribution along the ground is

$$\psi(x, z, t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \bar{M}(k, \lambda) e^{ikx} e^{i\lambda t} \times [h(z, k, \lambda) - F(z, k, \lambda)]\, dk\, d\lambda. \tag{22}$$

The solution is separated into two parts, ψ_1 and ψ_2, where $\psi = \psi_1 - \psi_2$,

$$\psi_1(x, z, t) = \int_{-\infty}^{+\infty} e^{i\lambda t}\, d\lambda \int_{-\infty}^{+\infty} \bar{M}(k, \lambda) e^{ikx} h(z, k, \lambda)\, dk, \tag{23}$$

and

$$\psi_2(x, z, t) = \int_{-\infty}^{+\infty} e^{i\lambda t}\, d\lambda \int_{-\infty}^{+\infty} \bar{M}(k, \lambda) e^{ikx} F(z, k, \lambda)\, dk. \tag{24}$$

The first component satisfies the heat conduction equation,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\psi_1 = b^2 \frac{\partial^2 \psi_1}{\partial z^2},$$

while the second component (ψ_2) satisfies the homogeneous differential equation obtained by setting

$H = 0$ in (12). Since $\psi(x, 0, t) = 0$ and $h(0, k, \lambda) = 1$, $F(0, k, \lambda) = 1$ and thus

$$\psi_2(x, 0, t) = M(x, t) = \int_{-\infty}^{+\infty} e^{i\lambda t}\, d\lambda \int_{-\infty}^{+\infty} \bar{M}(k, \lambda) e^{ikx}\, dk. \quad (25)$$

In the steady-state problem, $M(x)$ has been called the equivalent mountain function, since the component $\psi_2(x, z)$ is mathematically identical to the air flow over a mountain whose profile is given by $M(x)$. For the time-dependent problem, $M(x, t)$ cannot, strictly speaking, be referred to as an equivalent mountain, due to the difference between the stream function and the displacement of the streamlines. Thus, while $M(x, t)$ might rigorously be called the "equivalent ground-level stream function," we shall take the liberty of using the term "equivalent mountain function" for all cases, since this name gives physical insight into the dynamical effects produced by the heating.

It will be noted that (25) provides the boundary valve of the conduction component (ψ_1) as well as the mountain component (ψ_2); but aside from the fact that the first solution is necessary to satisfy the condition of zero vertical velocity at the ground, it is otherwise of little interest since it decreases exponentially with elevation. Thus, the problem of air flow over heated terrain is essentially reduced to the investigation of the equivalent mountain. In any particular problem, it is desirable that scale considerations be introduced to simplify the evaluation of the above integrals. For instance, table 1 indicates some important scale divisions and the approximations entailed in each.

TABLE 1. Scale considerations.

Scale	Characteristic horizontal length of heat source, L (km)	Simplifications in evaluating equivalent mountain
Small quasi-steady	$L \ll 2\pi f_0^{-1}\left(\frac{gs}{U^2}\right)^{-\frac{1}{2}}$	$f_0/k \ll 1$
	$L \ll 2\pi\lambda^{-1}\left(\frac{gs}{U^2}\right)^{-\frac{1}{2}}$	$\lambda/k \ll 1$
Middle unsteady	$L \approx 2\pi f_0^{-1}\left(\frac{gs}{U^2}\right)^{-\frac{1}{2}}$	$k^2 \ll 1$
	$L \approx 2\pi\lambda^{-1}\left(\frac{gs}{U^2}\right)^{-\frac{1}{2}}$	

5. The small-scale quasi-steady solution; Predictions on the sea breeze

We consider a flat coastal region, take the x-direction normal to the shore, and the origin at the shore. The surface temperature distribution is taken as a step function, so that the land exceeds the water temperature by τ deg. The temporal variation is represented by a finite number of Fourier harmonics, with the predominant one being the diurnal period ($P = 24$ hr); then $\lambda = (2\pi/P)(gs)^{-\frac{1}{2}}$ ($\lambda \approx 10^{-2}$) is the angular frequency of the temperature variation expressed in non-dimensional units. We now make the small-scale approximation (see table 1) and neglect λ/k and f_0/k in comparison with unity, this approximation being valid for heat sources whose characteristic horizontal length is much less than the distance traversed by a particle in one day if it travelled with the velocity of the wind. Equation (21) then becomes

$$\bar{M} = \frac{\bar{T}_0}{sT_m}\frac{1}{(ik/b^2) + 1 - k^2}.$$

If the ground temperature is given by

$$T_0(x, t) = \cos\lambda t \int_{-\infty}^{+\infty} \bar{T}_0(k)\, e^{ikx}\, dk,$$

the equivalent mountain is, from (25),

$$M(x, t) = \frac{\cos\lambda t}{sT_m}\int_{-\infty}^{+\infty} \frac{e^{ikx}\bar{T}_0(k)}{(ik/b^2) + 1 - k^2}\, dk. \quad (26)$$

This differs from the steady-state solution obtained by Stern and Malkus (1953) only in the presence of the $\cos\lambda t$ term, and we may use their results for the horizontal velocity perturbation, provided the present model is further restricted so that the undisturbed wind and stability remain constant to large heights. By expansion of their (46) about $x = 0$, and omission of higher-order terms which represent fine structure, the sea-breeze component perpendicular to the windward shore is found to be

$$u(x, z, t) = \frac{\tau}{sT_m}\cos\lambda t \frac{(gs)^{\frac{1}{2}}\cos z}{(a_2 - a_1)\pi} \times \int_{-a_1 x}^{-a_2 x} \frac{e^{-p}}{p}\, dp \quad (x \le 0), \quad (27)$$

provided x is small, and where $x = 0$ is at the windward shore,

$$\left.\begin{matrix} a_1 \\ a_2 \end{matrix}\right\} = \frac{1}{2b^2}\left[1 \pm (1 - 4b^4)^{\frac{1}{2}}\right],$$

$$b^2 = (K/U^2)(gs)^{\frac{1}{2}},$$

and $\tau\cos\lambda t$ = temperature excess of the island at a given time.

It is seen from (27) that the maximum sea-breeze component perpendicular to the coast is in phase with the maximum of the diurnal temperature wave. This is to be compared with the model of Haurwitz (1947), which gives a 90-deg phase difference when friction is not considered. Likewise, Defant (1951), by considering a model which is somewhat similar to the present one except that the classical eddy equation is used and the momentum-advection term is neglected completely, finds a phase difference of 4.7 hr. It is not surprising that by the addition of the appropriate amount of friction it is possible to reduce the phase discrepancy to agree with what is observed, since this friction takes the place of the linearized advection of horizontal momentum. While it is indeed possible to lump the non-linear terms on the right-hand side of an equation and call this a viscous stress, the preceding analysis suggests a simpler and more satisfactory explanation of the observed phase of the sea breeze.

The variation of $u(x, 0, 0)$ with the other atmospheric parameters is given in fig. 12 of the paper by Stern and Malkus. The quantities u and τ in this diagram may be interpreted as instantaneous values in the 24-hr cycle. From this diagram, it is seen that the ratio of the eddy conductivity to the square of the basic current is of great importance in determining the magnitude of the sea breeze. It will be noted that, on the scale considered, the Coriolis force has no influence on u; this is due to the fact that the horizontal pressure gradient and the inertial force $(U\,\partial u/\partial x)$ are much larger and control the motion in the x-z plane. However, in the balance of forces parallel to the coast, the Coriolis force is of importance, as will be seen in the following computation of v and the analysis of the sea-breeze hodograph. This subject has received considerable theoretical and observational attention, and it is generally accepted that the turning of the sea breeze is due to the earth's rotation. The purpose of the following development is (a) to show that the present theory of atmospheric heating is in accord with what is known about the sea breeze, and (b) to give theoretical relations showing how the various atmospheric parameters, such as static stability, eddy conductivity, and the undisturbed wind speed, affect the hodograph. To compute v, (2) is used, which in dimensionless coordinates becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)v = -f_0 u.$$

The solution of this equation is

$$v(x, t) = -f_0 \int_a^t u(x - t + \eta, \eta)\, d\eta, \qquad (28)$$

where a is a constant of integration, as may be verified by differentiation. This integral will be evaluated at $x = 0$, which is at the coast. Now, u varies as $\cos \lambda t$; hence at $t = -\pi/2\lambda$, $u = 0$. And we assume $v = 0$ at $t = t_0 - (\pi/2\lambda)$, where t_0 is small compared to $\pi/2\lambda$.[1] After transformation of the variable of integration, (28) becomes

$$\frac{v(0, t)}{-f_0} = \int_{t_0 - \pi/2\lambda - t}^{0} u(\eta, \eta + t)\, d\eta. \qquad (29)$$

The value of u is given by (27); but rather than to attempt to integrate this complicated expression, it is approximated by a simple exponential function which fits it in the vicinity of the windward shore. Thus, if u_0 is the amplitude of u given by (27) when $x = t = 0$, and G is the slope of this curve at the same point (*i.e.*, $G = \partial u/\partial \xi$),

$$u(x, t) \approx u_0 \left[\exp \frac{xG}{u_0 (gs/U^2)^{\frac{1}{2}}}\right] \cos \lambda t \quad (x \le 0),$$

where

$$u_0 = \frac{g\tau}{T_m} \frac{2}{\pi} \frac{K}{U^2} \ln \frac{U^2}{K (gs)^{\frac{1}{2}}},$$

and

$$G = \tau g/\pi T_m U.$$

We seek the behavior of v when $t \gg -\pi/2\lambda \approx -10^2$, *i.e.*, well after the time of onset of the sea breeze. With these magnitudes of t, it may be shown that $\dfrac{xG}{u_0(gs/U^2)^{\frac{1}{2}}} \ll 0$ when $x = t_0 - (\pi/2\lambda) - t$, and the lower limit in (29) may be extended to infinity. Therefore, (29) becomes

$$\frac{v(0, t)}{-f_0 u_0} = \mathrm{Re} \int_{-\infty}^{0} e^{i\lambda t} \exp\left[\frac{G\eta}{u_0 (gs/U^2)^{\frac{1}{2}}} + i\lambda\eta\right] d\eta$$

$$= \mathrm{Re} \frac{e^{i\lambda t}}{\dfrac{G}{u_0 (gs/U^2)^{\frac{1}{2}}} + i\lambda} \qquad (30)$$

where Re denotes "real part of."

From (30) it is seen that the sea-breeze component parallel to the coast is, in general, not of the same phase as the diurnal temperature wave. The quantity

[1] If one assumed that u and v were zero at the same time (at onset), t_0 would be zero. The present treatment is designed to weaken this restriction.

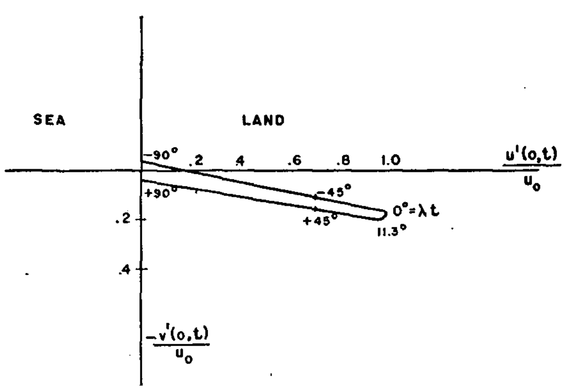

FIG. 2. Theoretically predicted hodograph of sea-breeze components (u') perpendicular to coast and (v') parallel to coast, when these are measured at windward shore. u_0 is maximum value of $u'(0, t)$ and occurs when land temperature reaches its maximum. Points placed on elliptic hodograph represent sea-breeze vector at various times, converted to angular degrees (0 deg corresponds to time of greatest temperature contrast and 90 deg is 6 hr later). This hodograph might correspond to sea breeze in middle latitudes under conditions of light prevailing wind, such that seaward extent of sea breeze was of order of 5 km.

$\frac{u_0(gs/U^2)^{\frac{1}{2}}}{G}$ is a measure of the depth of the sea breeze, or how far out to sea it extends. When this depth is small compared with the distance that an air parcel travelling with the speed of the undisturbed wind would cover in $24/2\pi$ hr, $v = -f_0 u_0(u_0/G)(gs/U^2)^{\frac{1}{2}} \cos \lambda t$. In this case, v is much smaller than u, and the hodograph of the sea-breeze components would be a straight line almost perpendicular to the coast. Fig. 2 shows the theoretical hodograph when $\frac{G}{u_0(gs/U^2)^{\frac{1}{2}}} = 0.05$ and $\lambda = f_0 = 10^{-2}$. This might correspond to a sea breeze that extended 5 km out to sea. It is readily shown from (30) that these hodographs are elliptic. The point labeled $\lambda t = 0$ in fig. 2 represents the time of maximum heating, *i.e.*, the temperature difference between land and sea is a maximum. For horizontally deeper sea breezes, $\frac{G}{u_0(gs/U^2)^{\frac{1}{2}}}$ decreases, thereby increasing both the amplitude and phase angle of v; hence, the hodograph would tend to become more circular. These relations between the depth of the sea breeze and the shape of the hodograph at the coast would appear to have some interesting practical applications.

In concluding this section, we present some results which are of interest from the observational point of view. By differentiating (27) with respect to x and then setting $x = 0$, it is possible to eliminate the parameters a_1 and a_2 which contain the eddy conductivity. We thereby obtain

$$\tau g/\pi G T_m U = 1, \tag{31}$$

where

$$G = \frac{\partial u}{\partial \xi} = \left(\frac{gs}{U^2}\right)^{\frac{1}{2}} \frac{\partial u}{\partial x}$$

is the horizontal wind divergence measured windward of the shore, and τ is the temperature excess at a given time. This formula poses a direct and simple test of the theory in terms of relatively easy measurements. Moreover, it is feasible to use these sea-breeze measurements to calculate the effective eddy conductivity, and thereby determine some of the gross features of the turbulent heating.

During the summer of 1952, the writer attempted some preliminary measurements on Nantucket Island, to see if this equation could be tested. Although this program had to be terminated before a conclusive amount of data could be obtained, they did indicate agreement with (31) within a factor of two, under circumstances which could not be expected to yield better results.[2] For future observational work, it is suggested that (31) be written in terms of the scalar pressure by means of the relation $U\,\partial u/\partial \xi = -\rho_m^{-1}\,\partial p/\partial \xi$ (for quasi-steady conditions). Thus, the equivalent of (31) is

$$\rho_m \tau g/\pi W T_m = 1, \tag{32}$$

where $W = -\partial p/\partial \xi$ is the perturbed pressure gradient at the windward shore. There seems to be little doubt that the pressure gradient associated with the sea breeze can be accurately measured. (See, for example, Leopold, 1949.)

The relations given above apply to a specific model with a discontinuity in temperature at the coastline. For comparison of theory with observation, it may be desirable to consider the actual profile of surface temperature, which is a continuously varying function of distance. The resulting formula for the sea breeze is given below without the derivation:

$$\frac{\pi T_m W(x)}{\rho_m g} \approx \int_0^\infty \frac{T_0(\eta)}{(\eta - x)^2}\, d\eta \qquad (x \le 0),$$

where $x = 0$ is at the coast, and $T_0(x)$ is zero for $x \le 0$. The major assumption in deriving this equation is that b^2 is large enough to justify replacing the equivalent mountain by $(sT_m)^{-1}$ times the surface temperature. This precludes the use of the formula with idealized profiles having jump discontinuities, since b^2 has an important effect here in rounding off the corners. On the other hand, it seems suited for actual measurements when the sea breeze is sufficiently developed.

6. Concluding remarks

Although the present theory is able to account for some of the small-scale phenomena that are associated with differential heating, it is apparent that any attempt to investigate the effect in less familiar meteorological problems is severely limited by the presence of those peculiar "constants" called the eddy coefficients. It is with the realization of this inadequacy that the present writer would like to suggest that there are certain quantitative conclusions regarding surface heating that are obtainable without inquiring into the details of the turbulent process. These are based on thermodynamic principles, certain theorems in differential equations, and simplifying assumptions on the physical model. In view of the generality of this approach, it will often happen that the only conclusions so drawn are trivial; but on the other hand, significant results may also be obtained.

Let the linearized equations of motion lead to the following equation connecting the mean vertical velocities with the heating function:

$$\mathfrak{L} w = H(x, z), \tag{33}$$

where $\mathfrak{L} = \mathfrak{L}(\partial/\partial x, \partial/\partial z)$ is a linear differential operator with constant coefficients (all the coordinates are

[2] These data are on file with the Woods Hole Oceanographic Institution.

now in their conventional dimensional units). Consider a region in which all the heat sources are located above the level $z = 0$, and in which there are no kinematical constraints (such as a ground surface). Then the particular solution of the above equation may be written in the form

$$w_1(x, z) = \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{z} H(x - \xi, z - \zeta)\, g(\xi, \zeta)\, d\zeta, \quad (34)$$

where $g(\xi, \zeta)$ is the Green's function associated with $\mathcal{L}$ and the stated boundary conditions. We consider the value of the particular integral at $z = 0$ and express w in terms of the displacement (ϕ) of the streamline from its equilibrium position. Also, H is written in terms of the temperature (T_e) by means of (13). These relations become:

$$w_1(x, z) = d\phi_1/dt = U\, \partial\phi_1/\partial x,$$

and

$$H(x - \xi, z - \zeta) = \frac{d}{dt} T_e(x - \xi, z - \zeta) = U \frac{\partial T_e}{\partial x}(x - \xi, z - \zeta) = U \frac{\partial T_e^*}{\partial x}(x - \xi, z - \zeta),$$

where $T_e^* = T_e - \alpha z$. The introduction of T_e^* in the above equation eliminates the constant of integration which appears subsequently. Substituting these in (34), one obtains

$$\phi_1(x, z) = \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{z} T_e^*(x - \xi, z - \zeta)\, g(\xi, \zeta)\, d\zeta \quad (35)$$

as the particular solution. For the problem of differential heating above flat ground, a solution of the homogeneous equation $\mathcal{L}\phi = 0$ must be subtracted from ϕ_1. By setting $z = 0$ in (35), we obtain the value of the equivalent mountain, $M(x)$:

$$M(x) = \int_{-\infty}^{+\infty} d\xi \int_{0}^{\infty} T_e^*(x - \xi, \zeta)\, g(\xi, -\zeta)\, d\zeta. \quad (36)$$

Now it is assumed that the surface temperature increases (or decreases) with x, and eventually approaches a constant value τ. Then, for "large" x and "small" values of ξ and ζ, $T_e^*(x - \xi, \zeta) = \tau$; while for large values of ξ and ζ, the Green's function approaches zero. Therefore, in those regions where the surface temperature is constant within a "large enough" interval, (36) becomes

$$\lim M(x) = \tau \int_{-\infty}^{+\infty} d\xi \int_{0}^{\infty} g(\xi, -\zeta)\, d\zeta. \quad (37)$$

This equation may be regarded as an asymptotic value of the equivalent mountain, and is independent of the distribution of heat sources in the vertical. In some cases, the integral of the Green's function is easily evaluated in terms of the coefficients of $\mathcal{L}$. Thus, if the operator is of even order in $\partial/\partial z$, and also satisfies the requisite convergence conditions,

$$M(x) = \tau/\mathcal{L}(0, 0),$$

where $\mathcal{L}(0, 0)$ is the constant obtained by setting the x- and z-derivatives equal to zero. It should be mentioned that for complicated operators it is necessary to evaluate this constant with caution given to the existence of a limit. In the model where the earth's rotation is omitted, the above equation becomes $M(x) = \tau/(\Gamma - \alpha)$. Therefore, the mean displacements of the streamlines at large distances from the heat source could be predicted without recourse to the details of the heating. How great an elevation is necessary will depend on the relative development of the turbulent region and the horizontal scale.

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