NOTE ON HEMISPHERIC NUMERICAL INTEGRATION OF THE BAROTROPIC MODEL

By Maj. Jacob F. Blackburn, USAF, and W. Lawrence Gates

Air Force Cambridge Research Center[1]

(Manuscript received 7 April 1955)

ABSTRACT

The computational procedures employed for the numerical integration of the barotropic model for two-dimensional non-divergent flow on a finite-difference grid covering all longitudes of the northern hemisphere south to approximately 10°N are described. The results of a pilot forecast are presented, and the implications of this work for both research and operational applications are discussed.

1. Introduction

In the past few years, research in numerical forecasting has gained much impetus. Efforts to integrate numerically a variety of non-linear dynamical models of the atmosphere have been made by a number of groups, such as the efforts reported by Charney (1954), Charney and Phillips (1953), Charney *et al* (1950), Staff Members of the University of Stockholm (1952), Bushby and Hinds (1954), and Thompson and Gates (1956). These researches on numerical integration represent an important phase of the development of the sytematic application of dynamical methods in forecasting practice, and at the same time have greatly contributed to the documentation of the performance of simple dynamical models in application to observed atmospheric flow. Among the common characteristics of each of these efforts is the employment of a finite-difference grid of limited lateral extent, centered more or less on the geographical area of most immediate synoptic concern to the researchers. The methods of solution employed have necessarily provided for the imposition of boundary conditions on the lateral edges of the grids. When carefully employed, this procedure is probably adequate to display the gross features of simple numerical prediction models and for the production of forecasts for a period of a day or so. These boundary conditions will, of course, affect the solutions in the immediate vicinity of the boundary; and under some circumstances of rapidly moving or highly developed disturbances near the edges of the grid, these effects may be strong enough to alter seriously the solutions thousands of miles into the interior of the forecast region. These boundary influences can be clearly seen in the extensive series of integrations recently completed at the Geophysics Research Directorate, Air Force Cambridge Research Center, and summarized by Thompson and Gates (1956).[2] To minimize these effects in an extensive program of integration and in operational application, it would appear desirable to place the grid's lateral boundaries in regions of relative synoptic inactivity, as indicated, for example, by the climatological areas of minimum standard deviation of pressure or isobaric contour height. By integration over a hemisphere, the observed small changes of atmospheric flow in low latitudes may be utilized to reduce the contamination of the middle- and high-latitude forecasts by purely boundary effects. It is the purpose of this article to describe briefly the computational techniques employed in hemispheric numerical integration and to indicate the character of preliminary results obtained with the barotropic model for two-dimensional non-divergent flow.

2. Method of solution and finite-difference grid

Under the usual assumptions of frictionless, adiabatic flow in hydrostatic and quasi-geostrophic equilibrium, the barotropic equation may be written in the familiar form

$$\nabla^2(\partial z/\partial t) - J(gf^{-1}\nabla^2 z + f, z) = 0, \qquad (1)$$

where z is the height of the 500-mb surface, t the time, g the acceleration of gravity, f the Coriolis parameter, $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ the plane Laplace operator, and $J(\alpha, \beta) \equiv (\partial\alpha/\partial x)(\partial\beta/\partial y) - (\partial\alpha/\partial y)(\partial\beta/\partial x)$ is the Jacobian or functional determinant of the variables α and β. After introduction of a system of non-dimensional variables in terms of a reference unit of length L, a reference unit of time Ω^{-1}, and a reference unit of isobaric contour height H, and when written for a square finite-difference grid, (1) becomes

$$\Delta_{ij}^2(\partial z/\partial t) = I(A_{ij}\,\Delta_{ij}^2 z + 4^{-1}f_{ij}, z_{ij}), \qquad (2)$$

[1] Maj. Blackburn's present duty station: U. S. Air Force Academy.

[2] A complete presentation and analysis of these results has been given by Gates *et al.*, *Geophys. Res. Pap.*, No. 46, 1956.

where the familiar finite-difference operators

$$\Delta_{ij}^2(\ ) = (\ )_{ij+1} + (\ )_{ij-1} + (\ )_{i+1j} + (\ )_{i-1j} - 4(\ )_{ij},$$

and

$$I(\alpha, \beta) = (\alpha_{i+1j} - \alpha_{i-1j})(\beta_{ij+1} - \beta_{ij-1}) - (\alpha_{ij+1} - \alpha_{ij-1})(\beta_{i+1j} - \beta_{i-1j})$$

have been introduced. In (2), the coefficient $A_{ij} = gH/4\Omega f_{ij} m_{ij}^2 l^2$, where $m_{ij} = (1 + \sin \varphi_{ij}) S/2$ is the scale or magnification factor of the polar stereographic projection (with φ_{ij} the latitude of a grid point and S the reference magnification of the working chart; in the present case, $S = 3 \times 10^7$), and where l is the mesh size of the finite-difference grid (in the present case, $l = 1.5$ cm).

Full account is thus taken in (2) of the variation of the Coriolis parameter f. And while a square grid was used for simplicity on the polar map, the variation of map scale and consequently of the distance between grid points was considered. In the calculations, L was taken as 1 grid internal ($L = ml$), Ω^{-1} was selected as 1 day, H was taken equal to 10 ft, and an appropriate scaling of initial contour height

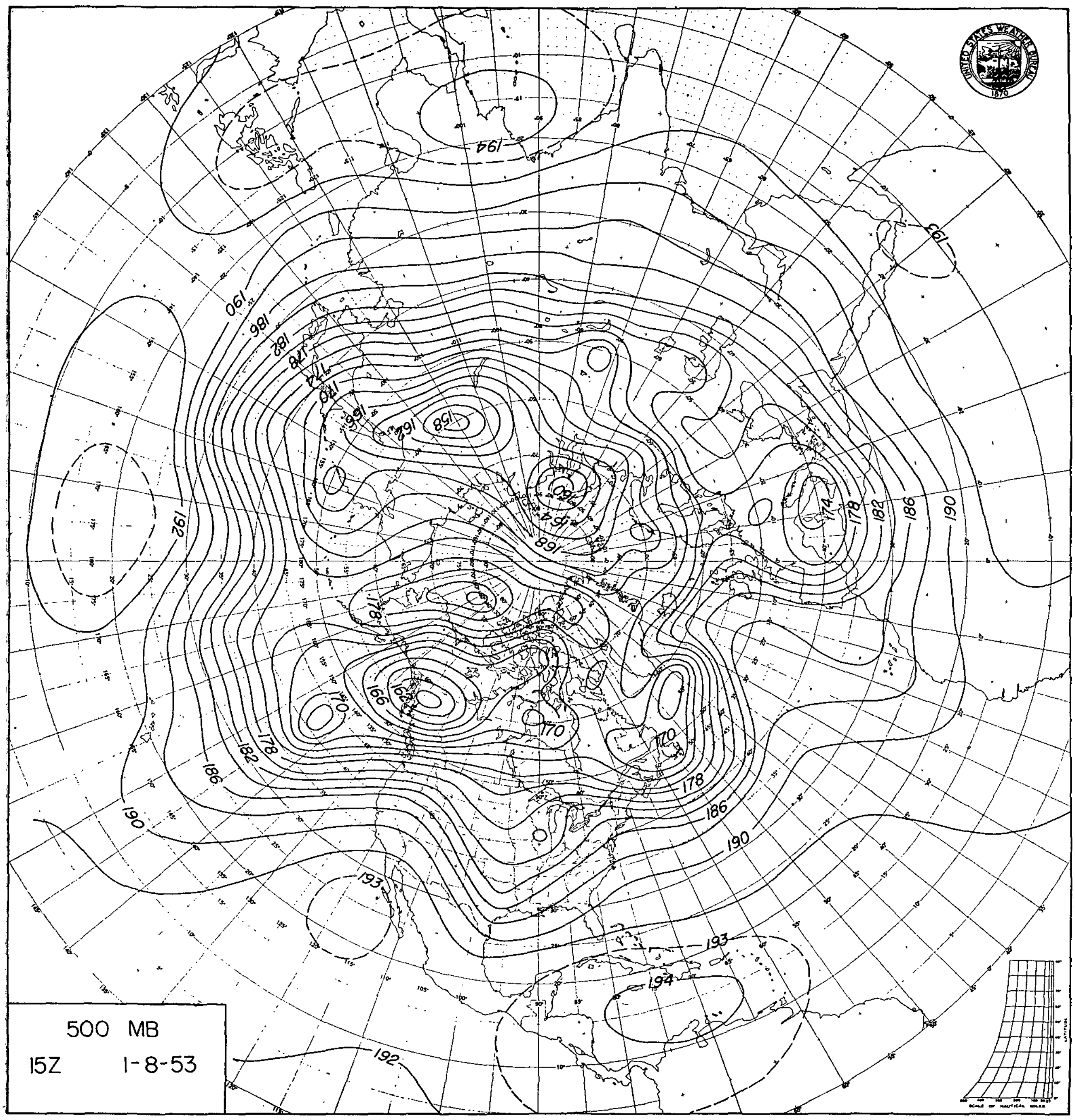

FIG. 1. Observed 500-mb flow, 1500 GCT 8 January 1953. Contours in hundreds of feet.

was introduced to avoid overflow or spill of the computer's internal memory.

In these exploratory integrations for hemispheric flow, the method of solution was similar to the scheme first outlined by Charney *et al* (1950). This procedure consists essentially of the cyclical calculation of the absolute vorticity advection at time t for each point of a finite-difference grid, the solution of the resulting finite-difference equations for $(\partial z/\partial t)$ at each point by a method of relaxation, and the calculation of the height z at time $t + \Delta t$ with use of centered differences over a short time interval, to regenerate the initial data. The method of relaxation actually used in the solution of (2) was the extrapolated Liebmann method, as described by Frankel (1950) and by Charney and Phillips (1953), wherein a set of residues or residuals is defined as the finite-difference Laplacian of the error of an initial approximation to the solution of (2), and is systematically reduced in successive row-by-row passages over the grid. The over-relaxation coefficient α in this process was ultimately selected close to the value given by Frankel for optimum convergence in a rectangular grid of approximately the same area as the present grid

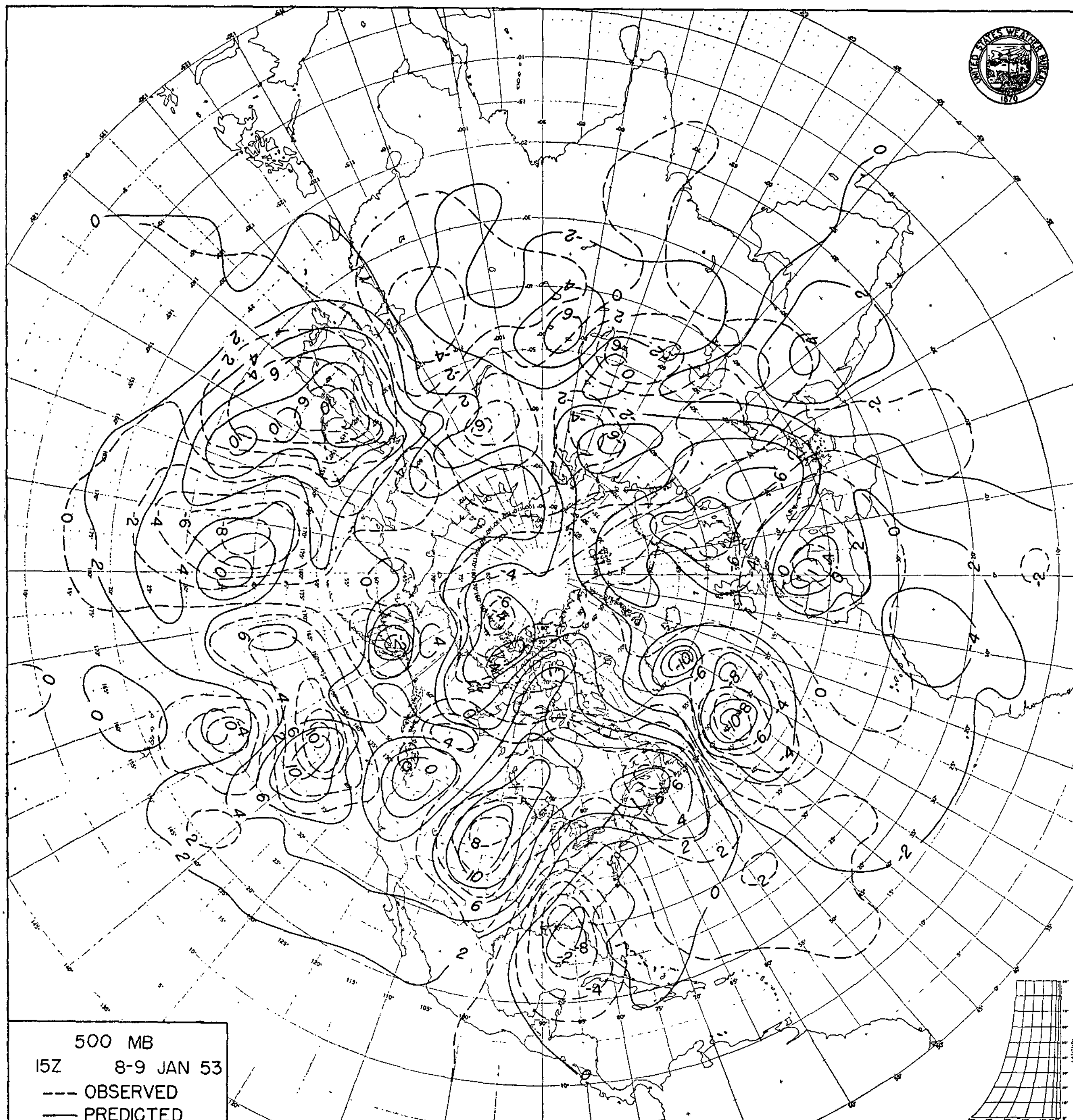

FIG. 2. Observed (dashed lines) and barotropically computed (solid lines) 24-hr changes at 500 mb, 1500 GCT 8 to 9 January 1953.

($\alpha = 0.438$); values of this coefficient lower than the Frankel value were found to lengthen significantly the relaxation process. The relaxation was carried out with a boundary condition $\partial z/\partial t = 0$ on the lateral edges of a grid of 1789 points, equally spaced on a polar stereographic projection of the northern hemisphere of scale $1:(3 \times 10^7)$, symmetrically arranged about the pole and extending south to approximately 10°N.

The solutions were obtained on an International Business Machines model 701 computer. Nearly the full capacity of this high-speed, digital machine was utilized, and extensive use was made of the magnetic drum units. The time required for solution on the computer was found to depend strongly on the character of the relaxation process, and particularly on the accuracy of the initial approximations in the relaxation process. For the first relaxation at $t = 0$, the most convenient initial approximation to the solution was $\partial z/\partial t = 0$ at all points. For this the computer required about 4 min to complete the relaxation for the 1529 interior points to an absolute residual corresponding to approximately 20 ft/day in height-tendency units. For later times, the previously obtained solutions were used in a linear extrapolation, the relaxation in each stage now requiring an average of 1.5 min for completion. Such a scheme has been successfully used for the barotropic equation by Charney and Phillips (1953), and by Thompson and Gates (1956), in numerical integrations over a somewhat smaller area. By using such procedures, a 24-hr hemispheric barotropic forecast was made in approximately 1.0 hr of machine time.[3]

3. Preliminary results and discussion

In the process of checking the code for machine integration a 24-hr forecast for 500 mb was made from the initial conditions at 1500 GCT 8 January 1953, shown in fig. 1. The computed and observed 24-hr height changes for this case are shown in fig. 2. The comparatively high quality of the forecasts over much of North America and the North Atlantic is evident, and over much of the Pacific and polar regions the barotropic forecast represents a major portion of the observed changes. The movement and change in shape of the major anticyclonic systems over the hemisphere was correctly predicted, with the exception of the subtropical high in the southwestern Pacific Ocean and portions of the blocking high in the eastern Atlantic; the barotropic predictions for these systems are probably due in some measure to a poorly documented analysis of the initial height field. The motion of the several well-developed cyclonic systems over the hemisphere was, in general, successfully predicted by the model (*e.g.*, those in the central Pacific and that in the northwestern Atlantic). The most marked failure of the model in a region of relatively dense data was the case of the growing cyclonic disturbance in the south-central United States. In this region, true baroclinic development — *i.e.*, a conversion of atmospheric potential energy into kinetic energy — was in progress, although the location of the region of development and the relative orientation of the surrounding regions of height rises were correctly forecast in this case. In general, the predictions are superior in regions of more adequate data coverage; over the western hemisphere (30°E–150°W), the correlation between the observed and predicted 24-hr height changes was 0.7, with an average of 0.8 over North America; the average correlation for the eastern hemisphere was only 0.3, mainly caused by apparent changes in the South Pacific, for which a re-analysis is in progress.

In analyzing the errors of such numerical prediction, attention must be given to the errors introduced by the use of approximate numerical procedures (*e.g.*, finite-differences and boundary conditions), the errors introduced by poor observations and their subjective analysis, and finally the errors inherent in the physical approximations of the model itself. From the limited evidence available at the present time, it appears that one of these errors — that introduced by the imposition of lateral boundary conditions — can be significantly reduced by use of a grid of hemispheric dimensions, whose boundaries are removed from the middle-latitude regions of primary interest. Also undoubtedly present are errors due to the use of finite-differences, and research is in progress to reduce the magnitude of the truncation errors from this source. The finite-difference error may systematically accumulate during the course of an integration, and will probably be of great importance in predictions for longer periods of time. In the use of the relaxation technique, care must also be taken to avoid the occurrence of a final pattern of residues of the same sign over large regions of the grid; this condition of "homogeneous residues" can lead to an appreciable systematic error (Gates, 1955).

The errors introduced in the present integrations by the poor data available over large portions of the northern hemisphere appear to be quite significant; even with the use of historical-series analyses, for which all available data are presumably used, the forecast errors appear to reflect strongly the uncertainty in the specification of either the initial conditions or of the observed or verification conditions. This error emphasizes the need for more

[3] This time has since been reduced to approximately 0.6 hr by use of a time increment of 2 hr, although such an interval may not prove computationally stable in all cases.

adequate upper-air data in many regions of the world, especially over oceanic areas.

The primary aim of the present integrations is, of course, to examine the behavior of the atmosphere when treated as a barotropic medium. As the above discussion shows, however, this is only possible in the face of several significant sources of error. It is believed that by use of a hemispheric grid, one of the heretofore troublesome sources of error can be effectively eliminated, and the isolation and eventual reduction of the other error sources carried out. This should permit a more accurate assessment of the applicability of the simple quasi-geostrophic barotropic model to the atmosphere.

It is, of course, recognized that reliable statistics on the performance of a hemispheric numerical prediction scheme cannot be obtained from a limited sample. The present barotropic model is being applied in the production of a series of hemispheric forecasts, the results of which will be presented in a future paper. Of particular interest will be the behavior of this model when applied to predictions of extended length.

REFERENCES

Bushby, F. H., and M. K. Hinds, 1954: The computation of forecast charts by application of the Sawyer-Bushby two-parameter model. *Quart. J. r. meteor. Soc.*, **80**, 165–173.

Charney, J. G., 1954: Numerical prediction of cyclogenesis. *Proc. natl. Acad. Sci.*, **40**, 99–110.

——, R. Fjørtoft, and J. von Neumann, 1950: Numerical integration of the barotropic vorticity equation. *Tellus*, **2**, 237–254.

Charney, J. G., and N. A. Phillips, 1953: Numerical integration of the quasi-geostrophic equations for barotropic and simple baroclinic flows. *J. Meteor.*, **10**, 71–99.

Frankel, S. P., 1950: Convergence rates of iterative treatments of partial differential equations. *Math. tables and other aids to computation*, **4**, 65–75.

Gates, W. L., 1955: *Note on convergence error in the relaxation method.* (Unpubl. manuscript.) Boston, A. F. Cambridge Res. Center, 8 pp.

Staff Members, Inst. Meteor., Univ. Stockholm, 1952: Preliminary report on the prognostic value of barotropic models in the forecasting of 500 mb height changes. *Tellus*, **4**, 21–30.

Thompson, P. D., and W. L. Gates, 1956: A test of numerical prediction methods based on the barotropic and two-parameter baroclinic models. *J. Meteor.*, **13**, to be published.