AN APPROXIMATE ANALYSIS OF THE SOLAR REFLECTANCE
AND TRANSMITTANCE OF A SNOW COVER

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(Original Manuscript received 23 November 1954; revised manuscript received 11 August 1955)

ABSTRACT

The transmittance and reflectance of a snow cover has been analyzed by use of the model of a diffusing medium introduced by Schuster. The general case of a slab of finite thickness, irradiated from both sides, is solved and reduced to the case of a semi-infinite slab irradiated from one direction, as an approximation of the radiative component of energy flow in a deep snow cover. The transmission factor of interest for heat transfer within the medium is shown to be different from that obtained experimentally.

1. Introduction

The determination of the heat flow within and from a snow cover is dependent to a great extent upon a reasonable description of the solar energy absorbed, transmitted and reflected. There are several factors which require a more adequate explanation of the observed experimental facts listed in the literature (see [11] for a review of the literature). The high reflectivity of snow is qualitatively attributed to multiple reflections from the small ice crystals composing the snow. There are a number of analyses of a diffusing medium; but, with one exception [3], none are applicable directly to this particular problem.

The transmission within the snow is usually expressed by an exponential attenuation, but experimental determinations do not result in consistent values for the attenuation factor. The non-homogeneity and variability of natural snow is responsible for much of the experimental differences; and, in addition, the experimental techniques utilized have not taken into consideration the energy flux passing through the medium in the direction opposite to that of incidence. This two-directional flow of energy arises from the multiple reflections which occur, and an understanding of the relationship between the two fluxes is essential to any analysis of the total solar energy absorbed at any depth, i.e., the radiative energy absorption at a specific depth.

The above considerations led to a more rigorous investigation of the basic problem and resulted in the analysis presented herein. Subsequent to completion of this analysis, a similar analysis by Dietzius [3] was brought to the attention of the writers. However, certain differences exist in the two analyses, particularly in interpretation and application.

The results presented are useful in that they begin to explain the large reflectivity observed for a snow cover and provide a method for determining certain physical parameters which may be used as a first approximation for correlating the transmission characteristics of various types of snow. The weaknesses of the analysis lie in the assumptions which were made to idealize the system and obtain a reasonable solution. A more complete derivation than presented here may be found in either [3] or [7].

2. Analysis

The method is based upon the model introduced by Schuster [10] for transmission of radiation through gaseous stars. The original work by Schuster has been expanded and modified by many writers, each one refining the techniques and methods involved. This is exemplified by Duntley [5], who introduced six physical parameters in the basic equations, and Hulbert [8], who considered the effects of angular incidence of the radiation. A different mathematical approach is that of Chandrasekhar [2], who started from the same differential equations but obtained a solution by means of integral methods. Refinements to this extent are not believed warranted in the particular problem considered here, but future experimental work may indicate that a more complicated analysis is necessary.

The following physical model is postulated:

1. The medium is homogeneous and of constant optical density.
2. The individual particles which compose the medium are such

![Fig. 1. Energy flux within medium.](image-url)
that the radiant energy within the medium is perfectly diffuse at any depth.
3. The reflection coefficient, \( r \), is considered distributed with depth (distance into the medium) and is a constant. This corresponds to an isotropic scattering coefficient, or can be considered to be due to reflection from a series of uniform laminae.
4. The absorption coefficient, \( k \), is also constant and independent of depth.
5. The medium is infinite in width and breadth, having only finite depth.

If a section of such a medium is considered, the following two differential equations result (see fig. 1):

\[
dY = -kY \, dx - rY \, dx + rZ \, dx,
\]
and
\[
dZ = kZ \, dx - rY \, dx + rZ \, dx.
\]

The boundary conditions are (see fig. 2)

at \( x = 0 \): \( Y_0 = (1 - r_0) Y_i + r_0 Z_0 \),

and

at \( x = b \): \( Z_b = (1 - r_b) Z_i + r_b Z_0 \).

The depth of the medium is taken as \( b \), with the upper and lower surfaces having a reflectivity \( r_0 \) and \( r_b \), respectively.

The solution of the equations for \( Y \) and \( Z \), with the boundary conditions given, are:

\[
Y = \left[ \left( 1 - r_0 \right) \left( a_2 - r_2 \right) Y_i \exp(\beta b) \right. \\
\left. \quad - (1 - r_b) \left( 1 - r_0 a_1 \right) Z_i \right] \exp(-\beta x) \\
+ \left[ (1 - r_b) \left( 1 - r_0 a_1 \right) Z_i - (1 - r_b) \right] \exp(\beta x) \\
\times (a_1 - r_2) Y_i \exp(-\beta b) \exp(\beta x) \\
\times \left[ D_1 \exp(\beta b) - D_2 \exp(-\beta b) \right]^{-1},
\]

and

\[
Z = a_1 \left[ (1 - r_0) \left( a_2 - r_2 \right) Y_i \exp(\beta b) \right. \\
\left. \quad - (1 - r_b) \left( 1 - r_0 a_1 \right) Z_i \right] \exp(-\beta x) \\
+ \left[ (1 - r_b) \left( 1 - r_0 a_1 \right) Z_i - (1 - r_b) \right] \exp(\beta x) \\
\times (a_1 - r_2) Y_i \exp(-\beta b) \exp(\beta x) \\
\times \left[ D_1 \exp(\beta b) - D_2 \exp(-\beta b) \right]^{-1},
\]

where the various constants are given by

\[
\alpha_1 = (k + r - \beta)/r, \quad \alpha_2 = (k + r + \beta)/r, \quad \beta = [(k + r)^2 - r^2]^{1/2},
\]

\[
D_1 = (a_2 - r_2)(1 - r_0 a_1), \quad D_2 = (a_1 - r_2)(1 - r_0 a_1).
\]

Equations (5) and (6) can be greatly simplified by assuming \( Z_i = 0 \). The equations then reduce to

\[
Y = \left[ (1 - r_0) Y_i \left( a_2 - r_2 \right) \exp(\beta b) \exp(-\beta b) \right. \\
\left. \quad - (1 - r_b) \exp(-\beta b) \right] \exp(\beta x) \\
\times \left[ D_1 \exp(\beta b) - D_2 \exp(-\beta b) \right]^{-1},
\]

and

\[
Z = \left[ (1 - r_0) Y_i \left( 1 - a_2 r_2 \right) \exp(\beta b) \exp(-\beta b) \right. \\
\left. \quad - (1 - r_b) \left( 1 - a_2 r_2 \right) \exp(-\beta b) \right] \exp(\beta x) \\
\times \left[ D_1 \exp(\beta b) - D_2 \exp(-\beta b) \right]^{-1}.
\]

For the case of a semi-infinite slab, i.e., \( Y \) approaches zero before \( x = b \), (12) and (13) become

\[
Y = \left[ \left( 1 - r_0 \right) Y_i \exp(-\beta x) \right] [1 - r_0 a_1]^{-1},
\]

and

\[
Z = \left[ \left( a_1 - r_2 \right) Y_i \exp(-\beta x) \right] [1 - r_0 a_1]^{-1}.
\]

Equations (12) and (13) correspond to a thin snow cover where the background may have an effect both on the albedo of the snow and the amount of solar energy absorbed. If the snow layer is deep enough, a negligible amount of energy penetrates to the ground surface, and (14) and (15) should be applicable.

The transmittance can be defined in two ways. If one is concerned with the problem of energy absorption as a function of depth, as in the case of a study of snow melting, the net radiant flux downward must be considered. According to this definition, the transmittance becomes

\[
T_1 = (Y - Z) (Y_i)^{-1}.
\]

For the deep snow cover, this reduces to

\[
T_1 = (1 - R_0) \exp(-\beta x).
\]

The difference between the transmittances at two depths represents the fraction of the incident radiation which is absorbed between these two depths.

An alternative definition of transmittance is in terms of the downward component of the radiation only, as given by Dietzius [3]. This second definition of transmittance probably corresponds more closely to the transmission as measured experimentally, due to the fact that most instruments are designed to detect only the downward component of the radiation. Such a measurement by itself does not provide suffi-
cient information to evaluate the total radiation exchange within the medium. Furthermore, the introduction of an instrument within the snow cover alters the radiation balance in the immediate vicinity of the instrument, tending to invalidate much of the data which have been reported from the field. This second definition of the transmittance is given by

$$T_2 = YY_i^{-1}. \quad (18)$$

This reduces, for a deep snow cover, to

$$T_2 = (1 - r_0)(1 - r_0a_1)^{-1} \exp(-\beta x). \quad (19)$$

The albedo, according to accepted usage, is the total reflectance for solar irradiation. This is made up of the energy reflected directly from the snow surface plus the energy transmitted upward through the surface, or (fig. 2)

$$R_0 = [r_0Y_i + (1 - r_0)Z_0]Y_i^{-1}. \quad (20)$$

Equation (20) is a general expression for the reflectance.

For the deep snow cover (semi-infinite medium), substitution into (20) yields the following equation for total reflectance or albedo:

$$R_0 = r_0 + a_1(1 - r_0)^2(1 - r_0a_1)^{-1}. \quad (21)$$

3. Distributed reflectivity for a series of slabs

The factor $r$ which was introduced in (1) and (2) represents the reflectivity per unit depth, or the number of reflections per unit depth times the reflectivity for each reflection. For a diffuse material, this would correspond to an isotropic scattering coefficient. For a series of $n$ slabs, omitting the first and last surfaces, the distributed reflectivity is defined by

$$r = [2(n - 1)r_s][n(\Delta x)]^{-1}, \quad (22)$$

where $\Delta x$ is the thickness of a single slab, and $n$ is the number of slabs. For a large number of slabs, the distributed reflectivity approaches

$$r = 2r_s(\Delta x)^{-1}. \quad (23)$$

Definition of the distributed reflectivity as in (22) allows the introduction of arbitrary reflectivities at the first and last surfaces of the series of slabs.

4. Comparison with the work of Benford

A different approach to the problem of the reflectance of a diffusing medium is that employed by Benford [1]. He considered the reflectance from a series of diffusing slabs, increasing the number of slabs until the reflectance approached a constant value for the infinite medium. The equations developed by Benford are in a form not easily adapted to varying boundary conditions. However, reflectances predicted by the two methods should show close agreement. A comparison of the two methods for a wide range of parameters is given in table 1. The values of the parameters used in the calculations were $r_s = r_0$; $k(\text{in}^{-1}) = 1, 10$ and 100; $r_0 = 0.01, 0.1$ and 0.9; $n = 2, 4$ and infinity.

It is interesting to note that there is a close agreement between the present analysis and that of Benford, although the form of the equations is very different.

5. Prediction of the albedo of an idealized snow

The analysis was also used to compute the spectral and total reflectance of a semi-infinite medium composed of thin layers of ice and irradiated by the sun. It is thought that the results of this analysis would be indicative of the effect of particle size and wavelength of incident radiation upon the reflectance from a deep, clean snow cover. The reflectivity of the ice ($r_0$ and $r_s$) was taken as a constant value of 0.018, based upon the data summarized by Dorsey [4] for the reflectance.

Table 1. Comparison of results obtained by analysis of Benford and method presented herein.

<table>
<thead>
<tr>
<th>$r_s$</th>
<th>$T$</th>
<th>Present analysis</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R_s$</td>
<td>$R$</td>
<td>$R$</td>
</tr>
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</table>

$$x = 0.010''; n = 2$$

$k = 1.0$

<table>
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</thead>
<tbody>
<tr>
<td></td>
<td>$R_s$</td>
<td>$R$</td>
<td>$R$</td>
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</tbody>
</table>

$k = 10$

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<th>Present analysis</th>
<th>$T$</th>
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<tbody>
<tr>
<td></td>
<td>$R_s$</td>
<td>$R$</td>
<td>$R$</td>
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</table>

$k = 100$

<table>
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<tbody>
<tr>
<td></td>
<td>$R_s$</td>
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</table>

$k = 1$

<table>
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$k = 100$

<table>
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<th>Present analysis</th>
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<tbody>
<tr>
<td></td>
<td>$R_s$</td>
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tivity and index of refraction of ice. The absorption coefficient for clear ice was obtained from the data of Sauberer [11] and is tabulated in Table 2. For particle sizes of 0.001, 0.01 and 0.1 in., the reflectance was calculated from (21). The spectral reflectances for these calculations are plotted in Fig. 3 as a function of wavelength. With use of the solar irradiation at sea level as summarized by Moon [9], the spectral values were integrated to give the albedo. This computed albedo is tabulated in Table 3.

6. Discussion

An analysis has been made of the reflection and transmission of diffusing media, the intent being to determine the factors which are important in the reflection and absorption of solar energy within snow. After the analysis was completed, it was found that essentially the same analysis had been made by Dietzus [3], who was primarily interested in the high reflectivity of clouds and fog. The present work has been extended to permit the introduction of arbitrary reflectivities on both surfaces of the medium, and it is thought that the equations are presented in a more convenient form.

Probably the most serious weakness in the analysis is the assumption of diffuse radiation within the medium, particularly at small depths. A more complex analysis, such as that given by Hulbert [8], would be needed to assess the effect of directional irradiation from the sun as compared to the assumption of diffuse irradiation. Since little is known about the detailed nature of the reflection and scattering within snow, it is thought that the simpler analysis given here will be of value in explaining the mechanisms of absorption and reflection within the snow and will be of value in correlating albedo measurements with the physical characteristics of snow.

The properties used in computing the reflectance of the idealized snow cover were those of pure ice. The reflectivity used for the ice was that for normal incidence. Many of the reflectances will be from crystals at angles far from the normal, and in some cases total internal reflection may occur. The effect of this would be to increase the reflectance for a given crystal size; hence for the same albedo, it would be expected that the size of the actual snow crystals would be larger than those utilized in the analysis. A further factor of importance is the presence of impurities in the snow, which would tend to increase the absorption coefficient and reduce the albedo.

The importance of the variation of reflectance of the snow cover with wavelength should be pointed out. The reflectance is fairly constant in the visible region, but drops rapidly in the infrared region due to the increase in the absorption coefficient with wavelength. It is seen that the use of radiation-measuring instruments which are selective may result in albedos which are considerably in error. The instruments used should have a fairly flat response curve out to about 2.5μ, but should cut off before 6μ so as to eliminate the snow and sky emission. The integrated values of albedo presented in Table 3 are approximate, as it was necessary to extrapolate the spectral reflectance beyond 1.3μ, the limit of Sauberer’s data.

Since the spectral distribution of the sun and sky radiation will vary with both time of day and weather conditions, it is expected that there will be some variation of albedo with sun altitude and weather conditions.

The theory presented here indicates that, as snow ages and the snow crystals grow, the albedo should tend to decrease from the initial high values. This conforms to physical observations. Further valid measurements are needed of the transmission of radiation within snows of different structural characteristics, and of the correlation of transmittance with the snow structure. Likewise, data on the spectral

Table 2. Absorption coefficients of ice [11].

<table>
<thead>
<tr>
<th>Wavelength (microns):</th>
<th>.313</th>
<th>.350</th>
<th>.400</th>
<th>.450</th>
<th>.500</th>
<th>.550</th>
<th>.600</th>
<th>.650</th>
<th>.700</th>
<th>.750</th>
<th>.800*</th>
<th>.850</th>
<th>.900</th>
<th>1.00</th>
<th>1.10</th>
<th>1.20</th>
<th>1.30</th>
</tr>
</thead>
<tbody>
<tr>
<td>k (cm⁻¹):</td>
<td>0.0011</td>
<td>0.0005</td>
<td>0.0004</td>
<td>0.0005</td>
<td>0.0008</td>
<td>0.0013</td>
<td>0.0020</td>
<td>0.0034</td>
<td>0.0060</td>
<td>0.011</td>
<td>0.024</td>
<td>0.094</td>
<td>0.32</td>
<td>0.16</td>
<td>1.50</td>
<td>1.70</td>
<td></td>
</tr>
</tbody>
</table>

* Average of two values given.

![Graph](image-url)

Fig. 3. Variation of spectral reflectance of snow cover with wavelength for three particle sizes.

Table 3. Albedo of an idealized snow cover.

<table>
<thead>
<tr>
<th>Particle size (inches):</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Albedo</td>
<td>0.80</td>
<td>0.72</td>
<td>0.62</td>
</tr>
</tbody>
</table>
reflectance of snow, particularly in the infrared region, are needed.

7. Conclusion

An analysis has been made of the transmission and reflection of a mathematical model approximating a snow cover. The degree of approximation must be tested experimentally and, if necessary, refinements and modifications made in the basic premises of the analysis. The intent of the analysis was to obtain a more reliable picture of the factors influencing the solar reflectance (albedo) and the energy absorbed which contributes to thawing or heat storage in a snow cover.

Acknowledgments.—The work was supported by the U. S. Army Corps of Engineers, Snow, Ice and Permafrost Research Establishment, under Contract DA–11–190–ENG–3. The suggestions of Dr. R. W. Gerdel were greatly appreciated and helped to provide a possible application not previously considered.

List of Symbols

\[ D_1 = \text{defined by (10)}. \]
\[ D_2 = \text{defined by (11)}. \]
\[ R_0 = \text{total reflectivity of medium (albedo); see (20)}. \]
\[ T_1 = \text{transmission of medium at depth } x \text{ as defined by (16)}. \]
\[ T_2 = \text{transmission of medium at depth } x \text{ as defined by (18)}. \]
\[ Y_i = \text{downward incident energy on upper face } (x = 0). \]
\[ Y_0 = \text{downward component of energy at } x = 0. \]
\[ Y = \text{downward component of energy at depth } x. \]
\[ Z_b = \text{upward component of energy at } x = b. \]
\[ Z_i = \text{upward incident energy on lower face } (x = b). \]
\[ Z_0 = \text{upward component of energy at } x = 0. \]
\[ Z = \text{upward component of energy at depth } x. \]
\[ b = \text{depth of lower boundary of medium}. \]
\[ k = \text{coefficient of absorption}. \]
\[ n = \text{number of slabs; see (22)}. \]
\[ r = \text{distributed reflection coefficient; see (22) and (23)}. \]
\[ r_b = \text{reflection coefficient at } x = b. \]
\[ r_0 = \text{reflection coefficient at } x = 0. \]
\[ r_s = \text{surface reflection coefficient of each particle}. \]
\[ x = \text{coordinate of depth}. \]
\[ \Delta x = \text{thickness of individual particles; see (22) and (23)}. \]
\[ \alpha_1, \alpha_2 = \text{constants; defined by equations; see (7) and (8)}. \]
\[ \beta = \text{constant; defined by (9)}. \]

References