ATMOSPHERIC TURBULENT DIFFUSION FROM INFINITE LINE SOURCES: AN ELECTRIC ANALOG SOLUTION

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ABSTRACT

The steady-state distribution of particles emitted by an infinite line source in the presence of winds and atmospheric turbulence is determined by means of an electric analog approach. A logarithmic wind profile and a linear eddy-diffusivity profile are assumed, and a series of normalized curves relating the horizontal wind velocity, the vertical eddy diffusivity, and the source magnitude to the concentration in the horizontal and vertical direction are presented. A method for extending the technique to the treatment of finite line sources and point sources is indicated.

1. Introduction

The classical treatment of the diffusion or scattering of suspended matter in a turbulent atmosphere is based upon the work of Taylor (1915) and Roberts (1923). According to their analysis, the concentration ψ may be determined from

$$\frac{\partial\psi}{\partial t} + u\frac{\partial\psi}{\partial x} + v\frac{\partial\psi}{\partial y} + w\frac{\partial\psi}{\partial z} = \frac{\partial}{\partial x}\left(K_x\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial\psi}{\partial z}\right), \quad (1)$$

where u, v and w are the wind-velocity components, and K_x, K_y and K_z the eddy diffusivities, in the x-, y- and z-directions respectively. Roberts then applied this equation to the evaluation of the diffusion from various idealized sources, assuming that the eddy diffusivities and wind-velocity components are constant. Numerous subsequent field observations have indicated that these assumptions are not tenable and that both parameters vary with height. In particular, under certain conditions, the wind velocity has been found to increase with height at a logarithmic rate while the eddy diffusivity appears to be directly proportional to height. The introduction of these functions in (1) makes that equation extremely difficult to solve, and numerous simplifying assumptions have been suggested. For example, Sutton (1953) and others have found it useful to approximate the logarithmic wind profile by a power relationship.

This article presents a solution for a special case of (1), that of diffusion from an infinite line source of constant magnitude. Such a problem is of interest, for example, in evaluating the air pollution due to automobile traffic along a highway. The method of solution involves the use of electrical analogs and is of sufficient flexibility to permit the solution of the pertinent differential equation to any desired accuracy for any desired wind-velocity and diffusivity profile.

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2. Mathematical formulation

In accordance with Sutton's (1953) notation, the problem is formulated in a Cartesian coordinate system in which the z-axis is in the vertical direction, and the infinite line source, emitting Q particles per unit length per unit time, is oriented in the y-direction. The wind is assumed to be entirely in the x-direction, so that $u = u(z)$, $v = 0$, and $w = 0$. If it is further assumed that the diffusion in the x-direction due to the wind is large compared to the diffusion due to the eddy diffusivity, the term $\partial(K_x\,\partial\psi/\partial x)/\partial x$ may be neglected. Under steady-state conditions, (1) then reduces to

$$u(z)\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial z}\left(K_z\frac{\partial\psi}{\partial z}\right). \quad (2)$$

The boundary conditions applying to this equation are

$$\psi \to 0 \quad \text{as } x \text{ and } z \to \infty, \quad (2a)$$

$$K_z\,\partial\psi/\partial z \to 0 \quad \text{as } z \to 0 \text{ for } x > 0, \quad (2b)$$

$$\psi \to \infty \quad \text{along } x = z = 0, \quad (2c)$$

and

$$\int_0^\infty u(z)\,\psi(x, z)\,dz = Q \quad \text{for} \quad x > 0. \quad (2d)$$

If the temperature gradient throughout the field is small, the condition of neutral stability is said to exist. The relationship between wind speed and height above the ground has been established, by Thornthwaite and Kaser (1943) and others, to be approximately expressed by

$$u(z) = A\log(z/z_0), \quad (3)$$

where A is a constant, and z_0 is identified as the roughness length and represents the height at which the wind velocity is zero. It is then customary to assume that the variation of eddy diffusivity is similar to the variation of eddy viscosity with height, so that

$$K_z = B(\partial u/\partial z)^{-1}, \tag{4}$$

where B is another constant, or

$$K_z = (B/A)z. \tag{5}$$

Equation (2) then becomes

$$\frac{A^2}{B}\left(\log\frac{z}{z_0}\right)\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial z}\left(z\frac{\partial\psi}{\partial z}\right). \tag{6}$$

3. Analog approach

The analog treatment proceeds from a recognition of the formal similarity between (6) and the diffusion equation. This similarity is made even more clear through the change in variable $x = A^2t/B$, so that (6) becomes

$$\left(\log\frac{z}{z_0}\right)\frac{\partial\psi}{\partial t} = z\frac{\partial^2\psi}{\partial z^2} + \frac{\partial\psi}{\partial z}. \tag{7}$$

The right side of (7) is now expanded in finite differences. The second partial derivative with respect to z is approximated by the second central difference, and the first partial derivative of ψ by the average of the forward and backward difference. This, in effect, breaks up the continuous z-coordinate into a series of discrete intervals separated by node points. For the nth node point, (7) becomes

$$\left(\log\frac{nh}{z_0}\right)\frac{\partial\psi}{\partial t} = \frac{\psi_{n+1}-\psi_n}{2h} + \frac{\psi_n-\psi_{n-1}}{2h} + nh\left(\frac{\psi_{n+1}-2\psi_n+\psi_{n-1}}{h^2}\right), \tag{8}$$

since $z = nh$; or

$$\left(\log\frac{nh}{z_0}\right)\frac{\partial\psi}{\partial t} = \frac{n+\frac{1}{2}}{h}(\psi_{n+1}-\psi_n) + \frac{n-\frac{1}{2}}{h}(\psi_{n-1}-\psi_n), \tag{9}$$

where h is the interval spacing.

Equation (9) is recognized to be analogous to the expression resulting from the application of Kirchhoff's current law ($\sum i = 0$) to the electrical circuit shown in fig. 1. Setting the current entering node n equal to the current leaving this node, we have

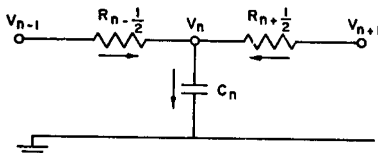

FIG. 1. Typical electric analog of finite-difference expansion.

$$C_n\frac{dV_n}{dt} = \frac{1}{R_{n+\frac{1}{2}}}[V_{n+1}-V_n] + \frac{1}{R_{n-\frac{1}{2}}}[V_{n-1}-V_n], \tag{10}$$

where V_{n+1}, V_{n-1} and V_n refer to the node voltages. A comparison of (9) and (10) indicates that the analogy will be complete if

$$R_{n+\frac{1}{2}} = h/(n+\tfrac{1}{2}),$$
$$R_{n-\frac{1}{2}} = h/(n-\tfrac{1}{2}), \tag{11}$$

and

$$C_n = \log(nh/z_0).$$

The atmosphere may therefore be represented by a resistance-capacitance ladder network. The transient voltage at any node, V_n, is then analogous to the concentration $\psi(z, x)$ at the corresponding point in the air. That is, a plot of the node voltage V_n *versus* time is analogous to the horizontal concentration profile at a height corresponding to $z = nh$. Since $t = Bx/A^2$, the correspondence between the t- and the x-coordinate is determined by the factor B/A^2. As the constants A and B do not appear explicitly in (7), this implies that one solution on the analog is applicable to all combinations of these constants, provided the x-coordinate is adjusted in an appropriate manner.

Boundary condition (2a) implies that the ladder network should extend to infinity. In practice, however, it develops that the concentration falls off rapidly in the z-direction and that it becomes effectively zero at some height. It is therefore feasible to terminate the ladder network after an appropriate number of sections. Boundary conditions (2b) to (2d) indicate that a current impulse of magnitude Q should be applied to node zero at time $t = 0$. This is best accomplished by charging the capacitor C_0 to a voltage Q/C_0 and then connecting it to the network at $t = 0$.

The finite-difference expansion given by (10) could equally well have been derived from physical considerations involving the lumping together of the distributed parameters of the problem. Furthermore, this type of reasoning gives the necessary insight into the physical significance of the mathematical approximations involved. In the present problem, it is advantageous to use a finer net spacing for smaller values of z, since this is the region of most rapid change of concentration. While this could be accomplished formally by modifying the finite-difference interval and the appropriate difference expressions, the physical point of view yields the answers more directly. Thus, the capacitors associated with any node n are given by

$$C_n = \int_{z_1}^{z_2}\log\frac{z}{z_0}\,dz, \tag{12}$$

where the limits of integration are from the mid-point of the preceding to the mid-point of the succeeding

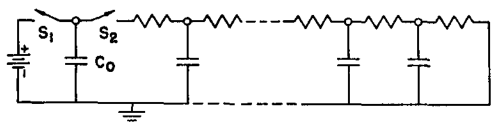

FIG. 2. Schematic diagram of analog network.

node interval. At the input node, the lower limit is taken as z_0. Likewise, the resistance between node n and $n+1$ may be calculated from

$$R_{n+\frac{1}{2}} = \int_{z_n}^{z_{n+1}} \frac{1}{z}\, dz. \qquad (13)$$

4. Experimental procedure and results

In order that a larger range in z and x could be accommodated, two electrical networks, of the form shown in fig. 2, corresponding to different interval spacings, were constructed. Appropriate factors were introduced in (10), so that reasonable values for both the circuit elements and the solution time would be obtained. All circuit elements used were accurate to within 1 per cent. Experimentally it was determined for both networks that the error due to the finite termination of the network was negligible for the first eight nodes of the network.

In operating the analog, all capacitors were initially uncharged. Relay S_1 was first closed to permit C_0 to charge, after which S_1 was opened and S_2 closed, permitting C_0 to discharge through the network. The voltage transient at each node was recorded by means of a two-coordinate plotter. To avoid any loading effect upon the network by the recorder, the operational amplifier circuit shown in fig. 3 was used. This circuit consists of two direct-current amplifiers with resistance feedback and resistance input of the magnitudes shown and has an effective input impedance in excess of 100 megohms.

To show clearly all ranges of the variables, three sets of curves were drawn and are shown in figs. 4 to 6. In these graphs, each ordinate and abscissa has been normalized to represent the dimensionless quantities $(A\psi\, z_0/Q)$ and $(B/A^2)(x/z_0)$, respectively. The graphs express the concentrations at discrete elevations as functions of the distance down-wind from the line source, but concentration at other values may be obtained by interpolation. Consistent units must, of course, be employed throughout.

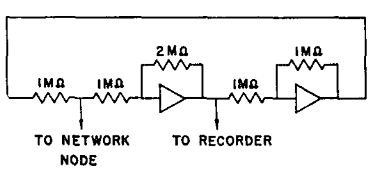

FIG. 3. High-impedance sensing circuit to prevent loading of network.

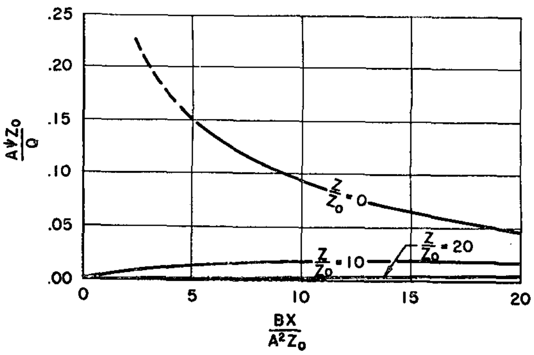

FIG. 4. Horizontal concentration profile near ground level. Both coordinates expressed as dimensionless ratios.

5. Extensions of the method

A major advantage of the analog approach lies in the ease with which it may be adapted and extended to other physical conditions. If, for example, in the present problem it is desired to determine the concentration distribution for a wind-velocity profile which is other than logarithmic or an eddy-diffusivity profile which is other than linear, it is only necessary to modify the magnitudes of the pertinent network capacitors and resistors.

In particular, Deacon (1949) has shown that, in the general case, the wind speed is best expressed by

$$u = A\left[\ln\left(\frac{z+z_0}{z_0}\right) + \frac{1-\beta}{2!}\ln^2\left(\frac{z+z_0}{z_0}\right) + \cdots\right], \qquad (14)$$

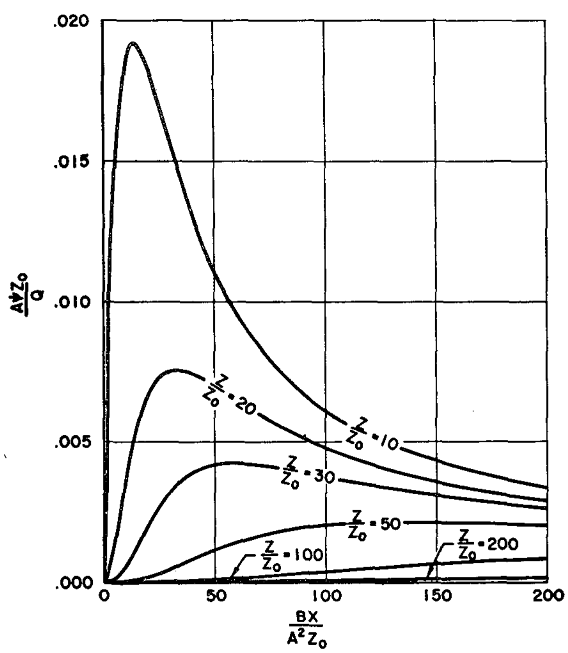

FIG. 5. Horizontal concentration profiles at intermediate elevations.

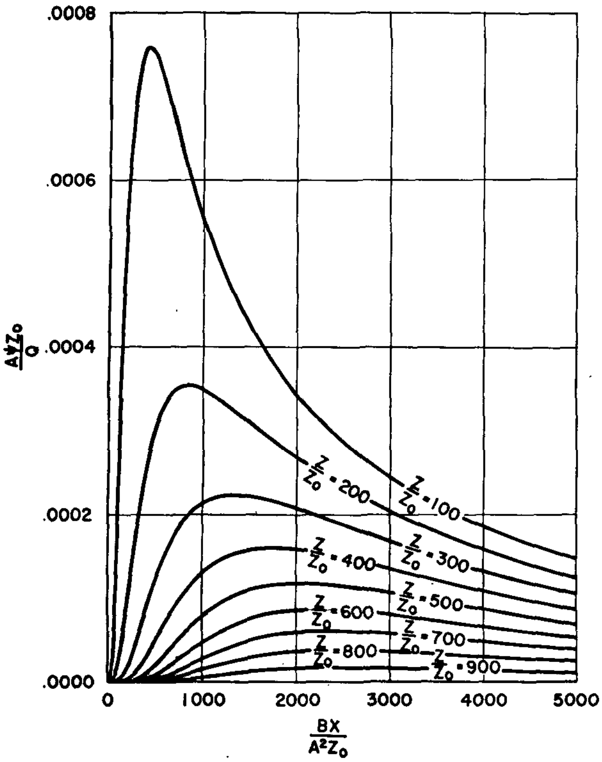

FIG. 6. Concentration profiles at high elevations.

where β is a constant determined by the vertical temperature gradient in the atmosphere. For typical inversion profiles, β is greater than unity, while it is less than unity for typical lapse profiles. The condition of neutral equilibrium treated above, with β equal to unity, is the transition point between these two conditions. To adapt the analog solution to the general case, the network capacitors are determined by

$$C_n = \ln \frac{nh}{z_0} + \frac{1-\beta}{2!} \ln^2 \frac{nh}{z_0} + \cdots. \tag{15}$$

In this procedure, it is possible to treat situations where β changes with height by employing appropriate values of β at the various nodes.

To extend the technique to point sources or finite line sources, it is necessary to consider the eddy diffusivity in the y-direction. This parameter, K_y, is in general a function of z but not of y. Accordingly, (1) becomes

$$u \frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z}\left(K_z \frac{\partial \psi}{\partial z}\right) + K_y \frac{\partial^2 \psi}{\partial y^2}. \tag{16}$$

A finite-difference expansion of this expression leads to a specification of the network resistor $R_{m+\frac{1}{2}}$ connecting node m and node $m + 1$ in the y-direction:

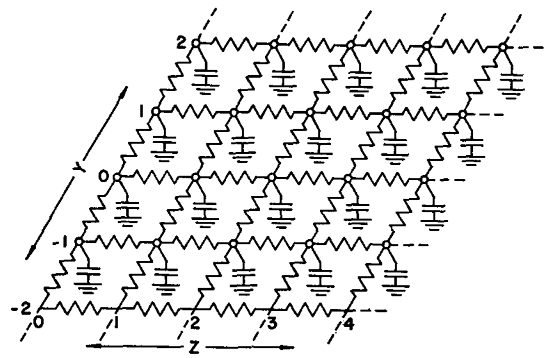

FIG. 7. Network analog for determining diffusion from finite line and point sources.

$$R_{m+\frac{1}{2}} = h^2/K_y. \tag{17}$$

A two-dimensional resistance-capacitance network of the type shown in fig. 7 must therefore be constructed. In effect, a number of identical ladder networks of the type shown in fig. 2 are placed side by side, and corresponding nodes are connected by resistors whose magnitudes are determined by (17). If K_y is a constant, all these resistors are identical; but if K_y is a function of z, their values are determined by the position in the z-plane of the nodes to which they are attached.

For the case of a point source, the capacitor located at ($y = 0$, $z = 0$) is given an initial charge proportional to the strength of the point source. If the source is distributed over a finite length, appropriate capacitors along the $z = 0$ axis are given a total initial charge corresponding to the total particle-emission rate. The voltage transient at any node is then analogous to the concentration profile in the x-direction at the corresponding position in the y- and z-planes. In either case, the network may be simplified by recognizing a plane of symmetry in the y-direction, eliminating half the network and reducing the source by a factor of two.

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