A COORDINATE SYSTEM HAVING SOME SPECIAL ADVANTAGES FOR NUMERICAL FORECASTING

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The coordinate system used to date in numerical forecasting schemes has been the x, y, p, t-system introduced by Sutcliffe and Godart [4] and also by Eliassen [3]. This system, in common with the ordinary x, y, z, t-system, has certain computational disadvantages in the vicinity of mountains, because the lower limit of the atmosphere is not a coordinate surface. The purpose of this brief note is to describe a modified coordinate system in which the ground is always a coordinate surface.

It is obtained by replacing the vertical coordinate p in the x, y, p, t-system by the independent variable $\sigma = p/\pi$, where $\pi = \pi(x, y, t)$ is the pressure at ground level. σ ranges monotonically from zero at the top of the atmosphere to unity at the ground. In describing the relation between this x, y, σ, t-system and the usual x, y, p, t-system, we will use a subscript p to indicate a derivative along a pressure surface. Differentiation in the new x, y, σ, t-system will have no subscripts.

The following relation holds, where ξ can be x, y, or t:

$$\left(\frac{\partial}{\partial \xi}\right)_p = \frac{\partial}{\partial \xi} - \frac{\sigma}{\pi}\frac{\partial \pi}{\partial \xi}\frac{\partial}{\partial \sigma}.$$

The horizontal equations of motion then become

$$\frac{du}{dt} = fv - \frac{\partial \phi}{\partial x} + \frac{\sigma}{\pi}\frac{\partial \phi}{\partial \sigma}\frac{\partial \pi}{\partial x} + F_x, \qquad (1)$$

and

$$\frac{dv}{dt} = -fu - \frac{\partial \phi}{\partial y} + \frac{\sigma}{\pi}\frac{\partial \phi}{\partial \sigma}\frac{\partial \pi}{\partial y} + F_y, \qquad (2)$$

where F_x and F_y are the horizontal components of the frictional force per unit mass, $u = dx/dt$, $v = dy/dt$, f is the Coriolis parameter, and ϕ is the geopotential. As is customary in most meteorological work, the Coriolis terms proportional to the cosine of the latitude have been neglected. Equations (1) and (2) differ from those in the x, y, p, t-system only by the inclusion of the terms in $\partial\phi/\partial\sigma$.

The operator d/dt is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \dot{\sigma}\frac{\partial}{\partial \sigma}, \qquad (3)$$

where $\dot{\sigma} = d\sigma/dt$.

The hydrostatic equation is obtained from the relation

$$\frac{\partial}{\partial p} = \frac{\partial \sigma}{\partial p}\frac{\partial}{\partial \sigma} = \frac{1}{\pi}\frac{\partial}{\partial \sigma},$$

and becomes, simply,

$$\partial\phi/\partial\sigma = -RT/\sigma. \qquad (4)$$

Here R is the gas constant, and T is the absolute temperature. The coefficient $\pi^{-1}\sigma(\partial\phi/\partial\sigma)$ appearing in (1) and (2) can thus be replaced by $-RT/\pi$. Since ϕ is known at $\sigma = 1$ (at the ground), a knowledge of $T(\sigma)$ will give $\phi(\sigma)$ from (4) by integration.

The equation of continuity in the x, y, p, t-system is

$$\nabla_p \cdot \mathbf{v} + \partial\omega/\partial p = 0,$$

where $\omega = dp/dt$, and $\mathbf{v}$ is the horizontal velocity. Introducing the relation $dp/dt = \pi\dot{\sigma} + \sigma(d\pi/dt)$, we obtain the continuity equation in the new system:

$$\nabla \cdot \pi\mathbf{v} + \pi\,\partial\dot{\sigma}/\partial\sigma + \partial\pi/\partial t = 0. \qquad (5)$$

Since $\dot{\sigma}$ is zero at the top of the atmosphere ($\sigma = 0$), integration of (5) with respect to σ gives

$$\pi\dot{\sigma} = -\int_0^\sigma \nabla \cdot \pi\mathbf{v}\, d\sigma - \sigma\frac{\partial \pi}{\partial t}. \qquad (6)$$

Extension of the integration all the way to the ground ($\sigma = 1$) gives the formula for $\partial\pi/\partial t$:

$$\frac{\partial \pi}{\partial t} = -\int_0^1 \nabla \cdot \pi\mathbf{v}\, d\sigma, \qquad (7)$$

since $\dot{\sigma} = 0$ at the ground.

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The first law of thermodynamics can be written as

$$\frac{d \ln \theta}{dt} = \frac{1}{c_p T} \dot{Q}, \qquad (8)$$

where θ is the potential temperature, c_p the specific heat at constant pressure, and $\dot{Q}$ is the non-adiabatic rate of heating per unit mass. When $\dot{Q}$ is proportional to dp/dt, as in the pseudo-adiabatic condensation process, dp/dt can be computed from the equation

$$\frac{dp}{dt} = \sigma \boldsymbol{v} \cdot \nabla \pi - \int_0^\sigma \nabla \cdot \pi \boldsymbol{v} \, d\sigma. \qquad (9)$$

Finally, the potential temperature θ is related to σ, π and T by the equation

$$\ln\theta = \ln T - \kappa(\ln\pi + \ln\sigma) + \kappa \ln P, \qquad (10)$$

where $\kappa = R/c_p$ and P is the standard pressure (normally 1000 mb) at which θ is defined.

Equations (1) to (10), in the dependent variables $\boldsymbol{v}$, ϕ, θ, T and π, would seem to have their greatest advantage in making a numerical forecast with the "primitive" equations of motion. Although they could undoubtedly also be used in formulating a system which incorporates either the quasi-geostrophic or the quasi-nondivergent assumption [1; 2], the somewhat more complicated forms of the pressure-force term in (1) and (2), and of the continuity equation (5), naturally result in more complicated vorticity and divergence equations. However, the new system does have the following very real advantages in the numerical process:

1. Vertical advection terms, *e.g.*, $\dot{\sigma}\, \partial u/\partial\sigma$, are identically zero at the top and bottom of the atmosphere.
2. The ground is a coordinate surface, so that the effect of orography can be introduced without leading to either (*a*) uncentered horizontal differences in the vicinity of mountains or, alternatively, (*b*) the assumption that the hypothetical flow patterns obtained by reduction to sea level actually exist.

The observations defining the initial state of the atmosphere in this system would, of course, have to be interpolated so as to apply at the various σ-levels used in the finite-difference forecast scheme rather than at the conventional standard pressure levels. Since the actual method used for this would probably depend on the forecast equations to be employed, and since the various possibilities for performing this interpolation are quite obvious, this aspect of the problem will not be discussed here.

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