DISPERSION IN THE UPPER ATMOSPHERE

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ABSTRACT

Transosonde balloon data are analyzed to provide a measure of the dispersion on a large scale in the upper atmosphere. An expression for the autocorrelation function is inferred from synoptic experience and is applied to autocorrelation values computed from the balloon data. Taylor's theorem is applied to this function to compute dispersion out to ten days. A comparison is made with direct computation at two days.

1. Introduction

In order to determine the efficacy of the large-scale motions of the high atmosphere to disperse floating particles, it appeared possible to utilize the constant-level balloon data of the Transosonde and Moby Dick programs. Data of this sort are ideally suited to an analysis based on Taylor's theory of diffusion by continuous movements [1]. The constant-level balloons are released singly and tracked by radio techniques for periods up to several days. The dispersion of the balloons can be derived from these data by superposing the initial points of individual trajectories and observing the distribution of positions after any given lapse of time. This procedure, if applied to balloons released throughout a specific season, provides an estimate of the distribution of end points appropriate to that season. It does *not* provide an estimate of the dispersion of a particular cluster released at one time and place.

Fig. 1 schematically describes the results that can be obtained by such a procedure. The probable end point of a particle at a given time after its release is described as a mean position and the expected variation about that mean. It is assumed that the distribution of end points is a bivariate normal distribution. An attempt will be made in the following sections to derive the parameters of this distribution. This could be done by simply plotting the end points and fitting the parameters of the distribution. It could also be done by following the arguments of Taylor's theorem. Both methods are used here, the first because of its simplicity and the second because it provides an insight into the motions which are responsible for the dispersions.

The data used were taken from transosonde flights at approximately 300 mb, for April and May, 1953, and January and February, 1956. These data consisted of four hourly fixes from which four hourly wind components were computed. To apply Taylor's technique, autocorrelations were obtained from these wind data. In order to smooth the data and to treat the problem analytically, a mathematical function was fitted to the autocorrelations. This function was then integrated to yield the desired parameters.

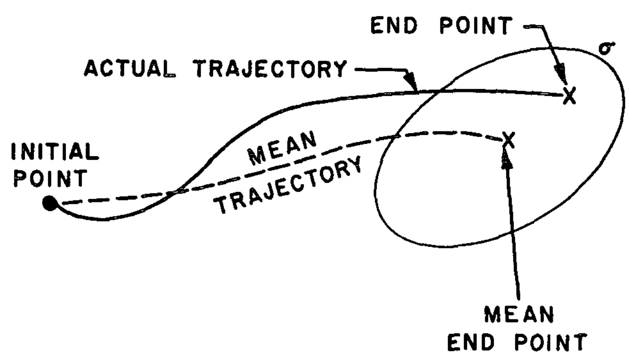

FIG. 1. Schematic representation of dispersion.

2. Theory

The balloon trajectories will be treated as though they are confined to a horizontal plane. Taylor's theorem as generalized for two components [2] may be written as follows:

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$$\overline{X_iX_j} = 2\sigma^2 \int_0^t (t - \xi)R_{u_iu_j}(\xi)d\xi \quad i, j = 1, 2, \quad (1)$$

where $\overline{X_iX_j}$ is the variance or covariance of the displacement during the time t; σ^2 is the variance of the wind, assumed independent of the direction and time; t is the time of flight of the particle; ξ is a time difference during the flight; $R_{u_iu_j}(\xi)$ is the autocorrelation function; and u_i and u_j are components of velocity. Both $R_{u_iu_j}(\xi)$ and $\overline{X_iX_j}$ were found to have very small values where $i \neq j$; therefore only $R_{u_iu_i}$ and $R_{u_ju_j}$ were used in the determination of the variance of the displacements. This is equivalent to assuming that the displacement patterns are symmetrical about the west-east and south-north axis.

3. Computation of $R(\xi)$

The south-north velocities, V, were computed by converting the latitudinal displacement between four hourly fixes to nautical miles. The west-east velocities, U, were computed by converting the longitudinal displacement into nautical miles using the midlatitude scale. The equation used for computing the autocorrelation was as follows:

$$R(\xi) = \frac{\overline{v(t)v(t+\xi)}}{\sqrt{\overline{v(t)^2}\,\overline{v(t+\xi)^2}}}, \quad (2)$$

where

$$v(t) = V(t) - \overline{V(t)},$$
$$v(t+\xi) = V(t+\xi) - \overline{V(t+\xi)}.$$

This form allows for variation along the trajectory of the variance of the wind. This variation in the variance was found to be small; therefore $\sqrt{\overline{v(t)^2}\,\overline{v(t+\xi)^2}}$ was assumed constant and set equal to σ^2 in all subsequent computations. The averages indicated in the equation were taken over all available balloon flights. In order to get a sufficient sample size for large ξ, each point on a trajectory was considered to be an acceptable initial point for a time difference.

The points on fig. 2 represent values of $R(\xi)$ for the u component and those on fig. 3 the values of $R(\xi)$ for the v component. In both instances the points appear to describe damped oscillations with periods of about two and one-half days. It should be noted, however, that the u component appears to have a period of about 52 hours, whereas the v component apparently has a longer period, about 62 hours. In addition to the regular oscillation, minor irregular changes also are apparent.

4. Construction of Model for $R(\xi)$ and Fit to Data

In order to develop analytic expressions for these autocorrelation functions, it is natural to consider the characteristics of the atmospheric flow at 300 mb,

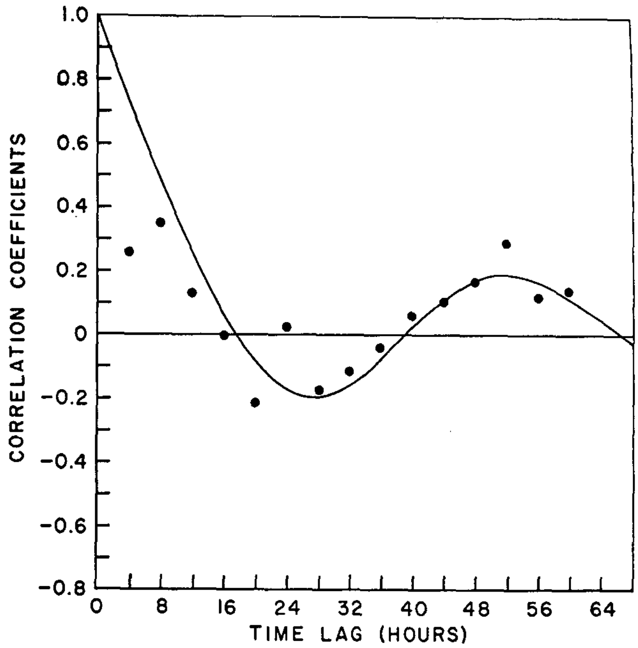

FIG. 2. Autocorrelation function, u component.

the level at which the Transosonde data were obtained. Long synoptic experience with the 300-mb map has shown that in midlatitudes this level is dominated by strong westerly winds on which are superposed long waves. These long waves generally move more slowly than the air, and they change their shape and dimensions but slowly. If a particle were introduced into this flow and there were no other perturbations present, its south-north motion might be described by

$$v = v_0 \cos(\omega t + \phi), \quad (3)$$

where v_0 is the amplitude, ω the frequency, and ϕ the phase angle. Different particles at different times will

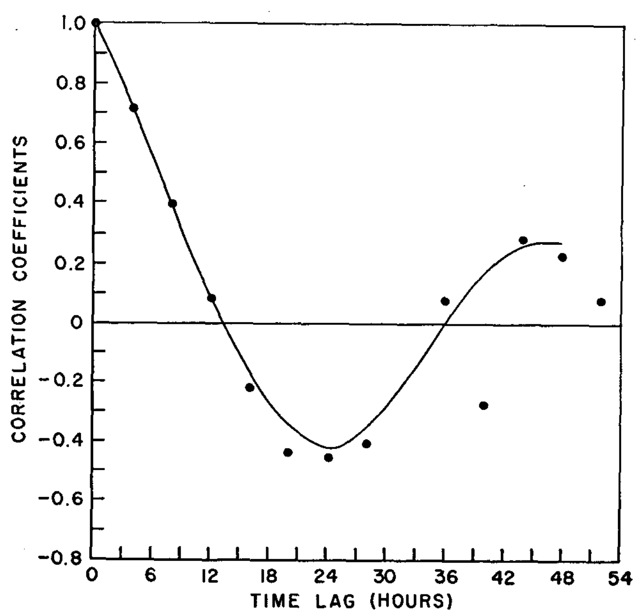

FIG. 3. Autocorrelation function, v component.

experience different amplitudes, frequencies, and phases. The motion of the particle, however, is not completely governed by these long waves; there are shorter, less regular perturbations present. In the data which were used, it was not always possible to distinguish the smaller perturbations from the errors of observation. A term must be added to the above expression to account for these less regular motions. Thus the motion of a single particle in this part of the atmosphere might be expressed by

$$v = v_0 \cos (\omega t + \phi) + v'(t), \tag{4}$$

where $v'(t)$ takes into account all the changes not due to the regular, long waves.

In order to compute the autocorrelation function, it is necessary to determine the value of $\overline{v(t)v(t+\xi)}$. The bar denotes that the average is taken over all possible phases, over a range of amplitudes and a range of frequencies. This is necessary because a given particle may be introduced into any part of a wave pattern possessing any of a range of frequencies and amplitudes. This averaging process may be performed in the following manner:

$$\overline{v(t)v(t+\xi)} = \frac{1}{2\pi}\frac{1}{2\Delta v_0}\frac{1}{2\Delta\omega} \times \int_0^{2\pi}\int_{v_0-\Delta v_0}^{v_0+\Delta v_0}\int_{\bar{\omega}-\Delta\omega}^{\bar{\omega}+\Delta\omega} v(t)v(t+\xi)d\phi d\omega dv_0. \tag{5}$$

When (4) is introduced in (5) and the integration performed, the following expression results:

$$\overline{v(t)v(t+\xi)} = \frac{2A^2}{\Delta\omega\xi}[\cos \bar{\omega}\xi \sin \Delta\omega\xi] + \overline{v'(t)v'(t+\xi)}, \tag{6}$$

where A^2 is a function of the amplitude and the range of amplitude.

If (6) is divided by the variance of the wind, it becomes an expression for the autocorrelation function. The last term in this expression, $\overline{v'(t)v'(t+\xi)}/\sigma^2$, involving only irregular motions and errors, is treated as a purely Gaussian random function and as such takes the form of a negative exponential [4]. The autocorrelation function, therefore, should be of the form:

$$R(\xi) = \frac{K^2}{\Delta\omega\xi}\cos \bar{\omega}\xi \sin \Delta\omega\xi + c^2 e^{-\mu\xi}, \tag{7}$$

where

$$K^2 = \frac{2A^2}{\sigma^2}$$

c^2 = a constant involving σ^2

μ = the exponential coefficient.

Where $\xi = 0$, $R(\xi) = 1$; and thus $K^2 + c^2 = 1$. Therefore K^2 and c^2 represent the relative significance of the two terms. It should be noted that the first term represents a damped oscillation, and that the sum of the two terms approaches zero at large ξ. This function, therefore, satisfies the requirements for an autocorrelation function.

To fit (7) to the data for the v component of the autocorrelation (shown in fig. 3), a trial-and-error method was used which involved adjusting K^2, c^2, $\bar{\omega}$, $\Delta\omega$, and μ. The result is shown as the solid curve in fig. 3. The arrangement of the data points for the autocorrelation of the u component, (fig. 2) suggested that the same form of the function might be applied to the u component as well. The solid curve in fig. 2 represents the fit of the function to these data. Table 1 gives the values of the parameters determined by these curves.

Table 1. Parameters of the autocorrelation functions.

Parameter	Units	u comp.	v comp.
K^2		0.28	0.49
c^2		0.72	0.51
μ	day^{-1}	2.41	3.51
$\bar{\omega}$	cycles/day	0.45	0.50
$\Delta\omega$	cycles/day	0.11	0.14

The values of K^2 and c^2 for the u and v components indicate the relative importance of the periodic and random terms. In the case of the south-north motion there appears to be an equipartition of this influence, while in the case of the zonal motions the partition appears to be three to one in favor of the random term. The values for μ indicate that the magnitude of the random term diminishes to less than 0.1 after one day.

The difference between the values of $\bar{\omega}$ for the u and v components of the velocity is small. The variation in $\Delta\omega$ for the u and v components also appears to be small. The mean frequencies indicate that a particle moving through a wave will perform a complete oscillation in about two days. The values of $\Delta\omega$ indicate that the range of periods is from about 1.3 to three days.

Comparing these results with flow patterns as they appear on the 300-mb chart, it is found from the above analysis that the particle's speed averaged over one complete journey through a typical wave, say 2500 miles in length, would be about 50 knots. This is in fair agreement with observed average velocities at this level.

5. Computation of the Dispersion

The dispersion may now be computed by inserting this expression for the autocorrelation function into Taylor's equation. If (1) is divided by $2\sigma^2$, it may be written as follows:

$$\frac{\overline{X_iX_i}}{2\sigma^2} = I^2 = \int_0^t (t-\xi)R(\xi)d\xi. \tag{8}$$

When $R(\xi)$ is substituted from (7) into this form of the Taylor theorem, the integration can be divided into three parts:

$$I^2 = \frac{K^2 t}{\Delta\omega}\int_0^t \frac{\cos\bar{\omega}\xi \sin\Delta\omega\xi}{\xi} d\xi - \frac{K^2}{\Delta\omega}\int_0^t \cos\bar{\omega}\xi \sin\Delta\omega\xi d\xi + c^2\int_0^t (t-\xi)e^{-\mu\xi}d\xi. \quad (9)$$

The first term on the right of (9) cannot be integrated analytically, but it can be put into the form of the sum of two integrals of the type $\int_0^x (\sin y)/y\, dy$. This integral has been extensively tabulated [5]. The last two terms in (9), on the other hand, are integrable and can be evaluated for any time in terms of the parameters of (7). The evaluation of all three terms was made for both the u and v components out to ten days. Figure 4 shows the value of I for both of the components for this period. To obtain the dispersion for any time, it is only necessary to multiply the I values of fig. 4 by $\sqrt{2}\sigma$.

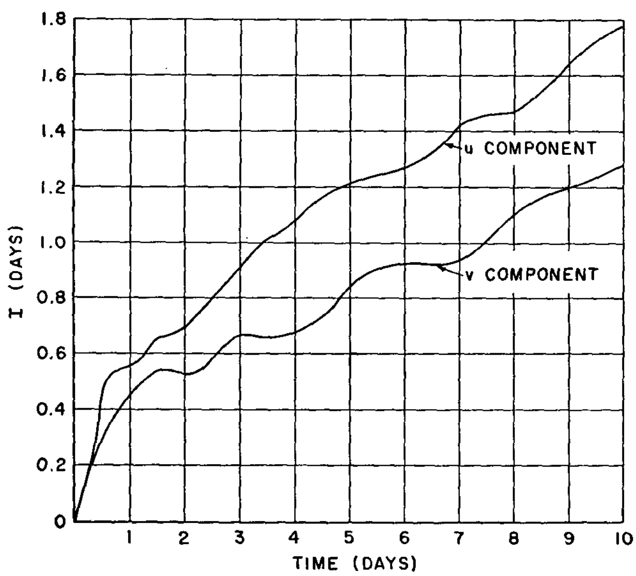

FIG. 4. I values as a function of time.

There are two types of standard deviations involved here. The first, which might be termed the local standard deviation, is computed from winds observed at fixed locations, *i.e.*, rawin data. The second, which might be termed the total standard deviation, is computed from transosonde winds. Since these winds are observed from a platform that follows the air motion, the computed standard deviation includes spatial as well as time variations. Two sources of data on the local standard deviation were available: the estimates available from Upper Winds Over the World [3]; and some spot computations due to Crutcher [6]. Table 2 summarizes these data, and the standard deviations computed from the transosonde flights.

TABLE 2. Standard deviations of wind component.

σ_u	σ_v	Source of data
49	51	Transosonde
30	30	Upper Winds over the World [3]
30.6	36.8	Calculated for Omaha, Neb. Crutcher [6]

The total standard deviations (transosonde) are seen to be larger by a substantial amount.

In order to obtain a valid comparison of the dispersions as calculated by the correlation method and the direct method, it is necessary to use the standard deviations of the Transosonde data. The results of these computations for t = two days are given in table 3. The direct computation involved the calculation of 29 sets of distances (balloon travel over a two-day interval) using the same system that was used for the four-hour winds. The standard deviation of these displacements were then calculated to give a direct measure of the dispersion. The tabulated values indicate there is still much to be desired in the way of agreement between the two methods of computation. The differences are, however, not unexpected in the light of the bias of the sample.

TABLE 3. Standard deviation of two-day positions.

	West-East	South-North
Direct computation	1598 n mi	677 n mi
Autocorrelation technique	1142 n mi	917 n mi

One of the interesting features of the method involving the autocorrelation function is its ability to extrapolate in time beyond the balloon data. To demonstrate this the dispersions were computed out to ten days for the hypothetical case of floating particles released at San Francisco at the 300-mb level during the months of December, January, and February. An estimate of the mean trajectory of the particles was determined from the December-January-February mean 300-mb chart from Ref. [3]. This result is shown in fig. 5. The dotted ovals in the figure outline the area within which 67 per cent of the floating particles would be expected to be found at the time indicated at the center of the oval. In these computations the standard deviations of the wind used also were obtained from Ref. [3]. In this type of example it seemed appropriate to use the local standard deviation in the computation of dispersion because the spacial variation was taken into account partially by using a varying mean current represented by the meandering mean trajectory. Interesting though such

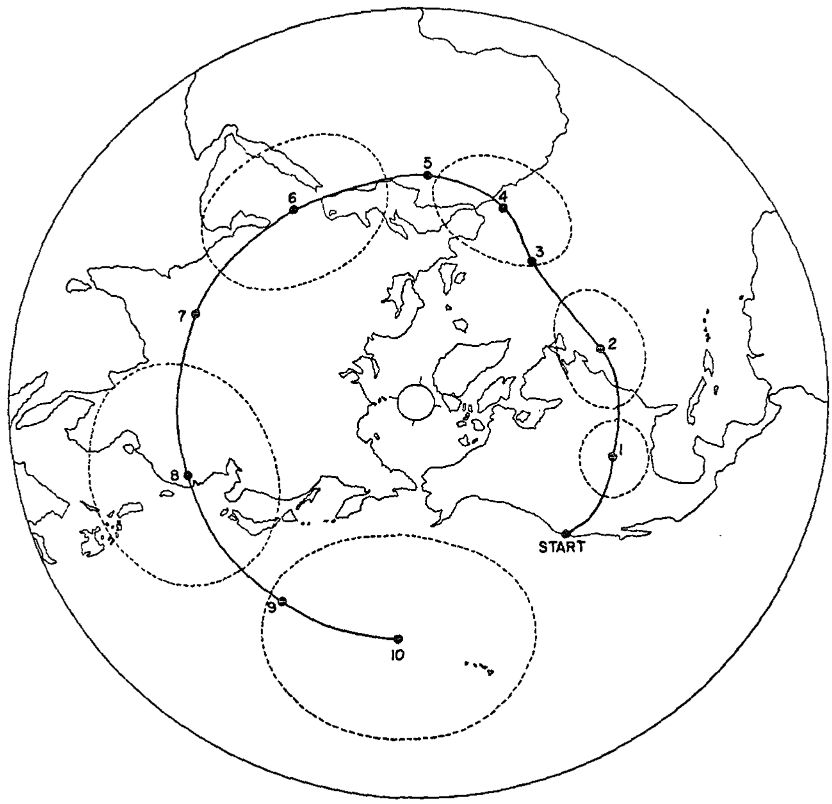

FIG. 5. An example of the dispersion for particles at 300 mb in winter, starting point San Francisco.

an extrapolation may be, it has inadequacies. Specifically, at eight to ten days it is noted that some particles are estimated to have moved south of the zone of the westerlies. Atmospheric flows at these near-equatorial latitudes differ substantially from those at midlatitudes where the Transosonde flights were made. The computed dispersion is, therefore, not representative for these tropical latitudes. However, the patterns provide a first estimate of how particles, floating at 300 mb in midlatitudes, tend to be dispersed as they circumnavigate the globe.

6. Critique and Conclusion

There are several limitations of the above technique. It should be pointed out that the statistical treatment assumed that the winds were blowing on a plane surface rather than on the spherical earth. This in itself would lead to distortions of the distribution function. Further, the statistics were accumulated only in the temperate latitudes for only four different months over a small portion of the northern hemisphere. This severely restricts the usefulness of the results. There is a more basic difficulty in that the assumption is made in Taylor's theorem that the standard deviation of the wind is constant along the trajectory; in other words, that the process is a stationary time series. The fact that these atmospheric motions can not be represented by a strictly stationary time series has been recognized in the computations of the autocorrelations. The deviations that the nonstationarity introduces are not obvious in the final results. Finally, the failure to consider the cross-correlations, R_{uv}, may have introduced unknown errors into the estimates of dispersion.

Despite these obvious disadvantages, the system does provide a method for estimating either the dispersion of successive flights or the probability of discovering a particle in a given area after a given lapse of time. Better estimates will depend upon larger and more representative samples and upon the elimination of some of the above limitations of the theory.

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