A NOTE ON OVERSTABILITY AND THE ELASTOID-INERTIA OSCILLATIONS OF KELVIN, SOLBERG, AND BJERKNES

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ABSTRACT

Results elucidating the mechanism by which overstable (oscillating) cellular convection occurs in rotating layers are given. There is a close connection with the elastoid-inertia oscillations in a rotating cylinder discussed by Kelvin, Solberg, and Bjerknes, and it is found that diffusion of heat by conduction acts in a destabilizing way in the motion. Simple experiments in a rotating cylinder on the elastoid-inertia oscillations show a remarkably close agreement with the free periods calculated theoretically. This, in the field of meteorological experiments, is important in establishing close agreement with meteorological theory in a situation where the flow can be strongly ageostrophic.

1. Introduction

Considerable progress has been made in recent years in the study of the unstable gravitational cellular convection of Benard and Rayleigh under more general conditions than were assumed in the extensive work prior to 1940. In particular, S. Chandrasekhar has investigated the marginal stability conditions for a variety of problems of general physical interest. In one group of papers (Chandrasekhar, 1953; Chandrasekhar and Elbert, 1955; Chandrasekhar, 1957), he has calculated the critical conditions for onset of convection in a horizontal layer of fluid which is rotating around a vertical axis and, in others (Chandrasekhar, 1952; Chandrasekhar, 1954 and 1956), the onset in a layer subject to a uniform magnetic field or both to rotation and a magnetic field. The phenomena to be discussed in this note are met primarily in the case of a rotating layer, but analogous phenomena are also present with magnetic fields and in more complicated contexts. For a rotating layer, Chandrasekhar finds that the critical Rayleigh number

\[ R_e = \frac{\alpha g \beta d^4}{\kappa \nu}, \]

where, in the usual notation, \( g \) is the acceleration of gravity, \( \alpha \) the temperature expansion coefficient, \( \beta \) the vertical adverse temperature gradient, \( d \) the depth, \( \kappa \) the thermometric conductivity, and \( \nu \) the kinematic viscosity, no longer has a single critical value for each set of boundary conditions as it does in the non-rotating problem. The critical value is a function of a rotation parameter which can be taken as the Taylor number

\[ T \equiv \frac{4\Omega^2 d^4}{\nu^2}, \]

where \( \Omega \) is the basic rotation rate (Chandrasekhar 1953) and of the Prandtl number

\[ \rho \equiv \nu/\kappa. \]

Moreover, and this is the point of major interest here, the convection in certain ranges may not begin as steady unidirectional cells but as cells oscillating in time. Chandrasekhar calls this "overstability," and it corresponds to the presence of a pure imaginary frequency at marginal stability.

The physical effects on the convection introduced by the rotational stability associated with \( T \) are, according to theory:

1. The critical Rayleigh number \( R_e \) is very much increased for high \( T \), with ultimately \( R_e \) increasing proportionally to \( T^4 \).

2. Owing to the inertial resistance to radial displacements in the cells, the horizontal sizes decrease or the horizontal wave numbers increase, markedly. With \( a^2 = d^2 K^2 \) where \( K^2 \) is the dimensional horizontal wave number squared, \(-\text{i.e.,} K^2 \equiv K_x^2 + K_y^2 \) where \( K_x, K_y \) are wave numbers in two orthogonal horizontal directions, \( a^2 \) is ultimately proportional to \( T^4 \) for high \( T \).

3. Because of the tendency to circulation conservation for fluid rings concentric with the cell axes, relative rotations about the cell axes develop.
4. Provided the Prandtl number \( P \) is less than a critical value (0.627 for boundary conditions in the theory such that both bottom and top are treated as though they were free liquid surfaces with no viscous stresses) and \( T \) is sufficiently high, overstable oscillating cells develop at marginal stability. The axial rotations in (3) then reverse periodically. This period \( \tau_p \), measured in units of the rotation period \( \tau_p \), increases with \( T \), ultimately as \( T^{1/6} \).

All of these theoretical indications have been verified with good to fair quantitative precision in convection experiments using water \((P \sim 7)\) and mercury \((P \sim 0.025)\) (Nakagawa and Frenzen, 1955; Fultz, Nakagawa, and Frenzen, 1954; Fultz and Nakagawa, 1955; Nakagawa, 1957; Dropkin and Globe, 1958).

A physical understanding of the reasons for the oscillating cells is intimately tied to the low Prandtl number condition on their occurrence at lower \( R_e \)'s than the ordinary cellular convection. A qualitative attempt at an explanation is given by Nakagawa and Frenzen (1955) that depends on the high thermal conductivity versus low viscosity for low \( P \)'s. It ascribes the oscillation primarily to a mechanism of the following kind: starting with, for example, rising fluid at the bottom near the cell center, this portion of the fluid acquires such a high buoyancy by conduction from the bottom that a rapid rise ensues and so rapid a spreading all the way to the periphery of the cell that the solenoids associated with the density stratification and the cell motion decelerates and reverses. By then, sufficient cooling at the top has occurred to reestablish instability and to drive the cell over-rapidly in the opposite direction. The rotational resistance to the converging and diverging motions with respect to the cell axis is regarded as a contributing factor only; their language, however, is very reserved on whether this could really be the case in view of the first objection below.

There are at least a couple of serious objections to the idea sketched above that occur on further thought:

1. If the conduction mechanism described is really sufficient, it is very difficult to see why, in a non-rotating layer, there should not be some limiting Prandtl number, perhaps extremely small, where the marginal stability criterion changes from that of ordinary convection to one of overstability.

2. In order to decelerate the cell motion, it is necessary in Nakagawa and Frenzen's mechanism to reverse the sense of the density-pressure solenoids in accordance with the Bjerknæs Circulation Theorem, and reflection shows that this must imply particle motions with amplitudes that exceed half the distance around the closed streamlines of the motion in meridional planes of the cells. This is more or less irreconcilable with the fundamental perturbation nature of the theory which implies, sufficiently close to marginal stability, small-amplitudes in all velocities and displacements for a flow periodic in time.

3. If the thermal conductivity is high, lateral diffusion of the perturbation temperature in the middle of the cells will be high, and it is not clear that this will not over-balance conduction effects at the top and bottom.

G. K. Batchelor pointed out to the writer two things which have led to great clarification of the oscillation mechanism and to the experiments reported in this note. His points were:

1. There ought to be a distinct stabilizing mechanism to give rise to the possibility of oscillations, and (consistent with small amplitudes) this can only be the inertial stability associated with the rotation.

2. To maintain or amplify the motions, there must be an energy transfer from the unstable density stratification in correct phase with the oscillations and sufficiently large both to compensate viscous dissipation and to provide for the parts of each cycle when the gravitational potential energy must increase. The high conductivity is what makes this possible.

Since the absolute motion is nearly a rigid rotation, point (1) is connected with the celebrated discussion by Rayleigh (1916) showing that axially symmetric perturbations (here relative to the cell axis) are stable in frictionless fluids provided the absolute circulation increases away from the axis. This, finally, suggested a strong connection between the oscillating cells and the elastoid-inertia oscillations discussed by V. Bjerknæs et al., in "Physikalische Hydrodynamik" (1933) as is detailed further below where point (2) will also be discussed.

2. Elastoid-inertia oscillations

Consider a right circular cylinder of radius \( r_0 \) containing frictionless, homogeneous, incompressible fluid (later to be compared with an individual convection cell) that is rotating rigidly about its axis of figure. Planes perpendicular to the rotation axis at distance \( \delta \) apart bound the fluid at top and bottom. The rotational stability gives rise to the possibility of axially symmetric standing-wave oscillations. The fundamental mode for these oscillations (see the next section for detailed expressions) consists of a periodic particle motion up and down the axis, an opposed oscillating motion parallel to the axis near the outer wall, and corresponding radial motions near top and bottom. The instantaneous streamlines of the motion in planes through the axis are closed curves (see fig. 1). The rotational stability arises from the variation of
the zonal or azimuthal motion in the regions of radial displacement in consequence of circulation conservation for fluid rings centered on the axis (Rayleigh, 1916). Fluid rings displaced outward slow up in the absolute motion (have easterly zonal components in relative coordinates) and, arriving in regions of increased equilibrium radial pressure gradient, are accelerated back toward their equilibrium position. Conversely, fluid rings displaced inward speed up, etc. The three-dimensional stream surfaces are toroids.

These standing oscillations have eigenfrequencies (see below) which are always less than 2\(\Omega\) and which depend, for a given normal mode, only on the ratio of the depth to the radius, \(\delta' = \delta/r_o\). \(2\Omega\), of course, is the so-called inertia circle frequency. The detailed theoretical expressions (given in the next section) were given for a number of cases, including the one we consider, in "Physikalische Hydrodynamik," (V. Bjerknes et al, 1933, p. 429 ff.) after initial work by H. Solberg (1928, p. 60 ff.) and V. Bjerknes and H. Solberg (1929).

The name "elastoid-inertia" oscillations was first used by these authors. Their special interest at the time was in what was thought to be the new feature of an internal oscillation of homogeneous fluid depending only on inertia and not on variable free surfaces, gravity, or density stratification. It turns out, however, that the essential part of the theory, including the frequency equation, was given by Kelvin (1880). Its significance in meteorological dynamics had to be rediscovered.

Experimental demonstrations of this type of motion have been fairly numerous though not well-known until fairly recently. Kelvin, himself, used a rapidly rotating spherical container to show that the rotational stability (rigidity of the absolute vortex tubes) almost prevented rising motion of a small buoyant sphere and it is very probable that he also caused the sphere to oscillate (Kelvin, 1881). More recently, both Sir G. Taylor\(^3\) and V. Bjerknes\(^4\) have used a small neutrally-buoyant sphere or cylinder immersed in fluid in a rotating cylinder to demonstrate the periodic motion on the axis when the floating object is disturbed.

Further, the waves in the wake of a sphere moving parallel to the axis in a rotating cylinder that have been very carefully studied by R. R. Long (1953) are essentially the progressive or stationary wave analogue of these standing oscillations (Squire, 1956).

None of the experimental work known to me, except for Long's which did not involve dealing with individual normal modes, has been carried to the point of determining the degree of quantitative agreement with the theoretical work. It therefore seemed worthwhile to do something along this line both for the intrinsic interest of the phenomenon and for more firmly establishing the connections with the cellular oscillations that will be given later.

\(^3\) Personal communication.

\(^4\) Done for the first time at a meeting of Norsk Geofys. Forening, on 3 October 1935 in Oslo. The cylinder used in these demonstrations had a \(\delta'\) ratio of about 2.5 and should therefore have given \(rs'\) values of about 1.6.

The linearized perturbation theory of these oscillations when the basic motion is a rigid rotation is given by Kelvin (1880, p. 154), V. Bjerknes and Sölberg (1929), and V. Bjerknes et al (1933, p. 434). Only the final solutions will be summarized here in a form suitable for comparison with experimental data. In a cylindrical \((r, \theta, z)\) coordinate system, the Eulerian expressions for the velocity components relative to the system rotating with the cylinder are

\[
\begin{align*}
\dot{u}_r &= -\frac{\dot{r}}{\tau_\Omega} = -\nu' A_1' J_1(\gamma r') \cos(n \pi z') \sin(\nu't') \\
\dot{u}_\theta &= -\frac{\dot{\theta}}{\tau_\Omega} = v A_1' J_1(\gamma r') \cos(n \pi z') \cos(\nu't') \\
\dot{u}_z &= -\frac{\dot{z}}{\tau_\Omega} = v A_1' \delta' \cdot \left( \frac{\gamma}{n \pi} \right) J_0(\gamma r') \sin(n \pi z') \sin(\nu't')
\end{align*}
\]

where, in this context, \(\nu'\) is a nondimensional frequency \(\equiv \nu/\Omega = 1/\tau_\Omega^0\) and \(\tau_\Omega^0 = \tau_0/\tau_\Omega\) is the oscillation period measured in units of the rotation period \(\tau_\Omega = 2\pi/\Omega\). \(A_1' = A_1/\tau_0\) is the arbitrary amplitude of the linear theory, \(\nu' = r/\tau_0\), \(\delta' = \delta/\tau_0\), \(z' = z/\delta\), and \(t' = \Omega t\).

\(J_0\) and \(J_1\) are the 0- and 1st-order Bessel functions of the first kind.\(\gamma_i\) and \(n\) are the indices that specify the possible normal modes of oscillation; \(\gamma_i\) is the \(i^{th}\) root of \(J_1\) so that the number of interior nodal radii for the radial and azimuthal velocities is \((i - 1)\), and \(n\) is the number of cells in the \(z\) direction so that the number of interior nodal planes of \(u_z\) is \((n - 1)\).

The Stokes stream function for the meridional motion is

\[
\psi' = v \left( \frac{A_1' \delta \nu}{n \pi} \right)^{-1} r' J_1(\gamma r') \sin(n \pi z') \sin(\nu't')
\]

with

\[
\begin{align*}
\dot{u}_r' &= \frac{1}{r'} \frac{\partial \psi'}{\partial r'} \\
\dot{u}_z' &= -\frac{1}{r'} \frac{\partial \psi'}{\partial z'}
\end{align*}
\]

Fig. 1 gives this stream-function for the fundamental mode (when the cylinder wall is placed at the radial position marked 1.0) and at the next radial mode. Modes with nodal planes will merely repeat this picture, changing signs alternately, and with each cell of height \(1/n\) in terms of \(z'\). Fig. 2 gives the radial amplitude dependences of the velocity components, namely \(J_0\) and \(J_1\), respectively, for the vertical and for the radial and azimuthal velocity components.

The frequency equation for the normal modes, obtained from the linearized equations for these solution forms, is a very simple one:

\[
\tau_\Omega^0 = \frac{1}{2} \left[ \left( \frac{\gamma_i}{n \pi} \right)^2 \delta_i^2 + 1 \right]^{1/2}.
\]

The eigen-period for a given mode depends only on \(\delta_i\) as mentioned earlier and in all cases approaches the half-pendulum day, \(i.e., \tau_\Omega^0 = \delta_i\), as the relative depth of the cylindrical layer decreases. The solid curves in fig. 3 give the theoretical \(\tau_\Omega^0\) vs. \(\delta_i\) relations for various of the normal modes \((\gamma_i, n)\).

4. Experimental arrangements and results

The experiments were carried out in the Cavendish Laboratory on a small turntable with excellent bear-
nings used earlier by Sir G. Taylor. The container was a small perspex (lucite) cylinder of mean radius \( r_0 = 4.24 \text{ cm} \) and height 13.3 cm. The whole assembly was leveled within less than 1° of arc by means of three leveling screws.

The turntable was driven by thread (No. 18 Barbour Linen Carpet) using various V-pulley combinations from a 220-v DC motor. The speed of the motor was varied over a fair range by means of two rheostats. In most cases, the rotation periods used were near two seconds so that, though a free surface was present at the top of the liquid (water), the equilibrium slopes were sufficiently small to ignore.

Part of the impetus to the experiments came from earlier observations of the following sort: if one stirs a cup of tea or coffee, waits for the vortex motion to smooth out, and then carefully pours in cream on the axis, a quasi-periodic pumping motion up and down on the axis and corresponding decelerations and accelerations of the circulatory motion at the top is observed to persist for some time. Some initial trials were run of oscillations generated both in this manner and with the cylinder rotating. Depths were from 1 \( \frac{1}{2} \) to 2 \( \frac{1}{2} \) times the radius, and the observations showed that the pulsations were indeed of the order of 1 to 1 \( \frac{1}{2} \) times the rough circulatory period as called for by equation (5) for the fundamental mode \((\gamma_1, 1)\), \((\gamma_1 = 3.83\)).

In view of these encouraging results, it was decided to force the oscillations by means of a small disk on the axis whose frequency could be controlled. A second DC motor was arranged to drive by various thread and pulley combinations a crank which oscillated a long brass lever. A small hollow brass shaft was passed thru a rotating support on the axis (with a tab arranged so that it rotated with the cylinder). The disk (usually of 1.9-cm diameter) was placed at the bottom end and a movable disk farther up arranged so that a yoke on the end of the lever forced it to move up and down. Both the position of the disk on the cylinder axis and its amplitude of oscillation could be set to any desired values. The ranges of disk motion used were from 0.5 to 2.0 cm but most often about 1 cm.

The drive system used was not ideal, especially since stabilized voltages were not available, but with care both the rotation and oscillation rates would remain constant for long enough to get quite good measurements. These were made with a 0.1-sec stopwatch alternately on the two rates. On the measurements used, repetitions had a reproducibility usually of 0.1 or 0.2 sec out of timing periods of 20 to 40 sec.

The fluid motions were detected by means of dye solution (methylene blue) inserted from a syringe thru a long piece of 1-mm hypodermic tubing. This tubing was passed down thru the hollow shaft holding the oscillating disk until its end was at any desired location above the base. Tap water was used for the working liquid and was generally at temperatures in the range 13°C to 20°C. It was found desirable to keep the difference between the water and air temperatures small as a convection preventative.

The principal measurements made were of the frequencies of the normal modes at a number of \( \delta' = \delta/r_0 \) ratios (0.50, 1.00, 1.50, 2.00, 2.50, and 3.50). It was found possible to detect with great sensitivity whether the disk was set at a resonant frequency in the following way: the disk was placed so that its mean height coincided with the topmost antinode of \( u_0' \) for the normal mode to be tested. The frequency of oscillation was then changed in a series of steps thru a range of values relative to the rotation frequency while small amounts of ink were inserted along the axis as needed. Criteria for resonance were: (a) the amplitude of up-and-down motion of the ink on the axis reached a quite sensitive maximum at the eigen-frequency, especially far away from the disk; (b) oscillatory motions of the water back and forth past the edge of the disk reached a quite sensitive minimum with a corresponding reduction in dispersion of the ink; (c) perhaps most useful of all was that the phase of the ink motion, at the maximum up-and-down position, preceded that of the disk for the corresponding extreme when the disk period was too long and lagged behind that of the disk when the disk period was too short.

The major result of these measurements was that the observed resonant frequencies agreed with those of the inviscid theory to a remarkable degree for all the modes which were checked. Table 1 gives a comparison of a selection of the measured values (all at rotation periods \( r_p \) between 1.8 and 2.1 sec) with the theoretical ones. The fourth and sixth columns give the nearest measured disk periods (in units of rotation period) which were, respectively, definitely less and definitely more than the eigenperiod. The fifth column gives the best measured values; often these could be seen by the tests mentioned earlier to be not quite exactly at resonance. The agreement is as good (\( \frac{1}{4} \) to 1 per cent) as the reproducibility of the frequency measurements. Had more accurately constant drives been available, I have the impression that the agreement would have reached the 0.1 per cent level. Fig. 3 shows (crosses) all the observed points for seven normal modes including as many as 3 nodal radii \((\gamma_1, 1)\), 2 nodal planes \((\gamma_1, 3)\), and the \((\gamma_2, 2)\) mode with one nodal radius and one nodal plane. It will be noted that the functional dependence in equation (5) of \( r_0' \) on \( \delta' \), ultimately linear, is fully confirmed.
Table 1. Comparison of a selection of the measured values (at rotation periods $\tau_2$ between 1.8 and 2.1 sec) with the theoretical. Depressed digits indicate the digit value is of doubtful significance mainly because of measurement uncertainties.

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>Theoretical</th>
<th>Definitely Less</th>
<th>Best Measured</th>
<th>Definitely Greater</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\gamma_1, 1)$</td>
<td>0.50</td>
<td>0.5856</td>
<td>0.575</td>
<td>0.575</td>
<td>0.595</td>
<td>1.05</td>
</tr>
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<td></td>
<td>1.50</td>
<td>1.0425</td>
<td>1.035</td>
<td>1.045</td>
<td>1.045</td>
<td>1.055</td>
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<tr>
<td></td>
<td>2.00</td>
<td>1.8045</td>
<td>1.855</td>
<td>1.625</td>
<td>1.625</td>
<td>1.655</td>
</tr>
<tr>
<td>$(\gamma_2, 1)$</td>
<td>0.50</td>
<td>0.7494</td>
<td>0.715</td>
<td>0.745</td>
<td>0.745</td>
<td>0.775</td>
</tr>
<tr>
<td></td>
<td>1.50</td>
<td>1.7479</td>
<td>1.665</td>
<td>1.725</td>
<td>1.725</td>
<td>1.795</td>
</tr>
<tr>
<td></td>
<td>3.50</td>
<td>3.9398</td>
<td>3.835</td>
<td>3.935</td>
<td>3.935</td>
<td>4.005</td>
</tr>
<tr>
<td>$(\gamma_3, 1)$</td>
<td>0.50</td>
<td>0.9515</td>
<td>0.915</td>
<td>0.955</td>
<td>1.025</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>1.6946</td>
<td>1.575</td>
<td>1.685</td>
<td>1.755</td>
<td>1.795</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>4.0787</td>
<td>4.025</td>
<td>4.115</td>
<td>4.255</td>
<td></td>
</tr>
<tr>
<td>$(\gamma_1, 2)$</td>
<td>2.00</td>
<td>4.2704</td>
<td>4.185</td>
<td>4.285</td>
<td>4.465</td>
<td></td>
</tr>
<tr>
<td>$(\gamma_2, 2)$</td>
<td>0.50</td>
<td>0.5273</td>
<td>0.515</td>
<td>0.525</td>
<td>0.545</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>0.91164</td>
<td>0.835</td>
<td>0.915</td>
<td>--</td>
<td></td>
</tr>
<tr>
<td>$(\gamma_1, 3)$</td>
<td>1.50</td>
<td>0.97534</td>
<td>0.935</td>
<td>0.985</td>
<td>0.995</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>1.48256</td>
<td>1.475</td>
<td>1.485</td>
<td>1.505</td>
<td></td>
</tr>
</tbody>
</table>

Because of the limitations of the dye method of tracing, it was not feasible to go much farther with quantitative verifications of the theory. The presence of nodal planes at the correct positions was easily detectable for the modes $(\gamma_1, 2)$, $(\gamma_1, 3)$, and $(\gamma_2, 2)$, among other things, from the alternations of phase of the axial motions at the center line. Also the correct qualitative alterations of the azimuthal motion with radial displacement were easily visible in details of the ink distribution. Presence of the nodal radii for $(\gamma_1, 1)$, $(\gamma_3, 1)$, $(\gamma_4, 1)$, and $(\gamma_2, 2)$ was partially indicated by the manner of spreading of the ink and by some special tests. It might be worthwhile to verify these in detail by different techniques.

The attempt to obtain good photographs illustrating features of the oscillations was considerably hampered by the instability of the motors since the equipment was not such that photographs could be taken rapidly. Fig. 4 illustrates the rapid diffusion radially of the ink with oscillations in the absence of rotation. The ink remains, however, fairly well confined to a dipole field near the disk for considerable periods of time.

In contrast to fig. 4, when the liquid is rotating, the ink becomes distributed in a column, closely related to Taylor ink walls (Taylor, 1921; Long, 1954), which is extremely persistent compared to the situation with zero rotation. The initial spread of the ink is partly brought about by the edge flows at the disk and by an extremely interesting variety of slow secondary flows associated with the oscillations (see next section). However, the spreading would have been very much less evident had it been possible to hold the disk frequency accurately at the normal mode values. Figs. 5 and 6 illustrate two extreme phases each of deformation of these ink columns for, respectively, the fundamental mode $(\gamma_1, 1)$ and the $(\gamma_1, 2)$ mode. Close comparison of the figures shows that the changes of shape are those to be expected from the theoretical pictures for these modes.

5. Conclusions

The striking agreement of measured periods with the frictionless theory is not too surprising when one considers the similar results with, for example, measurements of gravity wave frequencies. Viscous effects on the frequencies would be expected to be relatively very small since the values of $T$ range from $6 \times 10^6$ to $2 \times 10^9$, and the oscillating boundary layer at the bottom, for example, has a thickness of order $1/10$ to $1/80$ of the depth.

We return now to consideration of the over-stable oscillating cells and proceed to compare the fundamental mode in a finite circular cylinder with the motion in a single cell. The identification of the two types of motion now clarifies a number of the cellular convection results. The increase of the cellular values of $\tau_2$ with $T^{1/6}$ is seen to be primarily due to the increase of the most unstable wave number $a$, also
Fig. 5. Photographs of an ink column at alternate phases of the oscillation for the fundamental mode ($\gamma_1, 1$). Depth $\delta = 8.4_1$ cm, $\delta' = 2.00$ cm; oscillation period, 2.00 sec; rotation period, 1.63 sec; $r_0' = 1.30_1$ (theoretical $r_0' = 1.31_1$); disk range 1.5 cm. 5a. Disk near top (210158–2–17); photo taken 11 1/2 min after first dye release. 5b. Disk near bottom (210158–2–18); photo taken 11 1/2 min after fig. 5a. Note the variations of radii from 5a to 5b, especially in the outer ink envelope.

reversal of $u_0'$ at the middle and this was clearly apparent to direct visual observation of ink elements.

Fig. 6. Photographs of an ink column at alternate phases of the oscillation for the mode ($\gamma_1, 2$) with one nodal plane. Depth $\delta = 8.4_2$ cm, $\delta' = 2.00$ cm; oscillation period, 1.83 sec; rotation period, 2.31 sec, drifting toward higher values; $r_0' = 0.80_2$ (theoretical $r_0' = 0.7880$); disk range, 1.5 cm. 6a. Disk near bottom (210158–1–9); photo taken 10 min after first dye release. 6b. Disk near top (210158–1–10); photo taken 1 1/2 min after fig. 6a. Note especially the change in the outer ink envelope from barrel-shaped (6a) to hour-glass shaped (6b). The nodal surface for $u_0'$ is not directly shown by any features of the edge of the ink column. The thickening at the middle in fig. 6a and thinning in fig. 6b, taken with the conditions at top and bottom, however imply a
with $T^{1/6}$. The wave number $a$ is proportional to the ratio of depth to horizontal cell dimension and so the ultimately linear relation between $\tau_0'$ and $a$ for the cells is the same as that between $\tau_0'$ and $\delta'$ for the elastoid-inertia oscillations. The physical reason for the period increase in both cases as the relative height increases is that only radial motions are resisted by the Rayleigh rotational stability and the effective proportion of fluid undergoing radial displacement decreases with increasing $\delta'$ or $a$. The remaining motion, axially, is passive and increases the period by increasing the total inertia compared to the restoring forces (Høiland, 1939). This, of course, does not directly explain why the cellular values of $a$ have the dependence they do on $T$.

Quantitatively, the period results correspond fairly closely as may be seen very easily for the limit of large $T$ or $\delta'$. For the fundamental mode, then, of the inviscid elastoid-inertia oscillations,

$$\tau_0' = \frac{1}{2}(\gamma/\pi)\delta' = 0.61\delta'. \tag{6}$$

For the overstable cells (with two-free-surface boundary conditions), Chandrasekhar's asymptotic results as $P^4T \to \infty$ (Chandrasekhar and Elbert, 1955) lead to

$$\tau_0' = \frac{1}{2}(1 + 2P)\pi^2 \left(\frac{1}{\pi^2}\right) a. \tag{7}$$

If we consider square cells of the same area as a circle of radius $r_0$,

$$a^2 = \frac{\delta^2 \pi^2}{L^2} = \frac{\delta^2 \pi^2}{\pi^2 \delta^2},$$

where $L$ is the length of a side. Thus

$$a = (8\pi)^{1/2} \delta'. \tag{7}$$

(7) then becomes

$$\tau_0' \approx \frac{3}{2}(8\pi)(1 + 2P)\delta'. \tag{8}$$

Remembering that $P$ must be less than 0.67, the proportionality constant lies between 1.22 and 0.75 at very small $P$. The discrepancies between these values and 0.61 may be assigned to the effects of viscosity in reducing the inertial restoring forces, to the direct effects of the density stratification on the frequency, and possibly to the geometric difference between a square and a circle. In addition, at the other extreme of small $T$, if overstability occurs at all, an absolute lower limit of $T^\delta$ is $\frac{1}{2}$ and this, in fact, will not be closely approached because the cells do not reach indefinitely small depth to diameter ratios, and because of the viscous effects.

With the above identification of the basic oscillatory mechanism, it is now relatively easy to reason out how Batchelor's second point is met. Consider the upper portion of a cell in which a downward motion on the axis is just passing thru the equilibrium (unstable) configuration. As the motion continues past equilibrium, the density surfaces move downward on the axis, solenoids in axial planes develop which tend to accelerate the motion, and gravitational potential energy decreases. However, the development of relative rotations about the cell axis begins to resist and ultimately to overcome the solenoidal accelerations in the initial sense. The lower the viscosity, the stronger the relative rotations and the greater the restoring tendency. At the same time, the thermal conduction effects begin to diffuse the density perturbation sidewise, to reduce the total number of solenoids, and to increase the potential energy. In the over-stable condition, the inertial restoring forces become sufficiently strong to reverse the sense of cell motion and to return it in a direction opposed to the solenoids. However, because of the conduction preceding and during this reversing phase, the solenoids which oppose the return flow are less in number than those which at a corresponding phase accelerated the initially considered motion. One thus obtains a phase displacement of the energy transfer from the mass distribution which acts to maintain and accelerate the oscillation.

The particularly striking qualitative aspect of this mechanism is that the conductive diffusion of temperature or density, which one would ordinarily consider a damping factor, is the key to the instability of the oscillations. Thus, high values of $\kappa$ favor this type of motion. The theoretical dependence on low $P$ is now seen to depend on the enhanced stability (overstability) of the rotational motion as a result of small viscous damping of the azimuthal motion combined with an energy transfer mechanism depending on high lateral diffusion of the temperature by $\kappa$.

This completes our discussion of the relation between over-stable cellular convection and the elastoid-inertia oscillations. There are, in addition, however, some extremely interesting connections of the experiments themselves with other recent work that would be worth pursuing further in the future. The fact that Long's work (1953; Fraenkel, 1956) dealt with the progressive or stationary wave equivalent of these standing oscillations has already been mentioned. Long's measurements of wavelengths when translated into periods give values of $\tau_0'$ from 0.51 to 0.72 for half-wavelengths in units of cylinder radius (corresponding to $\delta'$) of 0.26 to 0.67 and thus correspond reasonably well to the fundamental mode. His frequency equation is the same as (8) (Long, 1953).

From a general theoretical point of view, it has been pointed out by H. Görtler (1944) that the
limiting frequency \(2\Omega (r\prime = \frac{1}{2})\) corresponds to a change of the linearized perturbation equations from elliptic \((r\prime < \frac{1}{2})\) to hyperbolic in type \((r\prime > \frac{1}{2})\).

Discussions of the solutions of the equations in the hyperbolic case for an oscillating disk in an infinite rotating fluid have been given by Morgan (1951), and more recently Oser (1957) has succeeded in obtaining explicit expressions for this case. The striking feature is that the real characteristic surfaces (which are cones) of the differential equation divide the fluid into regions of different solution type with finite or infinite discontinuities of some of the dependent variables on the cones. In particular, two characteristic cones emanate from the edge of the disk at angles \(\pm \phi\) to the plane of the disk such that \(\cos \phi = \frac{1}{2} r\prime\) = \(\frac{1}{2}(r\prime)^{-1}\). The presence of these characteristic discontinuity surfaces has been verified experimentally in a fairly large cylinder by observations of suspended aluminum-powder tracer (Görtler, 1957).

A great deal can undoubtedly be learned by systematic experimental and theoretical work on the relations between these discontinuous solutions in an infinite fluid and the continuous eigen-solutions in a finite cylinder. A similar need for thorough investigation exists for other forms of boundaries—a tilted rectangle, for example, where Holand (1957) has shown that the eigen-solutions may not form a complete set.

Some attempts were made to see whether evidence for the characteristic surfaces mentioned above could be obtained by releasing dye from around the rim of a modified disk both at frequencies in the elliptic and in the hyperbolic ranges. No strong indications of them were found, possibly because of a failure of the ink to become distributed properly or because the boundaries in the small cylinder were too close. In fact, near \(r\prime = \frac{1}{2}\) the ink distribution behaved rather similarly on both sides and gave evidence of a very interesting effect of secondary flow. Fig. 7 shows one example at \(r\prime = 0.23\) of the manner of spread of the ink from the disk edge. There is a slow growth in both directions from the disk edge of a trumpet-shaped envelope, the inner surface of which is particularly well-marked. The same trumpet forms on the hyperbolic side of \(r\prime = \frac{1}{2}\) but with some noticeable differences of shape and rate of growth. This phenomenon is undoubtedly due to a secondary flow of some kind and would be well worth a detailed study.

The connections of the preceding points with meteorology are perhaps more of a general theoretical nature than anything very specific. The possibility of overstable cellular motions in the atmosphere is not completely excluded, though no clearcut elementary examples occur to mind. However, the very existence of internal mechanisms capable of producing oscillations in convective flows is in itself very suggestive for meteorological problems, even though it may be that no important ones of exactly this kind in detail are taking place in the atmosphere, especially on a large scale. On the other hand, the elastoid-inertia oscillations have had an important place in the development of meteorological theory even though characteristic times of any such large-scale motions would be of the order of the half-pendulum day and would consequently lie in the range of tidal-type motions (Bjerknes et al., 1933). On smaller scales of motions, one might rather easily expect pumping oscillations and other phenomena similar to the type described at the beginning of section 4 in connection with tornados and even hurricanes. The extension of well-established quantitative experimental examples to this class of motions is an important step in the development of meteorological experiments. This is particularly so because of the markedly ageostrophic character of these flows by contrast with most of the previous experiments on large-scale motions which are most successful under quasi-ageostrophic conditions. Continued work is likely to increase the mutually valuable interactions between theory and experiment in, for
example, studying the influence of deviations from geostrophic flow.

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