A GENERALIZATION OF THE MIXING-LENGTH CONCEPT[1]

By J. A. Businger

University of Wisconsin[2]

(Original manuscript received 2 May 1958; revised manuscript received 7 April 1959)

ABSTRACT

A concept for the mixing length in diabatic conditions is introduced and elaborated. The basic idea is that convective energy has effect on the mixing length but not on the size of the largest eddies. The theory developed on this concept of the mixing length for the diabatic wind profile gives satisfactory agreement with observations over a wide stability range.

1. Introduction

In the last ten years, several papers dealing with turbulent transfer in the lowest layer of the atmosphere have been published. The progress made in that period is mainly experimental. Several techniques of collecting data were developed or improved. The two expeditions to O'Neill, Nebraska in 1953 and 1956, especially, resulted in a host of valuable data. Progress was made also on the mode of representation. Several studies, such as Deacon [7; 8], Priestley [23], Monin and Obukhov [18], Businger [4], Lettau [16], and Inoue [12], were carried out to achieve in part a more comprehensive representation of the data in order to test several possible theories. Although unanimity has not yet been reached on this point, the different representations are more or less equivalent and indicate the existence of a unique and universal relation describing the turbulent structure of the surface layer (*e.g.*, the wind profile in terms of a dimensionless height as a function of stability). On purely theoretical grounds, Batchelor [1] came to this same conclusion. However, until the present no satisfactory theory has been developed describing the observations in sufficient detail. The theories presented by Lettau [14; 15], Businger [4; 5], Halstead [10], Ogura [19], and Elliot [9] all introduce one or more assumptions which have an apparent lack of physical basis. This situation is somewhat surprising because of the strong limitations on the problem imposed by the general assumption of a steady condition throughout the surface layer—that is, that all time derivatives of mean quantities are zero, and that the surface is homogeneous and has a constant roughness parameter. In sections 3 and 4 of this paper, a theory is developed based on a model of turbulence which makes possible the introduction of a generalized mixing length. The agreement with observation, so far, is promising.

Although the question of whether or not the coefficients of eddy diffusivity for momentum and heat transport are equal is by no means settled, the equality of the two is assumed in most of the above mentioned theories and also in the present theory. It may be that this assumption is not true, but it simplifies the theoretical approach considerably and provides a result which eventually can be corrected because the analysis does not depend on it in any fundamental way. An empirical check of the equality of these coefficients is obtained by comparing the curvature of the wind and temperature profiles from the O'Neill data [27]. Comparison shows that these curvatures are different, and therefore it is concluded that the coefficients must be different. It is very likely, however, that the ratio of the coefficients is a function of the stability alone and does not affect the uniqueness of the above mentioned universal relation.

Another simplification is made here with regard to the zero displacement for the logarithmic profile. This zero displacement is not taken into consideration, so the level $z = 0$ is assumed to be where the wind velocity, according to the extrapolated logarithmic profile, is zero. Therefore, the theoretical results cannot be trusted below a level of $z = 10z_0$, for instance. For a more detailed discussion of this matter, reference is made to Bjorgum [2].

2. Dimensional characteristics and representation

Since there is sufficient experimental as well as theoretical (Batchelor [1]) evidence that the Richardson number, defined as

$$Ri = g\frac{\partial\bar{\theta}/\partial z}{T_m(\partial\bar{u}/\partial z)^2} \tag{1}$$

(where θ is potential temperature, u is horizontal mean wind, g is acceleration of gravity and T_m is the mean

[1] The research reported in this article has been made possible through support and sponsorship extended by the Geophysics Research Directorate, Air Force Cambridge Research Center, under contract No. AF 19(604)–1384.

[2] Present affiliation: University of Washington.

absolute temperature), determines the structure of the atmospheric turbulence in the surface layer, there must exist a unique relation describing this structure. Several studies [4; 7; 8; 12; 16; 18; 23] have been carried out in order to give a concise and comprehensive representation of this relation. Although most representations are basically equivalent, because they can be derived from each other, preference can be given to some representations above others. From a theoretical point of view, it is desirable to have a representation which is easy and reliable to derive from observations in order to be able to test the theory.

The first representation of this kind is that of Deacon [7], who plots a dimensionless parameter β, related to the curvature of the wind profile, defined as

$$\beta = -\frac{z\partial^2\bar{u}/\partial z^2}{\partial\bar{u}/\partial z} \tag{2}$$

versus Ri. Although Deacon's semi-empirical generalized wind profile has not been found in theory, the β *versus Ri* relation is still of practical interest. In recent studies, this relation has been used to compare some theories with observation. A great advantage of both the parameters β and Ri is that they can be derived from wind and temperature measurements without using the surface roughness. However, this relation is difficult to visualize in terms of the wind as a function of height—that is, in a more integrated form of the relation. Representations in this form are given by Monin and Obukhov [18] and also by Businger [5]. Monin and Obukhov represent the universal relation using a dimensionless wind difference Y and a dimensionless height ζ as

$$Y = U(\zeta) - U(\tfrac{1}{2}) \quad \text{and} \quad \zeta = kg\frac{\partial\bar{\theta}/\partial z(z + z_0)}{T_m\partial\bar{u}/\partial z u_*} \tag{3}$$

(where $U = k\bar{u}/u_*$ is the dimensionless wind, k is v. Karman's constant, u_* is the friction velocity, and z_0 is the roughness parameter). The representation $Y = f(\zeta)$ has the objection that it fails to show any detail for nearly neutral profiles; that means it is difficult to find the deviation from the neutral profile, as Lettau [17] has pointed out. This point is elaborated in section 4. Businger [5] used a representation which is easy to interpret. Here, U was plotted as a function of z/z_0 in a series of curves using the dimensionless stability number

$$Sn = \frac{g\partial\bar{\theta}/\partial z\cdot z_0}{T_m\partial\bar{u}/\partial z\cdot u_*}$$

as a parameter. Although this representation gives good detail for the nearly neutral profiles, it is not as concise as might be desired. A better representation can be introduced using a dimensionless velocity difference, ΔU;

$$\Delta U = U - \ln\frac{z + z_0}{z_0} \tag{4}$$

versus the dimensionless height ζ or *versus Ri.* An intermediate representation is given by Deacon [8]. He introduced a dimensionless coefficient of eddy transfer

$$\overset{*}{K}_m = \frac{K_m}{u_*2} = \frac{u_*}{z\partial\bar{u}/\partial z}$$

and made a plot of $\overset{*}{K}_m$ *versus Ri.* In section (4), the reciprocal value of $\overset{*}{K}_m$ is used as dimensionless wind gradient

$$P = \frac{kz}{u_*}\frac{\partial\bar{u}}{\partial z} = \frac{k}{\overset{*}{K}_m}.$$

In the following sections, both the representations

$$\beta = f(Ri) \tag{5}$$

and

$$\Delta U = h(Ri) \tag{6}$$

will be used.

3. Model of turbulence under neutral conditions

Turbulent structure under adiabatic conditions is now well understood. Prandtl's [20] derivation of the logarithmic law has found theoretical as well as experimental support. Therefore, the concept of the mixing length must be regarded as a valuable tool, with an exact definition in the neutral case, as was pointed out by Businger [4]. An interesting fact, furthermore, as shown by Inoue [11], is that the logarithmic profile is consistent with the similarity hypothesis of isotropic turbulence when the mixing length represents the size of the largest vertical eddies. A consequence of this result is that in the equation of energy decay, derived by Calder [6], and which under steady adiabatic conditions can be written as

$$\frac{\partial}{\partial z}\overline{w'E} - \overline{u'w'}\frac{\partial\bar{u}}{\partial z} - D = 0 \tag{7}$$

(u' and w' are the horizontal and vertical components of the turbulent wind respectively, E is the total turbulent energy and D is the dissipation rate of turbulent energy into heat), the term $\partial(\overline{w'E})/\partial z$, representing the divergence of the turbulent energy, is negligible in comparison to the two others.

The rate of decay of turbulent energy ϵ from one wave number to the next is independent of the wave number and equal to the dissipation rate. This means that

$$D = \epsilon \sim w^3k_0 \sim \frac{w^3}{l} \tag{8}$$

where k_0 is the wave number of the largest vertical eddies; $w = (\overline{w'^2})^{\frac{1}{2}}$ and l is the mixing length. The equations (7) and (8), together with the equation for the flux of momentum

$$\rho K \frac{\partial \bar{u}}{\partial z} = -\rho \overline{u'w'} \tag{9}$$

where the coefficient of eddy transfer K is defined by

$$K = wl, \tag{10}$$

define the logarithmic wind profile, when the mixing length $l \sim (1/k_0) \sim z$. The proportionality constant which enters because of relation (8) is v. Karman's constant.

It should be remarked that the mixing length as introduced here is a Eulerian concept. Prandtl's original concept of mixing length was "Lagrangian." But, in his first development where it was related to $\overline{u'w'}$, the transition to the Eulerian view was implied. Therefore, the explicit expression obtained for the mixing length in the neutral atmosphere,

$$l = k(z + z_0), \tag{11}$$

is a Eulerian concept. (Note the analogy with the kinetic gas theory!)

It is of interest now to present a different argument in Prandtl's original model. Consider an eddy with a vertical velocity w' with respect to its surroundings and a characteristic size l'. The question is now how far in the vertical will this eddy travel before it loses its identity. That distance will then be the mixing length for the eddy under consideration. To answer this question, we consider the energy of the eddy in the z-direction, which will be proportional to $\rho w'^2 l'^3$. This energy dissipates at a rate proportional to the eddy diffusivity of its surroundings K', the gradient in energy from the eddy to its surroundings, which is proportional to w'^2/l' and the surface of the eddy $\sim l'^2$. The dissipation then is proportional to $\rho K' w'^2 l'$. It may be assumed that K' is proportional to the overall eddy diffusivity K. By the time all the energy is consumed, the eddy has lost its identity and has travelled its mixing length l_n. In equation,

$$l_n \sim \frac{\rho w'^3 l'^3}{\rho K' w'^2 l'} \sim \frac{w' l'^2}{K}. \tag{12}$$

Averaging over all eddies, the left-hand side will give the mixing length in the Lagrangian form l_g, and the right-hand side will be proportional to $w l_0^2/K$ (where l_0 is the predominant eddy size), provided the size distribution of the eddies is independent of the intensity of the turbulence, which is implied in equation (8).

When we now consider all the eddies passing by a point of observation, we may assume that on the average these eddies have travelled in the vertical over a distance which is proportional to l_g. The mixing length l obtained in this fashion is a Eulerian concept. So we finally obtain, using (10),

$$l \sim l_g \sim \frac{w l_0^2}{K} \sim \frac{l_0^2}{l}$$

or

$$l \sim l_0 = \frac{1}{k_0} \sim z. \tag{13}$$

This argument may seem to be superfluous, but it will be shown in the next section that relation (12) provides a link to a generalization of the mixing-length concept. At the same time, it shows what assumptions are made in the present theory.

4. Generalization of the model for diabatic conditions

The previous section emphasized that there is considerable agreement even in detail between Prandtl's model and the exact theory as well as the observations. Therefore, it may be expected that a logical generalization of this model to diabatic conditions would provide satisfactory results. Equation (7) is a special case of the more general equation for diabatic conditions (see [6]).

$$\frac{\partial}{\partial z} \overline{w'E} - \overline{u'w'} \frac{\partial \bar{u}}{\partial z} + \frac{\alpha g}{T_m} \overline{w'\theta'} - D = 0,$$

$$D = w^3 k_0 \tag{14}$$

where α is a proportionality constant indicating the efficiency with which density differences contribute to the turbulent energy. It was Ogura [19] who emphasized the usefulness of this equation in the derivation of the diabatic wind profile. Next, there is the equation for the heat flux

$$K \frac{\partial \bar{\theta}}{\partial z} = -\overline{w'\theta'} \tag{15}$$

which is assumed to be similar to equation (9).

Further, there is some evidence from an analysis of the power spectrum of the vertical wind component that the turbulence behaves isotropic in the range of high wave numbers [26] under all stability conditions. This behavior must be related to the dissipating structure which tends to be isotropic even in homogeneous anisotropic turbulence, whereas the transition from homogeneous anisotropic to isotropic turbulence is very slow [28, p. 48]. The experimental evidence indicates that the turbulent energy from friction as well as from convection enters at the lowest wave numbers and that equation (8) is also true for diabatic conditions. The main difference between the diabatic and adiabatic case lies in the way the frictional and the convective energies enter into the turbulence. It may be said that the frictional energy enters the lowest vertical wave numbers in an isotropic way, because the horizontal turbulence extends to a lower frequency range than the vertical turbulence. However, the convective energy contributes only to the vertical wind component of the lowest vertical wave number.

Considering again an individual eddy, we see that besides a dissipation of energy there is also a source of energy proportional to

$$\frac{\rho g}{T_m}\theta' l'^3 w',$$

where θ' is the temperature difference between the eddy and its surroundings. In this case, instead of (12), we find

$$l_n \sim \frac{w' l'^2}{K - \dfrac{\alpha' g \theta' l'^2}{T_m w'}}. \qquad (16)$$

When averaging over all eddies and following a similar line of reasoning as in deriving (13) and considering that θ' may be replaced by $-l(\partial\bar{\theta}/\partial z)$, we find

$$l \sim \frac{1}{k_0}\left(1 + \frac{\alpha' g}{T_m}\frac{\partial\bar{\theta}}{\partial z}\frac{1}{(k_0 w)^2}\right)^{-\frac{1}{4}}. \qquad (17)$$

It is easy to see that, in this more general case, l is no longer proportional to z because $1/k_0 \sim z$ is still valid. The constant α' is not the same as the α in equation (14) because it contains also the constant of proportionality between $1/k_0$ and z. Under neutral conditions, relation (17) should reduce to equation (11); therefore, by replacing $1/k_0$ by $k(z+z_0)$, we find for (17) that

$$l = k(z+z_0)\left(1 + \frac{\alpha g}{T_m}\frac{\partial\bar{\theta}}{\partial z}\frac{k^2(z+z_0)^2}{w^2}\right)^{-\frac{1}{4}}, \qquad (18)$$

assuming that in this case α is the same in (14) and (18).

The equations (9), (10), (14), (15), and (18) now define the complete structure of the atmospheric surface layer. From these relations, it is possible to derive

$$\alpha\zeta = \frac{\alpha Ri}{1-\alpha Ri}\left\{\frac{\alpha}{2}Ri + \tfrac{1}{2}[(\alpha Ri)^2 + 4(1-\alpha Ri)^2]^{\frac{1}{2}}\right\}^{\frac{3}{4}}, \qquad (19)$$

using (1) and (3). From this equation, the relation (5) can be derived, considering that

$$\frac{d\ln Ri}{d\ln\zeta} = \beta,$$

by combining (1), (2), and (3), by differentiation of (19). A consequence of the present theory is that for $Ri = -\infty$, $\beta = \frac{4}{3}$, which is in agreement with Priestley's findings in the region of free convection [22].

From equation (19), the wind profile or equation (6) can also be derived. Writing the symbol for the part of equation (19) in brackets and considering that (see (1) and (3))

$$\frac{\partial U}{\partial\zeta} = \frac{1}{Ri},$$

it may be seen that

$$\frac{\partial U}{\partial\zeta} = \frac{P}{\zeta}$$

or, when a function $Q(\zeta) = (1/\zeta)(P-1)$ is introduced,

$$\frac{\partial U}{\partial\zeta} = \frac{1}{\zeta} + Q(\zeta)$$

and

$$U = \ln\frac{\zeta}{\zeta_0} + \int_{\zeta_0}^{\zeta} Q(\zeta)d\zeta \qquad (20)$$

where

$$\zeta_0 = kSn = \frac{kg\partial\bar{\theta}/\partial z z_0}{T_m u_* \partial\bar{u}/\partial z}.$$

Thus, equation (6) takes the form

$$\Delta U = \int_{\zeta_0}^{\zeta} Q(\zeta)d\zeta \qquad (21)$$

which can be obtained by graphical integration. The equations (20) and (21) indicate that U and ΔU are not only functions of ζ but also of ζ_0. So it is necessary to use, in this case, a two-parameter representation. However, in most cases

$$\frac{\zeta}{\zeta_0} = \frac{z+z_0}{z_0} \gg 1$$

and, because $Q(\zeta)$ is a limited positive function,

$$\int_{\zeta_0}^{\zeta} Q(\zeta)d\zeta \simeq \int_0^{\zeta} Q(\zeta)d\zeta.$$

So equation (6) can be represented as

$$\Delta U = \int_0^{\zeta} Q(\zeta)d\zeta \qquad (22)$$

provided $z \gg z_0$. Combining equation (22) with equation (19), ΔU can be written also as a function of Ri. Various representations of the universal function as derived by the present theory are illustrated in fig. 1.

It is now possible to compare equation (22) with equation (3). Equation (3) can be written when

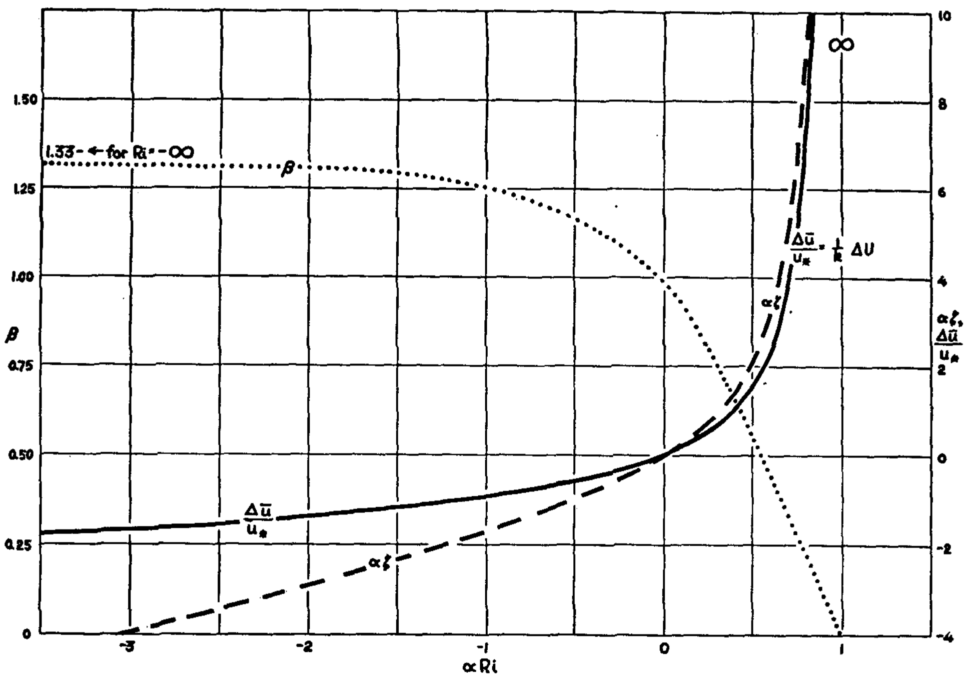

FIG. 1. Various forms of the universal function as derived by the present theory.

$\zeta > 0$ as

$$Y = \ln 2\zeta + \int_{\frac{1}{2}}^{\zeta} Q(\zeta)d\zeta$$

and when $\zeta < 0$ as

$$Y = \ln(-2\zeta) + \int_{-\frac{1}{2}}^{\zeta} Q(\zeta)d\zeta.$$

When ζ is very small, the absolute value of the integral in the right-hand side of these equations is small in comparison with $|\ln 2|\zeta||$. This explains, as was mentioned in regard to equation (2), why detail is lost in Monins and Obukhov's representation.

5. Comparison with experimental data

The theoretical relationship between β and αRi as given in fig. 1 is compared with two groups of data—*i.e.*, a series of 260 hr of simultaneous wind and temperature profiles, measured at Shirley, New Jersey, and a similar series of about 230 hr measured at O'Neill, Nebraska in 1953 and 1956. The O'Neill observations were carried out by the Johns Hopkins and MIT groups in 1953 and by the Texas A. and M. group in 1956. The data reduction was carried out by Lettau and the result was presented during the IUGG at Toronto in 1957. The curvature β and the Richardson number were divided in seven stability classes, which are defined in section 7.4.3 of [27]. For the method of data reduction, applied reference is made to [16]. The comparison of the theoretical curve with the two groups of data is presented in figs. 2 and 3, respectively. A reasonable fit is obtained for a value $\alpha = 3$ in fig. 2. In fig. 3, especially for nearly neutral conditions, a value of $\alpha = 10$ would fit better.

Also, the theoretical relation between ΔU and Ri has been compared with experimental results. To obtain

$$\Delta U = \frac{k\bar{u}}{u_*} - \ln\frac{z + z_0}{z_0}$$

from the profiles, it is necessary first to evaluate both u_* and z_0. The z_0 was obtained by using the logarithmic profile under neutral conditions, and u_* was obtained by assuming the logarithmic profile to be valid to the lowest level of wind observation. The data represented in fig. 4 are derived from 1953 observations by the Johns Hopkins group for the 4-m level and from 1956 observations by the Texas A. and M. group for the 8- and 16-m levels, all at O'Neill, Nebraska. The total was divided in ten classes of overall stability. In the more stable classes, the determination of u_* was not accurate because of the large deviation from the logarithmic profile. The theoretical curve shows also in this case a reasonable fit with the presented data when $\alpha = 3$.

6. Concluding remarks

Although the present theory is not exact, a fairly accurate description is obtained of the structure of the turbulence in the atmospheric surface layer from extremely unstable to stable conditions, provided the steady state is maintained.

In the very stable cases, the steady-state condition is usually not fulfilled because the transfer processes

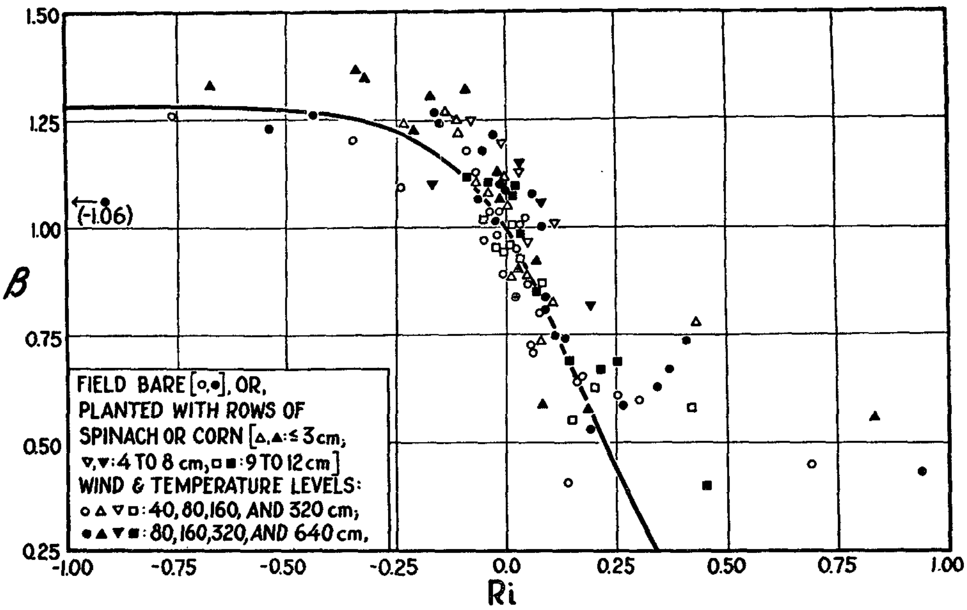

FIG. 2. Comparison of the theoretical β versus Ri relation for $\alpha = 3$ with observations obtained at Shirley, New Jersey.

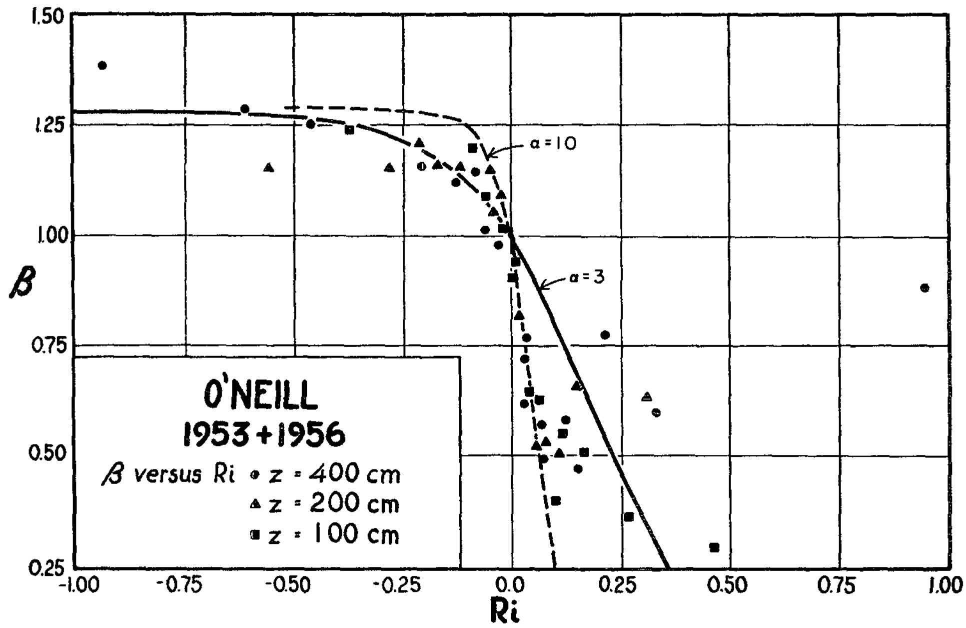

FIG. 3. Comparison of the theoretical β *versus* Ri relation for $\alpha = 3$ and $\alpha = 10$ with observations obtained at O'Neill, Nebraska.

become very slow. The result is a rather large scatter of the observed points in this region.

It is probably not possible to obtain a more precise theory with Prandtl's mixing length as a starting point. The constant of proportionality, α, makes it possible to adjust the theory to the observations. It is, of course, desirable to have a theory which predicts this constant and agrees with the observations. In order to arrive at such a theory, it will be necessary to have a better knowledge of

(a) The ratio of the coefficients of eddy transfer as a function of stability, and
(b) The conversion from Lagangian to Eulerian systems.

A remarkable feature of the data (especially the O'Neill data, see figs. 3 and 4) is the very rapid deviation from the neutral profile for only slightly diabatic cases. This corresponds to a fairly large value of α, which would not be directly expected from the theory presented here, and emphasizes the importance

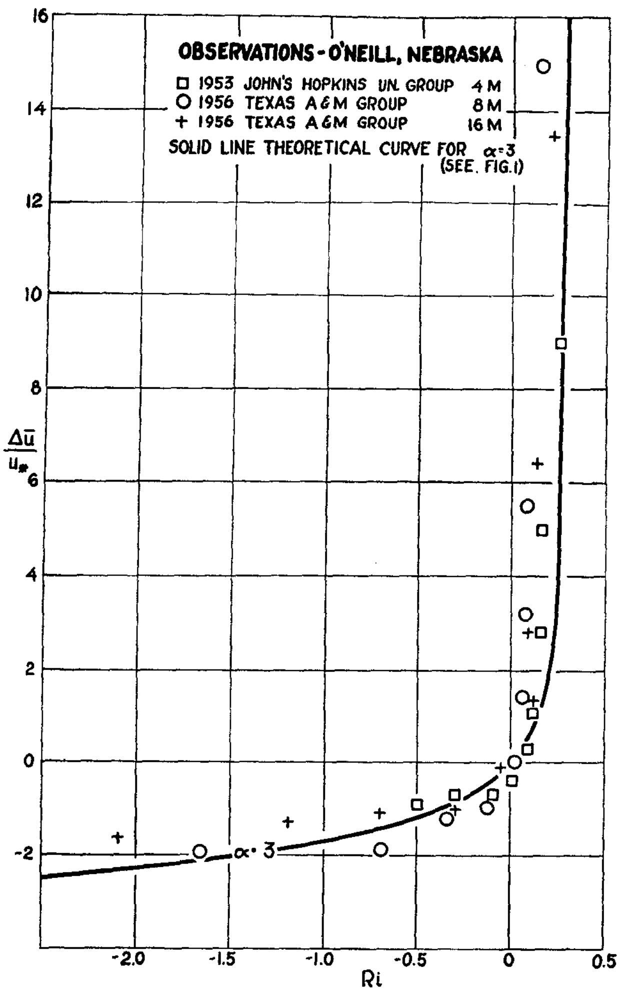

FIG. 4. Comparison of the theoretical ΔU *versus* Ri relation for $\alpha = 3$ with observations.

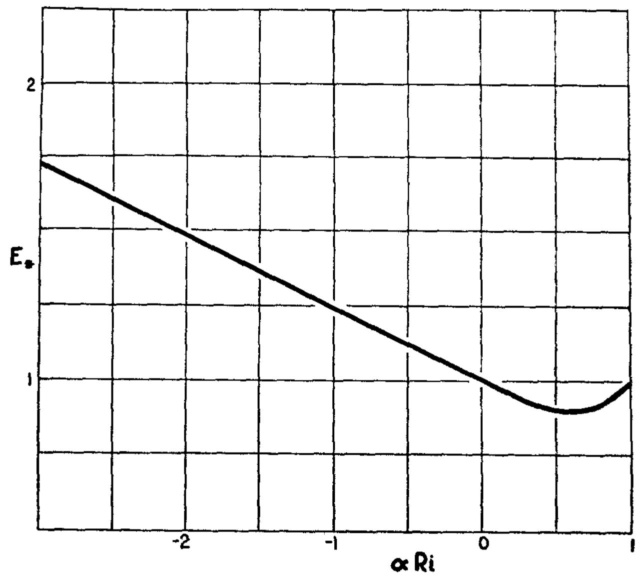

FIG. 5. The theoretical E^* *versus* Ri relation; $E^* = w^2/U_*^2$. This relation shows the variation of the vertical wind component with stability.

of this constant. The stage of free convection is reached already with small negative values of the Ri-number. $Ri \sim -0.1$. In this respect, it is of interest to refer to the models for free convection introduced by Priestley [24]. According to him, the transition from free to forced convection would take place at even smaller negative Ri numbers and, besides that, this transition would be discontinuous.

As a consequence, the theory predicts a profile of the vertical wind component. A convenient way to represent this profile is in terms of a dimensionless energy $E^* = w^2/u_*^2$. A combination of equations (9), (10), (14), and (18) gives the relation

$$E^* = \left\{\frac{\alpha}{2} Ri + \tfrac{1}{2}[(\alpha Ri)^2 + 4(1 - \alpha Ri)^2]^{\frac{1}{2}}\right\}. \quad (23)$$

A plot of E^* *versus* Ri is given in fig. 5. The values of E^* beyond the minimum under stable conditions are probably not realistic because in this region the molecular viscosity is no longer negligible in comparison to the eddy viscosity. It may be possible to test equation (23) by means of sonic anemometers. Experiments in this direction are planned.

The preceding study holds only for large Reynolds numbers. This requirement is fulfilled in most cases when a large area is used with a homogeneous surface. However, in the most extreme case of free convection, the size of the area is not the only requirement because the average wind is zero and, in spite of the size of the area, the Reynolds number can be small. The requirement for fully developed turbulence can be found in the thickness of the atmospheric layer participating in the free convection. In case this thickness is only a few meters, the free convection will occur mainly in laminar flow and the structure of the atmospheric surface layer will be entirely different. A discussion of this situation is given by Bryson [3].

Acknowledgment. The author is indebted to Dr. Franklin I. Badgley and Dr. Hans Panofsky for their interest in the presented study and for valuable comments.

REFERENCES

1. Batchelor, G. K., 1953: The conditions for dynamical similarity of motions of a frictionless perfect-gas atmosphere. *Quart. J. r. meteor. Soc.*, **79**, 224–235.
2. Bjorgum, O., 1953: On the velocity, temperature and humidity profiles. *Quart. J. r. meteor. Soc.*, **79**, 84–93.
3. Bryson, R. A., 1955: Convective heat transfer with light winds. *Trans. Amer. Geophys. Union*, **36**, 209–212.
4. Businger, J. A., 1954: Some aspects of the influence of the earth's surface on the atmosphere. *Meded. Verh. K. Ned. meteor. Inst.*, No. 61, 78 pp.
5. Businger, J. A., 1955: On the structure of the atmospheric surface layer. *J. Meteor.*, **12**, 553–561.
6. Calder, K. L., 1949: The criterion of turbulence in a fluid of variable density, with particular reference to conditions in the atmosphere. *Quart. J. r. meteor. Soc.*, **75**, 71.

7. Deacon, E. L., 1949: Vertical diffusion in the lowest layers of the atmosphere. *Quart. J. r. meteor. Soc.*, **75**, 89–103.
8. Deacon, E. L., 1955: *Turbulent transfer of momentum in the lowest layers of the atmosphere.* CSIRO Div. of Meteor., Phys. Tech. Pap. 4, 36 pp.
9. Elliot, W. P., 1957: A comparison of some approaches to the diabatic wind profile. *Trans. Amer. Geophys. Union*, **38**, 21–24.
10. Halstead, M. H., 1952: *The relationship between wind structure and turbulence near the ground.* Geophys. Res. Pap. 19, Internatl. Sympos. on Atmos. Turbulence in the Boundary Layer, 97–126.
11. Inoue, E., 1951: The structure of wind near the ground. *J. Meteor. Soc. Japan*, **29**, 14–19.
12. Inoue, E., 1957: Studies of the phenomena of waving plants ("Honami") caused by wind, Part 4: Turbulent transfer phenomena over the waving plants. *J. Meteor. Soc. Japan*, **35**, 137–144.
13. Lake, H., 1952: A comparison of the power law and a generalized logarithmic formula in micrometeorology. *Trans. Amer. Geophys. Union*, **33**, 661–668.
14. Lettau, H. 1949: *Isotropic and non-isotropic turbulence in the atmospheric surface layer.* Geophys. Res. Pap. No. 1, 13–84.
15. Lettau, H., 1952: *The present position of selected turbulence problems in the atmospheric boundary layer.* Geophys. Res. Pap. No. 19, 49–95.
16. Lettau, H., 1956: Note on the structure of the atmospheric surface layer. *J. Meteor.*, **13**, 507–509.
17. Lettau, H., Personal communication.
18. Monin, A. S., and A. M. Obukhov, 1954: Dimensionless characteristics of turbulence in the surface layer. *Akademia Nauk. SSSR. Geofis. Inst. Trudy*, No. 24 (151), 163–187.
19. Ogura, Y., 1952: Note on the wind velocity profile in the non-adiabatic atmosphere. *J. Meteor. Soc. Japan*, **30**, 329–342.
20. Prandtl, L., 1932: Meteorologische Anwendung der Stromungslehre. *Beitr. Physik fr. Atmos.*, **19**, 188–202.
21. Priestley, C. H. B., 1954: Vertical heat transfer from impressed temperature fluctuations. *Austr. J. Phys.*, **7**, 202–209.
22. Priestley, C. H. B., 1954: Convection from a large horizontal surface. *Austr. J. Phys.*, **7**, 176–201.
23. Priestley, C. H. B., 1955: Free and forced convection in the atmosphere near the ground. *Quart. J. r. meteor. Soc.*, **81**, 139–143.
24. Priestley, C. H. B., 1957: Convection from the earth's surface. *Proc. r. Soc.*, **238A**, 287–304.
25. Scorer, R. S., and J. H. Ludlam, 1953: Bubble theory of penetrative convection. *Quart. J. r. meteor. Soc.*, **79**, 94–103.
26. Businger, J. A., and V. E. Suomi, 1958: Variance spectra of the vertical wind component derived from observations with the sonic anemometer at O'Neill, Nebraska in 1953. *Arch. Meteor. Geophys. Biokl. Series A*, **10**, 415–425.
27. Lettau, H. H., and B. Davidson, 1957: *Exploring the atmosphere's first mile.* New York, Pergamon Press, Inc.
28. Townsend, A. A., 1956: *The structure of turbulent shear flow.* London, Cambridge University Press.