TRANSFER OF MOMENTUM VORTICITY AND THE MAINTENANCE OF ZONAL CIRCULATION IN THE ATMOSPHERE

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ABSTRACT

The properties of the volume integral of momentum vorticity are examined. These results are applied to the study of the maintenance of zonal circulation of a polar cap. It is shown that the rate of change of the vertical component of relative momentum vorticity is mainly due to (1) the effect of the convergence of meridional flux of angular momentum and its lateral boundary surface, (2) the frictional force at the earth's surface, and (3) the action of the mountains on the atmosphere. A model for the mean state of the atmosphere in the northern hemisphere, based on the distribution of the mean surface zonal wind, is studied; and the maintenance of the zonal circulation is discussed on the basis of the meridional transports of both angular momentum and momentum vorticity. It is shown that in the middle latitudes the meridional transfer of momentum vorticity is directed toward the north pole, whereas in the lower latitudes, as well as in the polar region, the transport is directed toward the equator. These results agree with the mean meridional transport of momentum vorticity in the month of January 1949, computed from the geostrophic winds.

1. Introduction

It is customary in hydrodynamics and meteorology to consider vorticity as one of the fundamental quantities in the study of the fluid motion, and indeed this concept is particularly useful in the case of a barotropic, non-viscous fluid because under these conditions vorticity is conserved. In the examination of the kinematic characteristics of a large volume of fluid, one may seek its properties in the volume integral of the vorticity.[1] However, in problems of large-scale atmospheric motion, the atmosphere is always baroclinic, and integration usually extends over a large air mass with great variation in density. It is certainly a loss of generality to treat the atmosphere as a barotropic fluid and assign to it an average density in evaluating the volume integral of the vorticity. However, if we are concerned with the transport of angular momentum or the interaction between the atmosphere and the earth's surface, the specific momentum is a quantity of basic importance. It is therefore natural to consider the momentum and its vorticity, hereafter called the "momentum vorticity," as the fundamental quantities. The objective of this paper is to examine the properties of the volume integral of momentum vorticity and to apply it to the study of the major agencies which control the zonal circulation in the atmosphere.

2. Total momentum vorticity and its rate of change

Let $\boldsymbol{\Omega}$ be the angular velocity of the earth, $\boldsymbol{r}$ the radius vector from the center of the earth, ρ the density, and $\boldsymbol{V}$, $\boldsymbol{V}_a$, respectively, the relative and absolute velocity. The relative momentum vorticity $\boldsymbol{q}$ and the absolute momentum vorticity $\boldsymbol{q}_a$ are defined respectively by $\nabla\times(\rho\boldsymbol{V})$ and $\nabla\times(\rho\boldsymbol{V}_a)$. The total relative momentum vorticity $\boldsymbol{Q}$, and the total absolute momentum vorticity $\boldsymbol{Q}_a$ within a volume V, are defined respectively by

$$\boldsymbol{Q}=\int_V \nabla\times(\rho\boldsymbol{V})d\tau=-\oint_S \rho\boldsymbol{V}\times d\boldsymbol{\sigma} \tag{1a}$$

and

$$\boldsymbol{Q}_a=\int_V \nabla\times(\rho\boldsymbol{V}_a)d\tau=\boldsymbol{Q}-\oint_S \rho(\boldsymbol{\Omega}\times\boldsymbol{r})\times d\boldsymbol{\sigma}. \tag{1b}$$

Here, Gauss's theorem has been employed. These are the fundamental equations of total relative and total absolute momentum vorticity for an arbitrary volume bounded by a surface S. Eq. (1a) and (1b) contain the following important information: *the total relative and total absolute momentum vorticities within an arbitrary volume are independent of the state of motion at all interior points of the volume.* In other words, they depend only on the values of ρ and $\boldsymbol{V}$ on the boundary surface of the volume. Eq. (1a) and (1b) are valid only if ρ and $\boldsymbol{V}$ are single-valued, continuous differentiable functions of position throughout the volume and are integrable upon the surface S. Hence, we cannot apply them to the atmosphere together with the hydrosphere because of the density discontinuity at the oceanic surface. It is interesting to apply equation (1a) to the entire atmosphere and hydrosphere separately. As the density of the air vanishes at the outer limit of the atmosphere and the velocity vanishes at the solid boundary of the earth's surface,

[1] For a non-rotating system of reference, the volume integral of vorticity has been studied by Poincaré (1893), Jeffe (1921), and Truesdell (1948).

we find that *the magnitude of the total relative momentum vorticity of the entire atmosphere (entire hydrosphere) is equal to the total momentum integrated over the boundary surface between the atmosphere and the hydrosphere.* As the velocity of air and water are equal at the boundary between them, whenever there is a change in the total relative momentum vorticity of the entire atmosphere, there is a corresponding change in that of the entire hydrosphere. Since there is very little change in the ocean currents and also in the densities of air and water in the neighborhood of the boundary surface, *the total relative momentum vorticity of the entire atmosphere (entire hydrosphere) may be considered as approximately constant.* However, both in atmospheric and oceanic motions, only the vertical component of the momentum vorticity is of major interest. We shall show later (section 3) that *for the entire atmosphere (hydrosphere) the total vertical component of the relative momentum vorticity is zero.* This result is useful, for, if we know the total relative momentum vorticity of a number of parts of the atmosphere (or hydrosphere), we can deduce that of the remaining part of the atmosphere (or hydrosphere). The above rule applies to the total absolute momentum vorticity also, if we confine ourselves to the vertical component and neglect the topographic corrugations of the earth's surface.

By combining the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0 \tag{2}$$

and the following identity

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + (\nabla \times V) \times V + \nabla (\tfrac{1}{2} V \cdot V), \tag{3}$$

we obtain

$$\rho \frac{dV}{dt} = \frac{\partial \rho V}{\partial t} + V \nabla \cdot (\rho V) + \rho (\nabla \times V) \times V + \tfrac{1}{2} \rho \nabla (V \cdot V). \tag{4}$$

By integrating the "curl" of this equation over a fixed volume and using Gauss's Theorem, we obtain the rate of change of total relative momentum vorticity of a fixed volume—

$$\frac{\partial Q}{\partial t} = \oint_S \rho [V \nabla \times V - \nabla \times V V] \cdot d\boldsymbol{\sigma} - \oint_S \left[\rho \frac{dV}{dt} - \tfrac{1}{2} \rho \nabla (V \cdot V) - V \nabla \cdot (\rho V) \right] \times d\boldsymbol{\sigma}. \tag{5a}$$

By use of (1b) and the equation of continuity, we derive for the rate of change of absolute momentum vorticity

$$\frac{\partial Q_a}{\partial t} = \frac{\partial Q}{\partial t} + \oint_S \nabla \cdot (\rho V)(\boldsymbol{\Omega} \times r) \times d\boldsymbol{\sigma}. \tag{5b}$$

In contrast with the conclusion drawn from the fundamental equations for total momentum vorticity, we have the following: *the rate of change of the total momentum vorticity of a fixed volume depends not only on the values of ρ, V, and V_a on the boundary surface but also in its neighborhood,* for the above two equations contain the velocity and density gradients across the surface.

In order to eliminate dV/dt from (5a) and (5b), we make use of the equation of motion

$$\frac{dV}{dt} = -2\boldsymbol{\Omega} \times V - \alpha \nabla p - \nabla \Phi + \alpha \boldsymbol{F}, \tag{6}$$

where α is specific volume, p pressure, Φ geopotential and $\boldsymbol{F}$ the total frictional force per unit volume. By substituting (6) into (5a) and (5b) and employing the equation of continuity, we obtain

$$\frac{\partial Q}{\partial t} = \oint_S \rho [V(2\boldsymbol{\Omega} + \nabla \times V) - (2\boldsymbol{\Omega} + \nabla \times V) V] \cdot d\boldsymbol{\sigma} + \oint_S [\nabla p + \rho \nabla (\Phi + \tfrac{1}{2} V \cdot V) + V \nabla \cdot (\rho V) - \boldsymbol{F}] \times d\boldsymbol{\sigma}. \tag{7}$$

Eq. (7) and (5b) are the general equations for the rate of change of the total momentum vorticity of a fixed volume.

3. Total vertical component of momentum vorticity and its rate of change

To a sufficiently close degree of approximation, the shape of the geopotential surfaces may be considered as spherical and we may use the spherical coordinates λ (the longitude) and the φ (latitude). Since the significant thickness[2] of the atmosphere is negligible compared with the radius of the earth, we may for the sake of calculation replace r by the mean radius, a, of the earth and introduce z as the vertical coordinate above the sea level. Unit vectors directed toward east, north, and zenith, are, respectively, $\boldsymbol{i} = a \cos \varphi \nabla \lambda$, $\boldsymbol{j} = a \nabla \varphi$, and $\boldsymbol{k} = \nabla z$.

The total vertical component of momentum vorticity, Q[3], of an arbitrary volume is defined as

$$Q = \int_V \nabla \times (\rho V) \cdot \boldsymbol{k} d\tau = \oint_S \rho V \times \boldsymbol{k} \cdot d\boldsymbol{\sigma}. \tag{8}$$

[2] Significant thickness of atmosphere refers to the average height of the atmosphere from the earth's surface to the top of the atmosphere where density is practically zero.

[3] Momentum vorticity Q refers to the total vertical component of relative momentum vorticity unless otherwise stated.

After expanding the vector product in the surface integral, we get

$$Q = \oint_S \rho(v\boldsymbol{i} - u\boldsymbol{j})\cdot d\boldsymbol{\sigma}; \tag{9a}$$

u and v are respectively the east and north components of the velocity. Similarly, the total vertical component of absolute momentum vorticity, Q_a, is

$$Q_a = \oint_S \rho\{v\boldsymbol{i} - (u + a\Omega\cos\varphi)\boldsymbol{j}\}\cdot d\boldsymbol{\sigma}. \tag{9b}$$

By examining (9a) and (9b), we find that *no two concentric spherical surfaces give any contribution to the total vertical component of relative and absolute momentum vorticity.* If we apply (9a) and (9b) to the entire atmosphere and recall that the velocity vanishes at the earth's surface and that the density vanishes at the outer limit of the atmosphere, it is seen that *the total vertical component of relative momentum vorticity of the entire atmosphere (or hydrosphere) is zero* and the total absolute momentum vorticity is due solely to the contribution of the density difference between the northern and southern sides of mountains, which is generally small. It is obvious that the total absolute momentum vorticity of the entire atmosphere would be zero if we neglect the topographic features of the earth's surface.

Now, since $\nabla\cdot\boldsymbol{k} \approx 0$, the rate of change of total vertical component of momentum vorticity for a fixed volume may be written as

$$\frac{\partial Q}{\partial t} = \int_V \nabla \times \frac{\partial\rho\boldsymbol{V}}{\partial t}\cdot\boldsymbol{k}d\tau = \int_V \nabla\cdot\left(\frac{\partial\rho\boldsymbol{V}}{\partial t}\times\boldsymbol{k}\right)d\tau.$$

By employing Gauss' theorem and making use of (4) and (6), we get

$$\begin{aligned}\frac{\partial Q}{\partial t} = \oint_S \Bigg\{&\rho[w(2\boldsymbol{\Omega}+\nabla\times\boldsymbol{V})-(2\boldsymbol{\Omega}+\nabla\times\boldsymbol{V})\cdot\boldsymbol{k}\boldsymbol{V}] \\ &-\frac{\boldsymbol{i}}{a}\left[\frac{\partial p}{\partial\varphi}+\rho\frac{\partial}{\partial\varphi}(\tfrac{1}{2}\boldsymbol{V}\cdot\boldsymbol{V})+av\nabla\cdot(\rho\boldsymbol{V})-aF_\varphi\right] \\ &+\frac{\boldsymbol{j}}{a\cos\varphi}\Bigg[\frac{\partial p}{\partial\lambda}+\rho\frac{\partial}{\partial\lambda}(\tfrac{1}{2}\boldsymbol{V}\cdot\boldsymbol{V}) \\ &+a\cos\varphi u\nabla\cdot(\rho\boldsymbol{V})-a\cos\varphi F_\lambda\Bigg]\Bigg\}\cdot d\boldsymbol{\sigma},\end{aligned} \tag{10a}$$

where w is the vertical component of the velocity, and F_φ and F_λ are the components of frictional force $\boldsymbol{F}$ in the directions of $\boldsymbol{i}$ and $\boldsymbol{j}$. The above equation shows that only those surface elements whose normals have horizontal components contribute to the change of the total momentum vorticity.

Similarly, the rate of change of the total vertical component of absolute momentum vorticity of a fixed volume may be written as

$$\frac{\partial Q_a}{\partial t} = \frac{\partial Q}{\partial t} + \frac{\partial}{\partial t}\int_V \nabla\times(\rho\boldsymbol{\Omega}\times\boldsymbol{r})\cdot\boldsymbol{k}d\tau.$$

After employing Gauss's theorem and the equation of continuity, we find that

$$\frac{\partial Q_a}{\partial t} = \frac{\partial Q}{\partial t} + a\Omega\oint_S \nabla\cdot(\rho\boldsymbol{V})\cos\varphi\boldsymbol{j}\cdot d\boldsymbol{\sigma}. \tag{10b}$$

Therefore, the contribution of the earth's rotation to the rate of change of total absolute momentum vorticity depends on the momentum divergence in the neighborhood of those surface elements which have meridional component.

For a material volume, the rate of change of total momentum vorticity is

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \oint_S \nabla\times(\rho\boldsymbol{V})\cdot\boldsymbol{k}\boldsymbol{V}\cdot d\boldsymbol{\sigma}.$$

4. Total vertical component of momentum vorticity of a polar cap and its rate of change

In examining hemispherical weather maps, one finds that the flow pattern exhibits a more or less zonal flow upon which perturbations are superimposed. The daily upper-air charts show a more pronounced zonal flow. Moreover, the zonal flow and these perturbations undergo changes, which are somewhat systematic. These changes give us important information about the propagation and development of atmospheric systems. In this section, we shall show that the study of total momentum vorticity of the polar cap leads to results which are equivalent to those obtained from the study of the mean zonal angular momentum.

If we apply (9a) to a polar cap bounded by a latitudinal surface at latitude φ, we find that the total momentum vorticity of a polar cap is

$$Q_\varphi = \int_{\sigma_\varphi}\rho u d\sigma = a\cos\varphi\int^{2\pi}\int_{z_0}^{H}\rho u dz d\lambda, \tag{11}$$

where σ_φ is the area of the latitudinal surface, z_0 is the height of the earth's surface above mean sea level, and H is the height where density is practically zero. Eq. (11) shows that *the total momentum vorticity of a polar cap is equal to the total zonal momentum integrated over the latitudinal boundary surface of the polar cap.* By using the hydrostatic equation, we may write (11) as

$$Q_\varphi = \frac{a\cos\varphi}{g}\int_0^{2\pi}\int_0^{P_0} u dp d\lambda = \frac{2\pi a}{g}\cos\varphi\bar{P}_0\bar{\tilde{u}}, \tag{12}$$

where

$$\bar{P}_0 \equiv \frac{1}{2\pi}\int_0^{2\pi}P_0 d\lambda, \qquad \tilde{u} \equiv \frac{1}{P_0}\int_0^{P_0} u dp$$

$$\bar{\tilde{u}} \equiv \frac{1}{2\pi}\int_0^{2\pi}\frac{P_0}{\bar{P}_0}d\lambda \approx \frac{1}{2\pi}\int_0^{2\pi}\tilde{u}d\lambda. \tag{13}$$

$\bar{P}_0$ is the mean surface pressure over a latitude circle at a fixed time, $\tilde{u}$ is the weighted mean zonal velocity for a given vertical at a fixed time, and $\bar{\bar{u}}$ is the weighted mean zonal velocity over the latitude circle. Equation (12) shows that the total momentum vorticity of a polar cap is equal to the product of the mean weighted zonal velocity and the total mass of a zonal ring of unit width along the line where the surface σ_φ meets the earth's surface. The quantities in (13) can be evaluated approximately from observed data.

Similarly, from (9b), the total absolute momentum vorticity of a polar cap is

$$\begin{aligned} Q_{a\varphi} &= \int_{\sigma_\varphi} \rho(u + a\Omega\cos\varphi)d\sigma \\ &= \frac{2\pi a\cos\varphi}{g}\bar{P}_0(\bar{\bar{u}} + a\Omega\cos\varphi). \qquad (14) \end{aligned}$$

Here, we have neglected the term due to the density difference between the northern and southern sides of mountains in the polar cap as it can be shown to be considerably smaller than the remaining terms in (14). Note that the contribution of earth's rotation to the total absolute momentum vorticity of a polar cap is equal to the total earth's momentum of the unit zonal ring defined by the latitudinal surface.

If we apply (10a) to a polar cap, we obtain the rate of change of total momentum vorticity of a polar cap

$$\begin{aligned} \frac{\partial Q_\varphi}{\partial t} = \int_{\sigma_\varphi} \Big\{ &\rho[-w(2\mathbf{\Omega} + \nabla\times V)\cdot \boldsymbol{j} \\ &+ v(2\mathbf{\Omega} + \nabla\times V)\cdot\boldsymbol{k}] \\ &- \frac{1}{a\cos\varphi}\Big[\frac{\partial p}{\partial\lambda} + \rho\frac{\partial}{\partial\lambda}(\tfrac{1}{2}V\cdot V) \\ &+ a\cos\varphi\{u\nabla\cdot(\rho V) - F_\lambda\}\Big]\Big\} d\sigma. \qquad (15) \end{aligned}$$

For atmospheric motion, it is generally found that the virtual stresses or Reynold's stresses are many thousand times as large as the viscous stresses. We may therefore express the frictional force along the latitude circle F_λ in terms of the virtual stresses and, further, to a good approximation, by $\partial\tau_{xz}/\partial z$ (Sutton 1951), where τ_{xz} is the eddy stress due to the effect of the velocity fluctuations in the vertical transport of the zonal momentum. In the spherical coordinates, the vorticity and mass divergence are

$$\begin{aligned} \nabla\times V = &\frac{\boldsymbol{i}}{a}\Big[\frac{\partial w}{\partial\varphi} - a\frac{\partial v}{\partial z}\Big] \\ &+ \frac{\boldsymbol{j}}{a\cos\varphi}\Big[a\frac{\partial}{\partial z}(u\cos\varphi) - \frac{\partial w}{\partial\lambda}\Big] \\ &+ \frac{\boldsymbol{k}}{a\cos\varphi}\Big[\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\varphi}(u\cos\varphi)\Big] \end{aligned}$$

and

$$\begin{aligned} \nabla\cdot(\rho V) = &\frac{1}{a\cos\varphi}\Big[\frac{\partial\rho u}{\partial\lambda} + \frac{\partial(\rho v\cos\varphi)}{\partial\varphi}\Big] \\ &+ \frac{\partial(\rho w)}{\partial z} + 2\frac{\rho w}{a}. \qquad (16) \end{aligned}$$

When these substitutions are made in (15) and the term $2\rho w/a$ is ignored, we get

$$\begin{aligned} \frac{\partial Q_\varphi}{\partial t} = \int_{\sigma_\varphi} \Big\{ &\rho[2\Omega(v\sin\varphi - w\cos\varphi)] \\ &+ \frac{1}{a}\Big(2\tan\varphi - \frac{\partial}{\partial\varphi}\Big)\rho uv \\ &- \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}(p + \rho u^2) \\ &+ \frac{\partial}{\partial z}(\tau_{xz} - \rho uw)\Big\} d\sigma. \qquad (17a) \end{aligned}$$

In the above equation, the terms involving meridional transport of relative momentum may be grouped in terms of meridional convergence of angular momentum. Thus,

$$\begin{aligned} \frac{\partial Q_\varphi}{\partial t} = &2\Omega\int_{\sigma_\varphi}\rho(v\sin\varphi - w\cos\varphi)d\sigma \\ &- \frac{1}{a^2\cos^2\varphi}\int_{\sigma_\varphi}\frac{\partial}{\partial\varphi}(a\cos^2\varphi\rho uv)d\sigma \\ &- \frac{1}{a\cos\varphi}\Big\{\int_{\underset{\sim}{\sigma_\varphi}}\frac{\partial}{\partial\lambda}(p + \rho uv)d\sigma \\ &+ \int_{\tilde{\sigma}_\varphi}\frac{\partial}{\partial\lambda}(p + \rho uv)d\sigma\Big\} \\ &+ \int_{\sigma_\varphi}\frac{\partial}{\partial z}(\tau_{xz} - \rho uwd\sigma, \qquad (17b)) \end{aligned}$$

where $\underset{\sim}{\sigma_\varphi}$ is the part of the latitudinal surface which is below the height h of the highest mountain in the latitudinal surface in question, and $\tilde{\sigma}_\varphi$ is the remaining part (see fig. 1). Now let us consider the last integral of (17b). We perform the integration first with respect to z and then with respecto to λ;

$$\begin{aligned} &\int_{\sigma_\varphi}\frac{\partial}{\partial z}(\tau_{xz} - \rho uw)d\sigma \\ &\quad = a\cos\varphi\int_0^{2\pi}\int_{z_0}^{H}\frac{\partial}{\partial z}(\tau_{xz} - \rho uw)dzd\lambda \\ &\quad = a\cos\varphi\int_0^{2\pi}\{(\tau_{xz})_H - (\tau_{xz})_{z_0} \\ &\qquad\qquad - (\rho uw)_H + (\rho uw)_{z_0}\}d\lambda. \end{aligned}$$

Since u and w vanish at the earth's surface and ρ vanishes at height H, the last two terms of the inte-

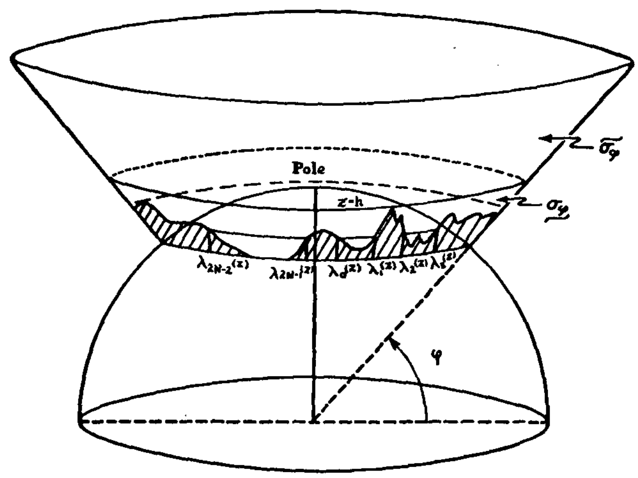


FIG. 1. Schematic diagram showing $\underset{\sim}{\sigma_\varphi}$, the latitudinal surface bounded below by the mountainous profile of the earth and above by the level $z = h$, and $\tilde{\sigma}_\varphi$, the latitudinal surface above $z = h$.

grand vanish. And, since the eddy stresses are caused by the transport of momentum, τ_{xz} vanishes at height H where density vanishes. The above integral therefore reduces to

$$-a \cos \varphi \int_0^{2\pi} (\tau_{xz})_{z0} d\lambda,$$

which is the total surface virtual stress integrated along the latitude circle. The fourth integral of (17b) is always zero. The third integral is to be taken over that area $\underset{\sim}{\sigma_\varphi}$ of the latitudinal surface bounded below by the mountainous profile of the earth and above by the level $z = h$. We have therefore to break it into a number of integrals, each of which operates between two neighboring mountains; thus,

$$-\frac{1}{a \cos \varphi} \int_{\underset{\sim}{\sigma_\varphi}} \frac{\partial}{\partial \lambda} (p + \rho u^2) d\tau$$
$$= -\int_{z_0}^{h} \left\{ \sum_{i=0}^{N-1} \int_{\lambda_{2i}}^{\lambda_{2i+1}} \frac{\partial}{\partial \lambda} (p + \rho u^2) d\lambda \right\} dz$$

where $N(z) - 1$ is the number of mountains at height z in the latitudinal surface, and $\lambda_{2i}(z)$, $\lambda_{2i+1}(z)$ are respectively the longitude of the zonal contour of the eastern and western sides of two neighboring mountains. The term $\int_{\underset{\sim}{\sigma_\varphi}} (\partial/\partial\lambda)(\rho u^2) d\sigma$ is identically zero, since the velocity vanishes at the earth's surface. The integration of the longitudinal pressure gradient between mountains gives the total pressure difference between neighboring mountains which may be named the "mountain-pressure" term. The mountain-pressure effect is positive if the total pressure over the eastern side of mountains is greater than that over the western sides, negative if the total pressure over the western side of the mountains is greater than that over the eastern sides. The second integral of the right-hand side of equation (17b) may be transformed into a more significant form if we can bring the partial differentiation out of the integral. To do this, we have to replace the surface element of integration $d\sigma$ by $a \cos \varphi d\lambda dz$ and also change the limits of integration since the area of latitudinal surface σ_φ is a function of latitude. After doing this, we return to the old surface element of integration. Thus,

$$-\frac{1}{a^2 \cos^2 \varphi} \int_{\sigma_\varphi} \frac{\partial}{\partial \varphi} (a \cos^2 \varphi \rho u v) d\sigma$$
$$= -\frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \int_{\sigma_\varphi} a \cos \varphi \rho u v d\sigma.$$

Eq. (17b) may now be expressed as

$$\frac{\partial Q_\varphi}{\partial t} = -\frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \int_{\sigma_\varphi} a \cos \varphi \rho u v d\sigma$$
$$- \int_{z_0}^{h} \left\{ \sum_{i=0}^{N-1} \int_{\lambda_{2i}}^{\lambda_{2i+1}} \frac{\partial p}{\partial \lambda} d\lambda \right\} dz$$
$$- a \cos \varphi \int_0^{2\pi} (\tau_{xz})_{z_0} d\lambda$$
$$+ 2\Omega \int_{\sigma_\varphi} \rho (v \sin \varphi - w \cos \varphi) d\sigma. \quad (18)$$

The rate of change of momentum vorticity of a polar cap can therefore be considered as due to the Coriolis effect, the effect of convergence of meridional flux of angular momentum, the mountain-pressure effect, and the surface virtual stress. In general, the Coriolis terms are two orders of magnitude smaller than that of the total convergence of meridional flux of angular momentum and may therefore be neglected. Thus, the speeding up and slowing down of the total momentum vorticity of the polar cap depends mainly on the balance of the last three terms of the right hand side of (18).

As a check of our formula (18), we may multiply it with $a \cos \varphi$, then integrate with respect to φ from latitude φ_1 to φ_2. We then get

$$\frac{\partial}{\partial t} \int_{\varphi_1}^{\varphi_2} \int_{\sigma_\varphi} a^2 \cos \varphi \rho u d\sigma d\varphi$$
$$= \int_{\sigma_{\varphi_1}} a \cos \varphi \rho u v d\sigma - \int_{\sigma_{\varphi_2}} a \cos \varphi \rho u v d\sigma$$
$$+ \int_{\varphi_1}^{\varphi_2} a \int_{z_0}^{h} \left\{ \sum_{i=0}^{N-1} \int_{\lambda_{2i}}^{\lambda_{2i+1}} \frac{\partial p}{\partial \lambda} d\lambda \right\} dz d\varphi$$
$$- \int_{\varphi_1}^{\varphi_3} a^3 \int_0^{2\pi} (\tau_{xz})_{z_0} \cos^2 \varphi d\lambda d\varphi$$
$$+ 2\Omega \int_{\varphi_1}^{\varphi_2} \int_{\sigma_\varphi} a^3 \rho \cos \varphi (v \sin \varphi - w \cos \varphi) d\sigma d\varphi. \quad (19)$$

This equation is equivalent to Jeffrey's equation for the rate of change of total zonal angular momentum of a zonal ring. This shows that the study of total momentum vorticity and of angular momentum are closely related to each other. However, equation (18) gives us information about the rate of change of the momentum vorticity of a polar cap, and therefore its application at two latitudinal walls would give us the rate of change of momentum vorticity for the enclosed zonal ring.

By use of eq. (14), we have

$$\frac{\partial Q_{a\varphi}}{\partial t} = \frac{\partial Q_\varphi}{\partial t} + \frac{a^2 \cos^2 \varphi \Omega}{g} \int_0^{2\pi} \frac{\partial p_0}{\partial t} d\lambda.$$

This equation states that the rate of change of total absolute momentum of a polar cap is equal to the rate of change of total relative momentum vorticity of the polar cap plus the rate of change of the total momentum of the mass of the latitudinal ring of unit width which encircles the polar cap and rotates with the earth.

5. The maintenance of zonal circulation in relation to meridional transfer of momentum vorticity and angular momentum

The physical principle concerning the general circulation as related to the global balance of angular momentum, which had drawn the attention of Jeffreys (1926), has been re-examined and investigated by Bjerknes (1948), Starr (1948), Priestley (1948), Mintz (1949), White (1949), Widger (1949), and Palmén (1949, 1950). The role of vorticity transport in atmospheric circulation has been discussed by Eady (1950), Kuo (1951), and Yeh (1951).

Here we shall apply the results obtained in the last section to the study of the maintenance of zonal circulation. For brevity, let us denote the terms in (18) due to the effect of convergence of meridional transfer of angular momentum, mountain-pressure, surface stress, and Coriolis force respectively by A, M, T, and C. Eq. (18) may therefore be written as

$$\frac{\partial Q_\varphi}{\partial t} = \frac{\partial}{\partial t} \int_{V \text{ (polar cap)}} \nabla \times (\rho V) \cdot k d\tau \qquad (20)$$

$$= C + A + M + T.$$

The more-or-less systematic day-to-day changes of basic zonal flow as pointed out in the beginning of section 4 suggest to us to express each term, say β, of the above equation as the sum of the mean β and its deviation β' from the time mean

$$\beta = [\beta] + \beta', \qquad [\beta] = \frac{1}{\tau} \int_{\tau_0}^{\tau_0+\tau} \beta dt$$

where τ is the time interval (*e.g.*, month, season, year, *etc.*). By doing so, eq. (17) may be written as

$$\frac{\partial}{\partial t}[Q_\varphi] + \frac{\partial Q_\varphi'}{\partial t} = [C] + [A] + [M] + [T] + C' + A' + M' + T'. \qquad (21)$$

Since the time mean of total momentum vorticity of a polar cap may be assumed as a constant over a sufficiently long-time interval, we have

$$[C] + [A] + [M] + [T] = 0, \qquad (22)$$

$$\frac{\partial Q_\varphi}{\partial t} = \frac{\partial Q_\varphi'}{\partial t} = C' + A' + M' + T'. \qquad (23)$$

For a sufficiently long-time interval, the total meridional mass transport through the entire latitudinal surface will approach zero. We will also assume that the Coriolis term due to ρw integrated over the latitudinal surface is small. Therefore, the time mean of the two Coriolis terms may be taken as zero, and equation (22) becomes

$$[A] + [M] + [T] = 0. \qquad (24)$$

Eq. (24) shows that for a long-time interval *the mean convergence of meridional flux of angular momentum on the latitudinal surface is balanced by the joint effects of surface friction and mountain-pressure along the longitudinal circle.* The physical implication of (24) will be discussed later.

It is a well-known physical principle that any dynamic system cannot change its absolute angular momentum except through external agencies. For the atmosphere, the only significant external agency is the action of the surface of the earth. Eq. (19) shows that this action is the joint effect of skin friction and mountain-pressure. According to computations by White (1949), the mean total surface stress and mountain-pressure effects have a high positive correlation, although the magnitude of the surface-stress term is generally greater than that of the mountain-pressure term except in the regions where the mean zonal component of surface winds are small.[4] The surface frictional force may therefore be considered as the essential agency. For simplicity of discussion, we may consider mainly the effect of surface friction which includes the action of mountains on surface pressure systems.

The effect of skin friction of the earth tends to decrease the momentum relative to the earth. From the viewpoint of absolute angular momentum, surface friction works toward the creation of absolute angular momentum in regions of surface easterly wind

[4] The physical justification of the positive correlation between the surface stress and mountain-pressure may be obtained from the mass-accumulation or pressure-increase on the windward sides of mountains.

and the destruction of absolute angular momentum in the regions of surface westerly wind. Since, in the long run, the total angular momentum of the atmosphere remains unchanged, the angular momentum generated from the source regions (mean easterly wind) must return to the sink regions (mean westerly winds) of the earth's surface. The physical meaning of (24) therefore implies that *the amount of mean angular momentum created by skin friction and mountain-pressure effect must be equal to that lost by divergence from the region, and that the amount of mean angular momentum gained by convergence into a region must be equal to that destroyed by the skin friction and the mountain-pressure effect. The zones where the joint effects of surface friction and mountain-pressure vanish should be the zones of non-divergence of angular momentum or zones where the transport of absolute angular momentum is either a maximum or a minimum.*

Now we proceed to show that for the mean state of the atmosphere the term of convergence of meridional transport of absolute angular momentum in (24) is equivalent to the transport of momentum vorticity or approximately equal to the transport of mass-vorticity ($\rho\nabla \times V\cdot k$).[5] The total transport of momentum vorticity across a latitudinal surface may be written as

$$\int_{\sigma_\varphi} v\nabla \times (\rho V)\cdot k d\sigma \approx \int_{\sigma_\varphi} \rho v \nabla \times V\cdot k d\sigma$$
$$= \int_{\sigma_\varphi} \frac{\rho v}{a \cos \varphi} \left\{ \frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \varphi}(u \cos \varphi) \right\} d\sigma.$$

For the mean state of the atmosphere, mass divergence is zero and the last integral of the above equation becomes

$$\int_{\sigma_\varphi} \frac{\rho v}{a \cos \varphi} \left\{ \frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \varphi}(u \cos \varphi) \right\} d\sigma$$
$$= \frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \int_{\sigma_\varphi} a \cos \varphi \rho u v d\sigma$$
$$- \frac{1}{a \cos \varphi} \int_{\sigma_\varphi} \left\{ \frac{\rho}{2} \frac{\partial v^2}{\partial \lambda} - \frac{1}{2\rho} \frac{\partial (\rho u)^2}{\partial \lambda} - au \cos \varphi \frac{\partial \rho w}{\partial z} \right\} d\sigma.$$

Over a sufficiently long-time interval, the vertical velocity and the density variation along a latitudinal circle may be taken as small. We may therefore neglect the second term against the first one of the right-hand side of the above equation. Thus,

$$\left[\int_{\sigma_\varphi} v\nabla \times (\rho V)\cdot k d\sigma \right] \approx \left[\int_{\sigma_\varphi} \rho v \nabla \times V\cdot k d\sigma \right]$$
$$\approx \left[\frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \int_{\sigma_\varphi} a \cos \varphi \rho u v d\sigma \right]. \quad (25)$$

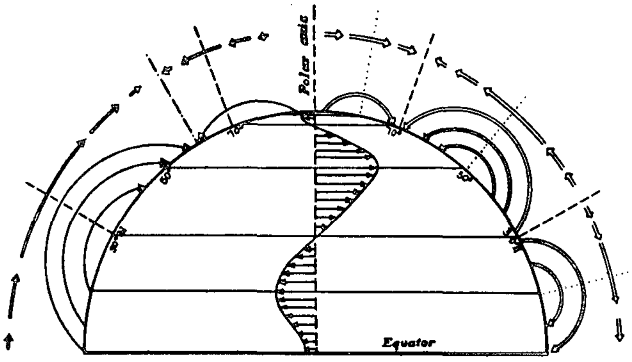
The mean convergence of angular momentum can thus be interpreted as the mean transport of momentum vorticity across the latitudinal surface. This is useful in the determination of the direction of mean meridional transport of momentum vorticity in the atmosphere, as we shall see in the following discussion.

Following the reasoning that skin friction tends to decrease the momentum of the atmosphere relative to the earth's surface, we find that in the region of anticyclonic momentum vorticity the surface friction works toward the creation (destruction) of cyclonic (anticyclonic) momentum vorticity, whereas in the regions of cyclonic momentum vorticity it works toward the destruction (creation) of cyclonic (anticyclonic) momentum vorticity. As we have shown in section 4 that the total momentum vorticity of the entire atmosphere is zero, it follows that, in order to maintain the zonal circulation, the cyclonic momentum vorticity generated by the earth's surface in the regions of anticyclonic momentum vorticity must return to the earth in the regions of cyclonic momentum vorticity. From the viewpoint of transport of momentum vorticity, (24) implies the following: *for the mean state of the atmosphere, the amount of momentum vorticity of a polar cap generated by the earth's surface must be equal to the amount of momentum vorticity transported out through the latitudinal surface.* Further, from (24) we infer that, *at the latitude circle where the mean effects of skin friction and mountain-pressure cancel each other, the mean meridional transfer of momentum vorticity is zero.* Since surface friction has high positive correlation with mountain-pressure effect, *the latitude of zero meridional transfer of momentum vorticity must be in the neighborhood of the latitude of zero surface zonal velocity;* by (25), *this is also the latitude of non-divergence of meridional transfer of angular momentum.*

For the convenience of discussion, let us draw a schematic diagram of mean surface zonal-wind distribution of the northern hemisphere as shown in fig. 2, with light easterly zonal wind at the equator, the mean easterly component increasing with latitude to a maximum at 15N, then decreasing to zero at 30N. Farther north, the mean zonal surface winds are westerly, increasing in strength with latitude to a maximum at 50N, then decreasing to zero again at 70N. North of 70N, there are easterlies again, with a maximum at about 80N and becoming zero at the pole.

According to the mean surface zonal wind distribution, we find that the corresponding source regions of

[5] On account of weak horizontal density gradient, it can be shown that the magnitude of momentum vorticity of moderately large horizontal area is practically equal to that of the mass-vorticity. For simplicity of calculation, we may use the latter quantity instead of the former.



Transport of momentum vorticity.

Transport of angular momentum.

Surface of zero transport of momentum vorticity and non-divergence of angular momentum.

Surface of maximum transport of momentum vorticity.

Surface of zero transport of angular momentum.

FIG. 2. Schematic diagram showing the mean zonal surface wind distribution in the northern hemisphere and the corresponding meridional transports of the vertical component of momentum vorticity and angular momentum.

angular momentum are between the equator and 30N and between 70N and the north pole; the sink region is between 30N and 70N. For momentum vorticity,[6] the source regions are from 15N to 50N and from 80N to the north pole; the sink regions are from 50N to 80N and from 15N southward, terminating at the latitude where surface momentum vorticity changes its sign. By (25), we find that the surfaces of zero transport[7] of momentum vorticity and non-divergence of meridional transport of angular momentum are in the vicinity of 30N and 70N, whereas at the north pole this surface degenerates into a line. The direction of meridional transport of momentum vorticity changes its sign across these surfaces, and the maximum northward and southward transports of angular momentum occur respectively on the latitudinal surfaces 30N and 70N. The surface of non-transport of angular momentum must occur in the poleward part of the region between 30N and 70N, since the strength of northward transport in this region is stronger than the transport southward from the polar region. Between 15N and 50N and between 80N and the north pole are the regions of meridional divergence of momentum vorticity, whereas between 50N and 80N and between the equator and 15N are the regions of meridional convergence of momentum vorticity. The mean meridional transport of momentum vorticity in the month of January 1949 has been computed (Kao, 1953) from the upper geostrophic winds, and the results give independent verification of the theoretical pattern of vorticity flux. By (25), we find that the regions of divergence of meridional transfer of angular momentum occur between the equator and 30N and between 70N and the north pole. In these regions, at the earth's surface the angular momentum is being created, whereas in the regions of meridional convergence (between 30N and 70N) the angular momentum is being destroyed.

[6] Momentum vorticity refers to the vertical component of cyclonic momentum vorticity unless otherwise stated.
[7] Refers to meridional transport.

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