GEOSTROPHIC VORTEX MOTION

G. K. Morikawa

New York University

ABSTRACT

The concept of a geostrophic point vortex results from incorporating the geostrophic approximation in the equations of motion of the one-layer homogeneous atmosphere with a free surface. A consistent, systematic approximation procedure is used, yielding higher-order approximations without requiring additional physical assumptions. Application to the prediction of hurricane tracking and upper-air cyclogenesis is discussed.

This paper is based on Report IMM–NYU 226, "On the Theory of Large-Scale Nearly-Horizontal Motions in the Atmosphere" (February 1956) written under the auspices of the Geophysical Research Directorate, U. S. Air Force Cambridge Research Center and the U. S. Office of Naval Research.

1. Introduction

The simplest mathematical model of large-scale, nearly-horizontal atmospheric (or oceanic) motion, incorporating the important predominant forces of gravity and the earth's rotation (Coriolis), is the one-layer homogeneous atmosphere with a free surface (described in §2). In this long wave (or shallow water) theory,[1] a well-established meteorological observation is implicit: the atmosphere is assumed to be in nearly-hydrostatic equilibrium; *i.e.*, the vertical component of the inertia force is negligibly smaller than the gravity force which, consequently, is balanced by the vertical pressure-gradient force. Real gas effects such as compressibility, heat conduction, and viscosity are not considered.

Another well-known observation is the geostrophic approximation: in the nearly-horizontal motion (primarily in the middle latitudes) of the upper atmosphere where ground effects are small, the Coriolis force and the horizontal pressure-gradient force are approximately in balance, and the horizontal component of the inertia force is small. One of the primary motives in this paper is to demonstrate that the geostrophic approximation may be introduced in this long-wave model using a consistent, systematic "ordering approximation procedure" (§3 and Appendix I). This procedure yields higher-order approximations without requiring additional physical assumptions. To the lowest order of approximation, a "geostrophic conservation equation" (§3 and 4) results, yielding a precise concept of an atmospheric vortex (described in §5). This geostrophic point vortex (singular solution) is the generalization of the ordinary potential (logarithmic) vortex of two-dimensional, incompressible flow when gravity and Coriolis forces are introduced.

Recently, several meteorologists, aided by some ideas and suggestions of Rossby (1940), have made important contributions in attempting to obtain simpler formulations of atmospheric motion by introducing these (and other) physical approximations into the complicated equations of gas dynamics. Charney (1948), Thompson (1952), Bolin (1955) and others have discussed rather thoroughly the orders of magnitude of the physical variables and have obtained approximate formulations which are being subjected to the high-speed numerical methods required for short-range weather forecasting (a few days) in the middle latitudes. One of the first fruitful ideas was the concept of filtering out the relatively unessential characteristic waves, such as those small disturbances moving with the speed of gravity, and sound waves, from the lowest-order approximate description of large-scale atmospheric motions. This is, of course, consistent with the observation that the flow velocities usually encountered in the atmosphere are an order of magnitude smaller than these characteristic speeds. In §3, we shall see just how this low-velocity approximation is related to the geostrophic approximation.

A relatively novel feature of the approximate mathematical description of large-scale atmospheric motions is already indicated by the two basic meteorological observations of nearly-hydrostatic equilibrium and nearly-geostrophic, low-velocity flow. In such problems of non-linear motion, an asymptotic time-dependent formulation, which describes phenomena closer to the final equilibrium state than the linear description can adequately give, may be necessary. The implication here is that many of the relevant atmospheric motions are inherently "second order" (approximations) in nature; a linear approximation is best-suited for describing perturbations not too far removed from the initial state. In other surface-wave problems, a like situation has already been encountered—for example, in the investigation of the solitary

[1] A systematic derivation is displayed in Stoker's treatise on water waves (1956). Also, *cf.* Prandtl (1952).

wave by Friedrichs (1948) and Keller (1948) and in the study by the author (1957) of flood waves moving in slow rivers.

In meteorology, the concept of geostrophic vortex motion may prove to be of more than aesthetic interest. Two possible applications are: (1) the short-range prediction of fully-developed hurricane tracks (discussed in §5) over the ocean approaching land areas. In many important situations, the available meteorological data indicates the feasibility of avoiding the necessity of considering non-steady, real-fluid processes such as viscous decay of hurricane strength and interaction of hurricanes with fronts. (2) the study of upper-air cyclogenesis by representing closed high- and low-pressure systems by a distribution of geostrophic point vortices and following their motion on a high-speed computer.[2] Some success has already been achieved in the first application by E. Isaacson, D. Levine and the author.

2. Formulation

In a completely systematic derivation of approximate meteorological equations, one would begin with the known mathematical formulation of atmospheric motion on a nearly-spherical earth based on the equations of gas dynamics.[3] However, such an ambitious program would involve additional physical approximations, primarily of a thermodynamic nature, which do not seem to be so well-established as the hydrostatic and geostrophic approximations. So, in this paper, we content ourselves with more modest aims and take the one-layer, inviscid, homogeneous atmosphere (on a horizontal plane) with a free surface as our original, primitive model. Thus, some overall physical assumptions are immediately implied: (1) the hydrostatic approximation (mentioned in §1), (2) a tangent plane approximation to flow over a nearly-spherical rotating earth, and (3) approximation by a mechanical model, implying that thermodynamical and other real-gas effects are of lesser consequence.

The one-layer, homogeneous atmosphere is described by the conservation equations of mass (continuity) and momentum in terms of the depth $h(x, y, t)$ and the horizontal velocity components $u(x, y, t)$ and $v(x, y, t)$ in the x and y directions respectively;

$$\frac{dh}{dt} + h(u_x + v_y) = 0, \tag{1a}$$

$$\frac{du}{dt} + gh_x - fv = 0, \tag{2a}$$

$$\frac{dv}{dt} + gh_y + fu = 0 \tag{2b}$$

where

$$\frac{d}{dt}(\ ) = (\ )_t + u\cdot(\ )_x + v\cdot(\ )_y$$

means differentiation along particle paths in (x, y, t) space. g is the acceleration of gravity, and $f = 2\Omega \sin \phi$ is the Coriolis parameter in terms of the earth's angular velocity Ω and the latitude angle ϕ measured from the equator. f is fixed at the desired point of tangency to the earth's surface. Subscripts denote partial differentiation. In this long-wave theory (hydrostatic approximation), the pressure p and vertical velocity w have become subsidiary relations which vary linearly in the vertical direction z;

$$p = g\rho(h - z) \tag{3a}$$

where ρ is the density (assumed to be constant) and

$$w = -z(u_x + v_y). \tag{3b}$$

Equations (1a) and (2) are a system of three first-order non-linear partial differential equations for (h, u, v) of *hyperbolic type* (like the wave equation). That is, the characteristic manifold, $\Phi(x, y, t) =$ constant, along which certain discontinuities can propagate is defined by

$$\frac{d\Phi}{dt}\left\{\left(\frac{d\Phi}{dt}\right)^2 - gh(\Phi_x^2 + \Phi_y^2)\right\} = 0. \tag{4}$$

The first factor, $d\Phi/dt = 0$, represents the particle paths in (x, y, t) space, and the quadratic quantity within the bracket describes the Monge cone[4] (*cf.* Courant and Hilbert) along which surfaces small disturbances propagate with the characteristic gravity wave velocity $(gh)^{\frac{1}{2}}$.

In meteorological applications, the continuity equation (1a) is usually replaced by the vorticity equation obtained by the curl operator on the momentum equation (2) and eliminating the divergence $(u_x + v_y)$ between the resulting equation and (1a). Thus, we get

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0 \tag{1b}$$

where $\zeta = (v_x - u_y)$ is the vertical component of vorticity. The system of equations (1b) and (2) is more directly descriptive of atmospheric motions than, but not essentially different from, that of equations (1a) and (2).[5] We defer to §4 the discussion of initial and boundary conditions necessary to complete the formulation.

[2] Suggested by F. H. Clauser of Johns Hopkins University.

[3] Just such a derivation has been attempted with partial success in an unpublished paper by J. B. Keller and Lu Ting.

[4] If the system of equations (1) and (2) is linearized, the non-linear Monge cone becomes the linearized Froude (or Mach) cone.

[5] This rather-qualitative remark is also strikingly demonstrated if one tries to carry through the formal derivation of the geostrophic conservation equation (§3) using (1a) (in place of (1b)) and (2). The final result is, of course, the same, but the amount of additional calculation necessary is quite appreciable.

3. The geostrophic and low-velocity approximations

The ordering approximation procedure is apparently a useful way of systematically introducing physical approximations into a well-formulated but complicated mathematical description of a physical system. The more manageable approximate systems which result are asymptotic representations in some sense depending on the particular problem considered (*cf.* §1 and Morikawa, 1957). The method consists of combining the following two formal mathematical devices: (1) transforming each variable (dependent and independent) by multiplying it with a small parameter ($\alpha \ll 1$, say) raised to an arbitrary exponent, and (2) making a power series expansion with respect to α of each of the dependent variables. By introducing a sufficient number of consistent physical approximations, the exponents can be evaluated. The second device, alone, is the widely-used linear perturbation method which always retains the underlying local geometrical properties of the original system. The first device, by appropriately scaling each variable, allows an ordering of the terms in the original system, dictated by the physical approximations; this is essentially Prandtl's boundary layer procedure which can be traced still earlier to Poincaré's asymptotic method. However, *α need not occur explicitly in the original differential equation system.* In fact, here (§4) and in Morikawa (1957), α occurs in the initial or boundary conditions. The ordering approximation procedure which combines these two devices is particularly suitable for those physical systems which are described by complicated differential (or integral) equations but in which the orders of magnitude of the terms for the particular phenomenon of interest are known. This is a weak, but often natural and sufficient, condition on the properties of the desired approximate solution so that the approximate system which it satisfies can be derived. This method is an extension in viewpoint of that used by Friedrichs (1948) and Keller (1948). In addition, our later experience (unpublished) shows that the asymptotic behavior of solutions of the appropriate linearized problem can be helpful in determining the unknown exponents (also, *cf.* Morikawa, 1957).

We now introduce the geostrophic and low-velocity approximations into the system of equations (1b) (or (1a), but see footnote 5) and (2). The geostrophic approximation states that the inertia term (actually, three terms), in each of equations (2a) and (2b), is of higher order in α (or lower order of magnitude) than the pressure-gradient term which is of the same size as the Coriolis term (to the lowest order to approximation). The low-velocity approximation implies that we retain the particle derivative $d(\ )/dt$ intact (see Appendix I) and make a perturbation expansion on the atmosphere at rest—*i.e.*, on the solution

$$(h, u, v) = (h_0, 0, 0), \quad h_0 = \text{constant}. \tag{5}$$

We note that (5) is an exact solution of (1a) (or (1b)) and (2). The formal evaluation of the unknown exponents for each variable is relegated to Appendix I. The simple result is that the *time t is the only necessary scaled variable* (*with respect to $\alpha \ll 1$*). This means that *the combined geostrophic and low-velocity approximations imply an asymptotic representation for large time.* That is, to lowest order of approximation, the flow which would *initially move in the direction of negative pressure gradient* is directed, *after a sufficiently long time, normal to this direction by the horizontal Coriolis force when the description is given in terms of the appropriate scaled time τ.* From Appendix I, the scaled time[6] is

$$\tau = \alpha t \tag{6a}$$

and the system of equations (1a) or (1b) and (2) is transformed by replacing $d(\ )/dt$ by $d(\ )/d\tau$ where

$$\frac{d}{d\tau}(\ ) = \alpha\cdot(\ )_\tau + u\cdot(\ )_x + v\cdot(\ )_y. \tag{6b}$$

The solution (h, u, v) and the characteristic Φ are expanded as follows:

$$h(x, y, \tau) = h_0 + \alpha h^{(1)}(x, y, \tau) + \alpha^2 h^{(2)}(x, y, \tau) + \cdots, \tag{7a}$$

$$u(x, y, \tau) = \alpha u^{(1)}(x, y, \tau) + \alpha^2 u^{(2)}(x, y, \tau) + \cdots, \tag{7b}$$

$$v(x, y, \tau) = \alpha v^{(1)}(x, y, \tau) + \alpha^2 v^{(2)}(x, y, \tau) + \cdots, \tag{7c}$$

$$\Phi(x, y, \tau) = \alpha \Phi^{(1)}(x, y, \tau) + \alpha^2 \Phi^{(2)}(x, y, \tau) + \cdots. \tag{7d}$$

We carry out the approximation by putting (7a, b, c) into the transformed system of equations (from (6) and (1b) and (2)) and equating coefficients of like powers of α in the usual way. We do the same with (7d) and the transformed characteristic equation corresponding to (4).

The lowest order (first order in α) approximate solution $(h^{(1)}, u^{(1)}, v^{(1)})$ satisfies

$$\frac{d^{(1)}}{d\tau}\left(\zeta^{(1)} - \frac{fh^{(1)}}{h_0}\right) = 0, \quad f = \text{constant},^7 \tag{8a}$$

[6] In a recent review article of Soviet progress in short- and long-range weather forecasting, Blinova and Kibel (1957) mention a similar scale transformation of time combined with a perturbation expansion with respect to a small parameter. However, no motivation or derivation such as we give in Appendix I is presented there. A more detailed exposition of the Soviet work is awaited with interest.

[7] Actually, this condition is $J(f, \psi) = u^{(1)}f_x + v^{(1)}f_y = 0$; but we consider $(u^{(1)}, v^{(1)})$ not identically zero. The Jacobian $J(a, b) = a_x b_y - a_y b_x$, and ψ is the stream function (*cf.* §4).

$$gh_x^{(1)} - fv^{(1)} = 0, \tag{8b}$$

$$gh_y^{(1)} + fu^{(1)} = 0 \tag{8c}$$

where

$$\frac{d^{(1)}}{d\tau} = (\ )_\tau + u^{(1)}\cdot(\ )_x + v^{(1)}\cdot(\ )_y \tag{8d}$$

and the vorticity $\zeta^{(1)} = v_x^{(1)} - u_y^{(1)}$. Since f = constant, the geostrophic conditions (8b, c) give

$$\zeta^{(1)} = \frac{g}{f}\Delta h^{(1)} \tag{8e}$$

where the Laplacian $\Delta = (\ )_{xx} + (\ )_{yy}$ and the divergence is zero,

$$u_x^{(1)} + v_y^{(1)} = 0, \tag{8f}$$

to this order of approximation. Thus, by (3b), the first-order vertical velocity $w^{(1)} = 0$. The characteristic solution $\Phi^{(1)}$ satisfies

$$\left(\frac{d^{(1)}\Phi^{(1)}}{d\tau}\right)\{(gh_0)[(\Phi_x^{(1)})^2 + (\Phi_y^{(1)})^2]\} = 0. \tag{8g}$$

(8a) is the non-linear conservation equation consistent with the geostrophic conditions (8b, c). The characteristic equation (8g) states that only the particle paths $(d^{(1)}\Phi^{(1)}/d\tau) = 0$ remain as true characteristics (to first order in α) since the Monge cone has degenerated into a plane τ = constant (the expression in the bracket); this is the precise statement of the low-velocity approximation. This degeneracy of the characteristics is further evidence of the asymptotic nature of the approximation. The restriction that f = constant is apparently a consequence of the tangent plane approximation (*cf.* §2). The only consistent way which we can see to relax this condition is to consider a better approximation to flow over a nearly-spherical earth.

The second-order (in α) approximate solution $(h^{(2)}, u^{(2)}, v^{(2)})$ satisfies

$$\begin{aligned}\frac{d^{(1)}}{d\tau}&\left(\zeta^{(2)} - \frac{fh^{(2)}}{h_0}\right) + \left(u^{(2)}\frac{\partial}{\partial x} + v^{(2)}\frac{\partial}{\partial y}\right)\left(\zeta^{(1)} - \frac{fh^{(1)}}{h_0}\right)\\ &= -\left(\frac{h^{(1)}}{h_0}\right)^2\frac{d^{(1)}}{d\tau}\left(\frac{h_0\zeta^{(1)}}{h^{(1)}}\right)\\ &= \frac{h_\tau^{(1)}}{h_0}\left(\zeta^{(1)} - \frac{fh^{(1)}}{h_0}\right),\end{aligned} \tag{9a}$$

$$gh_x^{(2)} - fv^{(2)} + \frac{d^{(1)}u^{(1)}}{d\tau} = 0, \tag{9b}$$

$$gh_y^{(2)} + fv^{(2)} + \frac{d^{(1)}v^{(1)}}{d\tau} = 0. \tag{9c}$$

From (9b, c), the second-order vorticity $\zeta^{(2)} = v_x^{(2)} - u_y^{(2)}$ becomes[8]

$$\zeta^{(2)} = \frac{g}{f}\Delta h^{(2)} - \frac{2}{f}J(u^{(1)}, v^{(1)}) \tag{9d}$$

where the Jacobian $J(a, b) = a_x b_y - a_y b_x$ and the second-order divergence is

$$u_x^{(2)} + v_y^{(2)} = -\frac{1}{f}\frac{d^{(1)}}{d\tau}\zeta^{(1)} = -\frac{1}{h_0}\frac{d^{(1)}}{d\tau}h^{(1)} = -\frac{h_\tau^{(1)}}{h_0}. \tag{9e}$$

By (3b), the second-order vertical velocity is

$$w^{(2)} = -z(u_x^{(2)} + v_y^{(2)}) = \frac{zh_\tau^{(1)}}{h_0}. \tag{9f}$$

(9a, b, c) describe a system of non-homogeneous linear differential equations with non-constant coefficients in terms of the lowest-order solution $(h^{(1)}, u^{(1)}, v^{(1)})$. The second-order characteristic solution $\Phi^{(2)}$ satisfies

$$\left\{\frac{d^{(1)}\Phi^{(2)}}{d\tau} + \left(u^{(2)}\frac{\partial}{\partial x} + v^{(2)}\frac{\partial}{\partial y}\right)\Phi^{(1)}\right\} \cdot\{(gh_0)[(\Phi_x^{(1)})^2 + (\Phi_y^{(1)})^2]\} = 0. \tag{9g}$$

The first bracketed factor shows that the second-order disturbances are perturbed off the first-order particle paths by the additional factor $\left(u^{(2)}\frac{\partial}{\partial x} + v^{(2)}\frac{\partial}{\partial y}\right)\Phi^{(1)}$. The second bracketed factor is identical to the bracketed factor in (8g), showing that the Monge cone is still the degenerate plane τ = constant. Although we do not study this *non-geostrophic system* further in this paper, meteorologists may be interested in a comparison of this linear second-order system to their approximate formulation using the non-linear "balance equation" (*cf., e.g.,* Charney (1955), Thompson (1956), or Bolin (1956)).

4. The geostrophic conservation equation

Although the lowest-order approximate system of equations (8) is still non-linear, it is appreciably simpler than the parent equations (1b) and (2). As we shall see, we have apparently retained the basic structure of the desired solution, expressing the full implications of the geostrophic and low-velocity approximations, while stripping the original system of unessentials.

The conservation equation can be regarded as a single third-order non-linear differential equation for

[8] The approximate vorticity relations (8e) and (9d) are most readily found by applying (6) and (7) to the equation resulting from taking the divergence of the momentum equations (2).

the *perturbed depth* $h^{(1)}$ after eliminating $u^{(1)}$, $v^{(1)}$, and $\zeta^{(1)}$ by (8b, c, d, e) and by redefining $h^{(1)}$ as follows:

$$\psi \equiv \frac{g}{f} h^{(1)}. \tag{10}$$

The system of equations (8a, b, c) can be written in a neater form as follows:

$$\frac{d^{(1)}}{d\tau}(\Delta - \kappa^2)\psi = 0, \tag{11a}$$

$$u^{(1)} = -\psi_y, \tag{11b}$$

$$v^{(1)} = \psi_x \tag{11c}$$

where $\kappa^2 = f^2/gh_0$ (= constant). We call (11a) (or (8a)), *the geostrophic conservation equation.*[9] The geostrophic conditions (11b, c) show that $\psi(x, y, \tau)$ is actually the stream function and $d^{(1)}\psi/d\tau = \psi_\tau$. Of course, this follows immediately from (8) since *the flow is divergence-free.*[10] Then, since the vertical velocity $w^{(1)} = 0$ (from (3b)), there is *no time-wise interchange of but a definite balance between kinetic and potential energy to this order of approximation; this interchange is described by the non-geostrophic second-order approximate equations* (9).

Further evidence of the asymptotic nature of the approximate equations (8) or (11) is given by considering the initial conditions[11] necessary to complete the mathematical formulation. For (11), it is necessary to give only ψ (or $h^{(1)}$) initially, while, for the original hyperbolic system (1) and (2), all three quantities (h, u, v) must be specified initially. Thus, (11) more closely resembles an elliptic or parabolic system—*e.g.*, like the heat equation. The loss of two initial conditions for (11)—*i.e.*, the initial flow velocities need not be specified—is a familiar asymptotic property (see Friedrichs, 1955) and coincides neatly with the physical observation that velocities are more difficult to measure in the atmosphere than pressure (or depth $h^{(1)}$).

In setting up a systematic program of studying the geostrophic conservation equation (11), three possible approaches come to mind: (1) linearization—many meteorologists have studied similar (and more complicated) equations, so we abstain from such considerations here; (2) numerical method—again, numerous meteorologists have made such studies in applying similar equations to short-range weather forecasting on high-speed computers (*e.g.*, Charney, Fjörtoft, and von Neumann (1950) and Bolin (1956)); and (3) *geostrophic point vortices*, which we present in detail in §5 and Appendix II—except in very simple situations, we apparently must resort to numerical methods here, even though the problem can be reduced to a study of *ordinary differential equations.* Also, the formal generalization of (11) to a two-layer (or more) model is a simple matter; the resulting equations are summarized in Appendix III.

The form of the geostrophic conservation equation closely resembles another hydrodynamical equation which has received much attention—the equations of two-dimensional incompressible *rotational flow.* In this case, the flow is described by the continuity equation,

$$u_x + v_y = 0, \tag{12a}$$

which implies the existence of a stream function Ψ such that the velocity components $u = -\Psi_y$ and $v = \Psi_x$ and the vorticity $\zeta = v_x - u_y = \Delta\Psi$ is conserved along particle paths

$$\frac{d}{dt}\Delta\Psi = 0 \tag{12b}$$

where the particle derivative

$$\frac{d}{dt}(\ ) = (\ )_t + u\cdot(\ )_x + v\cdot(\ )_y. \tag{12c}$$

(11) may be regarded as a generalization of (14) when gravity and Coriolis forces are introduced, and (11) approaches (14) in the limit $\kappa \to 0$ (or $f \to 0$). The analytical difficulties of studying the geostrophic conservation equation (11a) are similar to those of studying (12b). However, in a large class of flow phenomena, experience has shown that a reasonable approximation is that the flow is irrotational ($\zeta = \Delta\Psi = 0$) almost everywhere except in small, isolated regions which can often be approximated by point singularities (or line singularities in three-dimensions). Then this approximation gives a good representation of the physical picture except in the immediate vicinity of the singularities. Well-known examples are von Karman's representation of the "vortex street" flow behind a circular cylinder and Prandtl's "horseshoe vortex" representation of a finite-span airfoil in uniform flow. Thus, in §5, we study the motion of geostrophic point vortices. Of course, meteorologists have studied atmospheric vortex motion for many years but in a more heuristic way—*e.g.*, James (1950). Related to the existence (Appendix II) of singular point vortex solutions of (11) and (12) is the observation that if we try to specialize to one space dimension (x or y), both (11) and (12) become time-independent. In fact, we have apparently reduced the number of possible time-varying solutions by a considerable factor in replacing (1) and (2) by the approximate system (11).

Some remarks are appropriate here about possible ways of choosing the parameter α for the approaches

[9] The operator $(\Delta - \kappa^2)$ might well be called the Helmholtzian.

[10] This result differs from conclusions reached by Bolin (1955) using heuristic arguments. The simplified two-layer model (attributed to Rossby) which he studies is included in the results of Appendix III, although he does not require f = constant.

[11] Of course, in applications, boundary conditions must also be considered. A thorough mathematical analysis has recently been performed by C. Sensenig (unpublished NYU doctoral thesis, 1958).

outlined above. So far, the only evident restriction on α is that $\alpha \ll 1$ and as $\alpha \to 0$ the solution $(h, u, v) \to (h_0, 0, 0)$, the flow at rest. Thus, we apparently have some freedom in the physical interpretation of α. Since α is related not only to the basic physical parameters f and g which appear as coefficients of the original equations (1) and (2), the physical interpretation must come from the initial and boundary conditions. For (1) linearizations, or (2) numerical methods, we can relate α to the maximum initial velocity; *e.g.*,

$$\alpha = \frac{f}{g}\max\{[u^{(1)}(x,y,0)]^2 + [v^{(1)}(x,y,0)]^2\}^{\frac{1}{2}} \quad (13a)$$
$$= \max|\text{grad } h^{(1)}|$$

or, as an alternative,

$$\alpha = (gh_0)^{-\frac{1}{2}}\cdot\max\{[u^{(1)}(x,y,0)]^2 + [v^{(1)}(x,y,0)]^2\}^{\frac{1}{2}}. \quad (13b)$$

For geostrophic point vortices (*cf.* §5) of strength γ_i, $i = 1, \cdots, n$ we can take

$$\alpha = \frac{f}{gh_0}\max|\gamma_i|. \quad (13c)$$

Since our approximation procedure is based on the introduction of a characteristic vertical depth h_0, and not a characteristic horizontal length as is frequently done in meteorological problems, it seems evident that h_0 is a more natural length scale in (13) than, for example, the mean radius of a cyclone. This preference can be illustrated by the following arguments: let us first introduce a horizontal length S as well as a characteristic vertical length H $(\equiv h_0)$ and a characteristic horizontal velocity V, and non-dimensionalize the momentum equation (2); then the low-velocity and geostrophic approximations state that

$$V^2 \ll gH = fSV$$

which implies

$$\frac{V^2}{gH} \ll 1 \quad \text{and} \quad \frac{V}{fS} \ll 1;$$

since we have introduced the depth h_0 in a rather natural way (*cf.* (5) and (7a) in §3), the first inequality (small Froude number) rather than the second (small Rossby number) is emphasized here.

Before actually using the geostrophic conservation equation (11) in specific problems, we can regard both the small parameter α and the depth h_0 as physical parameters which are at our disposal to help us fit as closely as we can this idealized model[12] of the one-layer homogeneous atmosphere to the actual physical problem. Exactly how to do this is a rather delicate question in itself; *e.g.*, Charney (1949), Bolin (1955), and others have studied this problem of fitting. *The primary role of α is to fix the time-scale* (see (6a)). However, we note that in applying (11) to physical problems, it is not necessary to give a physical interpretation to the parameter α since (11) can be expressed in terms of the actual time and the perturbation velocity $(\alpha u^{(1)}, \alpha v^{(1)})$; nevertheless, for the second-order approximate equations (9), an estimate for α is needed. The primary role of h_0 in (11) is to fix the balance of kinetic to potential energy (*cf.* remarks following (11)).

[12] Of course, later experience might show that the two-layer (or more) model given in Appendix III is necessary in some cases (*cf.* Phillips, 1951); some meteorologists have used from three to five layers.

5. Motion of geostrophic point vortices

We study the motion of n geostrophic point vortices in the entire x,y-plane without solid boundaries. First, consider a single vortex at the origin. The stream function ψ of this vortex of strength γ is the solution, which vanishes at ∞, of

$$(\Delta - \kappa^2)\psi = \gamma\delta(r) \quad (14a)$$

where $\delta(r)$ is the two-dimensional delta function and $r = (x^2 + y^2)^{\frac{1}{2}}$; the solution[13] of (14a) is

$$\psi = \frac{-\gamma}{2\pi}K_0(\kappa r) \quad (14b)$$

where $K_0(\kappa r)$ is the modified Bessel function of the second kind and zero order. The strength γ is obtained by integrating (14a) over an arbitrary region R with outer boundary C enclosing the vortex as follows:

$$\gamma \iint_R \delta(r)dxdy = \iint_R (\Delta - \kappa^2)\psi dxdy \quad (14c)$$

or

$$\gamma = \oint_C \psi_\nu ds - \kappa^2 \iint_R \psi dxdy \quad (14d)$$

where the subscript ν denotes differentiation in the direction of the outward-drawn normal to the contour C. The contour integral is the circulation, and the sign of γ has been so chosen that positive γ corresponds to a counter-clockwise rotating vortex (cyclonic). Since $(\Delta - \kappa^2)\psi = 0$ everywhere except at the origin, the vorticity distribution of a geostrophic vortex is

$$\zeta^{(1)} = \Delta\psi = \kappa^2\psi = -\frac{\gamma\kappa^2}{2\pi}K_0(\kappa r). \quad (14e)$$

We note that the vorticity $\zeta^{(1)}$ and stream function ψ have the same sign. The tangential velocity distribu-

[13] This derivation can be carried out in a more systematic way by transforming (11) to polar coordinates and seeking singular solutions satisfying the condition of zero radial flow.

tion is

$$\psi_r = -\frac{\gamma\kappa}{2\pi}K_0'(\kappa r) = \frac{\gamma\kappa}{2\pi}K_1(\kappa r). \qquad (14f)$$

Near the origin ($r \to 0$),

$$\psi_r \sim \frac{1}{r} \qquad (14g)$$

which behaves like the ordinary logarithmic vortex. At a large distance from the origin ($r \to \infty$),

$$\psi_r \sim (\kappa r)^{-\frac{1}{2}} e^{-\kappa r} \qquad (14h)$$

which damps out faster than the ordinary vortex. Equations (14) show that the geostrophic vortex has the qualitative features of a closed high- or low-pressure system in the atmosphere such as, for example, a hurricane (or typhoon). Clearly, the approximate representation of a hurricane by a geostrophic vortex breaks down completely in the immediate vicinity of the vortex point, corresponding to the eye of the hurricane; but this deviation is to be expected, even with more realistic solutions of (11), since both the hydrostatic and geostrophic approximations become invalid there. *The vortex (14) is the simplest possible closed model of geostrophic flow* and differs from the ordinary potential vortex in several respects: (1) the direction of rotation is coupled to the sign of the perturbed depth $h^{(1)}$ which corresponds to the perturbed pressure, (2) the motion is everywhere rotational as expressed by (14e), and (3) comparing vortices of the same strength, the influence of a geostrophic vortex given by (14b) is shorter-ranged than that of an ordinary vortex.

The motions of ordinary vortices have been studied extensively since the early work of Helmholtz and Kirchhoff—*e.g.*, see Lin (1943), and Lamb. However, nearly all the analyses[14] with which we are familiar rest on the implications of irrotational motion and Bernoulli's law, without direct reference to the vorticity equation (12b). To obtain the equations of motion of geostrophic vortices, we must of necessity base the derivation directly on the geostrophic conservation equation (11) without the aid of subsidiary conditions. This derivation, which of course also holds for ordinary vortices, (replacing $K_0(\kappa r)$ by ln (r)) is carried out in Appendix II by initially considering lumped, finite-area, symmetrical, strength distributions which do not overlap for all time $\tau \geq 0$. We summarize the results here.

For n vortices at points (x_i, y_i), the stream function ψ satisfies

$$(\Delta - \kappa^2)\psi = \sum_{i=1}^{n} \gamma_i \delta(|r - r_i(\tau)|). \qquad (15a)$$

[14] With the possible exception of a rather involved paper by Lichtenstein (1930).

Thus,

$$\psi = -\frac{1}{2\pi}\sum_{i=1}^{n} \gamma_i K_0(\kappa|r - r_i|) \qquad (15b)$$

where $(r - r_i) = [(x - x_i)^2 + (y - y_i)^2]^{\frac{1}{2}}$. The velocity of the k-th vortex is obtained by differentiating

$$\psi_{(k)}(x, y, \tau) = -\frac{1}{2\pi}\sum_{\substack{i=1\\ i\neq k}}^{n} \gamma_i K_0(\kappa|r - r_i|), \qquad (16a)$$

—that is, the *regular* (*non-singular*) *part* of (15b) at the vortex point (x_k, y_k), giving

$$\underset{(k)}{\overset{(1)}{u}} = -\frac{\partial}{\partial y}\psi_{(k)}(x_k, y_k, \tau) = \frac{1}{2\pi}\sum_{\substack{i=1\\ i\neq k}}^{n} \gamma_i \frac{(y_k - y_i)}{|r_k - r_i|} K_0'(\kappa|r_k - r_i|) = \frac{dx_k}{d\tau} \qquad (16b)$$

and

$$\underset{(k)}{\overset{(1)}{v}} = \frac{\partial}{\partial x}\psi_{(k)}(x_k, y_k, \tau) = -\frac{1}{2\pi}\sum_{\substack{i=1\\ i\neq k}}^{n} \gamma_i \frac{(x_k - x_i)}{|r_k - r_i|} K_0'(\kappa|r_k - r_i|) = \frac{dy_k}{d\tau}. \qquad (16c)$$

Kirchhoff has shown that (16) can be expressed in a more elegant way, and the motion of the k-th vortex is given by

$$\gamma_k \frac{dx_k}{d\tau} = \frac{\partial W}{\partial y_k}, \qquad (17a)$$

$$\gamma_k \frac{dy_k}{d\tau} = -\frac{\partial W}{\partial x_k} \qquad (17b)$$

where the Kirchhoff function W for geostrophic vortices is

$$W = \frac{1}{4\pi}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \gamma_i\gamma_j K_0(\kappa|r_j - r_i|). \qquad (17c)$$

This system of $2n$ first-order, ordinary, non-linear differential equations is somewhat more susceptible to analytical or computational treatment than (11), especially if the number of vortices is small.

The motion of a *single* geostrophic vortex embedded in a continuous-flow field is of some interest, particularly in view of possible application to the problem of predicting the motion (tracking) of a hurricane.[15] We

[15] A series of interesting papers on the prediction of hurricane motion (with some success) has been presented by Japanese meteorologists including Sasaki and Miyakoda (1955), Syono (1955), and recently Kasahara (1957). We seem to have some common general viewpoints, but their approach and method are

write the stream function ψ as the sum of an axially-symmetric function ψ_0 which may be singular and a continuous field ψ_1 as follows:

$$\psi = \psi_0(|r - r_0|) + \psi_1(x, y, \tau) \quad (18a)$$

where $[r - r_0(\tau)] = \{[x - x_0(\tau)]^2 + [y - y_0(\tau)]^2\}^{\frac{1}{2}}$ and ψ_0 is the particular solution of

$$(\Delta - \kappa^2)\psi = F_0(|r - r_0|). \quad (18b)$$

By substituting (18a) into (11), we obtain

$$\frac{d^{(1)}}{d\tau}(\Delta - \kappa^2)\psi_1 + \frac{F_0'}{|r - r_0|}\left\{(x - x_0)\left(u^{(1)} - \frac{dx_0}{d\tau}\right) + (y - y_0)\left(v^{(1)} - \frac{dy_0}{d\tau}\right)\right\} = 0 \quad (19a)$$

where

$$u^{(1)} = -(\psi_0 + \psi_1)_y, \quad v^{(1)} = (\psi_0 + \psi_1)_x. \quad (19b)$$

By following the same reasoning as in Appendix II, we choose

$$F_0(|r - r_0|) = \delta(|r - r_0|) \quad (20)$$

and

$$\frac{dx_0}{d\tau} = u^{(1)}_{(0)} = -(\psi_1)_y, \quad (21a)$$

$$\frac{dy_0}{d\tau} = v^{(1)}_{(0)} = (\psi_1)_x \quad (21b)$$

where $\left(u^{(1)}_{(0)}, v^{(1)}_{(0)}\right)$ is the velocity evaluated at the vortex point (x_0, y_0) using the regular part of ψ, namely ψ_1. Then the expression within the bracket in (19a) vanishes, and we obtain

$$\frac{d^{(1)}}{d\tau}(\Delta - \kappa^2)\psi_1 = 0. \quad (21c)$$

Equations (21a, b, c) are a system of three equations to be solved for $\psi_1(x, y, \tau)$, $x_0(\tau)$, and $y_0(\tau)$, given the initial (and boundary) conditions. Although we must take $\psi_0 = -(\gamma/2\pi)K_0(\kappa|r - r_0|)$ in this formulation, we believe that a non-singular ψ_0 may be used to good approximation, especially if ψ_0 is a rather sharply-peaked function as compared to ψ_1.

Care and ingenuity must be used in applying (and interpreting) the results of this paper to actual flows. For example, flows with large curvature such as near the eye of a hurricane can deviate considerably from geostrophic flow so that we certainly cannot expect good quantitative agreement in such regions. Also, in representing closed highs and lows by a distribution of geostrophic vortices, (*cf.* §1) there is some arbitrariness in choosing the distribution of both position (x_i, y_i) and strength γ_i. As a closing general remark, we point out that, as long as the flow region over the entire earth (or one hemisphere, at least) is not considered, the problem of specifying the correct boundary conditions will remain an extremely delicate one. This is true not only of (11) but of the original equations (1) and (2).

The author wishes to thank J. J. Stoker for introducing him to meteorological problems and E. Isaacson for innumerable suggestions and criticisms.

APPENDIX I. Ordering approximation procedure

For (1b) and (2), describing the one-layer, homogeneous atmosphere, consider the scale transformations (x, y, t) (original) $\rightarrow (\xi, \eta, \tau)$ (transformed) and $(\bar{h}, \bar{u}, \bar{v})$ (original) $\rightarrow (h, u, v)$ (transformed) with respect to a small parameter $\alpha \ll 1$:

$$\xi = \alpha^a x, \quad \eta = \alpha^b y, \quad \tau = \alpha^c t, \qquad \bar{u} = \alpha^l u, \quad \bar{v} = \alpha^m v, \quad \bar{h} = \alpha^n h, \quad (I.1)$$

combined with the perturbation expansion of (h, u, v)

$$h = h_0 + \alpha h^{(1)}(\xi, \eta, \tau) + \alpha^2 h^{(2)}(\xi, \eta, \tau) + \cdots,$$
$$u = \alpha u^{(1)}(\xi, \eta, \tau) + \alpha^2 u^{(2)}(\xi, \eta, \tau) + \cdots, \quad (I.2)$$
$$v = \alpha v^{(1)}(\xi, \eta, \tau) + \alpha^2 v^{(2)}(\xi, \eta, \tau) + \cdots.$$

To determine the unknown exponents a, b, c, l, m and n, we demand that $(h^{(1)}, u^{(1)}, v^{(1)})$ satisfy the following conditions:

(1) *Particle paths are invariant.*

$$\alpha^c \cdot \frac{d^{(1)}(\ )}{d\tau} = \alpha^c \cdot (\ )_\tau + \alpha^{a+l+1} \cdot u^{(1)} \cdot (\ )_\xi + \alpha^{b+m+1} \cdot v^{(1)} \cdot (\ )_\eta$$

yields

$$c = a + l + 1 = b + m + 1. \quad (I.3a)$$

(2) *Vorticity is invariant.*

$$\alpha^{a+m} \cdot v_\xi^{(1)} - g^{b+l} \cdot u_\eta^{(1)}$$

yields

$$a + m = b + l. \quad (I.3b)$$

(3) *Geostrophic approximation.*

$$\alpha^{c+l} \frac{d^{(1)}u^{(1)}}{d\tau} + \alpha^{a+n} \cdot gh_\xi^{(1)} - \alpha^m \cdot fv^{(1)} = 0$$

and

$$\alpha^{c+m} \frac{d^{(1)}v^{(1)}}{d\tau} + \alpha^{b+n} \cdot gh_\eta^{(1)} + \alpha^l \cdot fu^{(1)} = 0$$

yields

$$a + n = m, \quad c + l > m \quad (I.3c)$$

and

$$b + n = l, \quad c + m > l. \quad (I.3d)$$

of a more heuristic nature. In their formulation, they assume that the vortex does not influence the motion of the continuous flow field.

(I.3) are four *independent* equations for six unknowns; by solving in terms of c and n,

$$a=b=\tfrac{1}{2}(c-n-1), \quad l=m=\tfrac{1}{2}(c+n-1). \quad \text{(I.4a)}$$

The inequalities, equations (I.3c, d), reduce to

$$c > 0. \quad \text{(I.4b)}$$

The simplest scale transformation satisfying (I.4) is given by

$$n = 0, \quad c = 1 \quad \text{(I.4c)}$$

which makes $a = b = l = m = 0$. We obtain the extremely simple result that *time is the only scaled variable*, the transformation being $\tau = \alpha t$.

We note that if we attempt to incorporate the above conditions (1), (2), and (3) in (1b) and (2) by *scale transformation only* (*i.e., without a perturbation expansion*), we end up with two alternatives: (i) either f = constant and $h_\tau = 0$ *or* (*ii*) h progresses without change of shape in the *east-west direction* with a velocity $gh(f^{-1})_\eta$ which is an order of magnitude larger than the gravity wave velocity $(gh)^{\frac{1}{2}}$. (ii) is physically unacceptable, but (i) is consistent with our basic solution on which we perturb—*i.e.*, the atmosphere at rest, $(h_0, 0, 0)$. For completeness, the stretching transformation for this case is defined by the following relationships between exponents:

$$a = b = \tfrac{1}{2}(c - n), \quad l = m = \tfrac{1}{2}(c + n) \quad \text{(I.5a)}$$

and the inequality

$$c > 0. \quad \text{(I.5b)}$$

For example, we can choose

$$n = 0, \quad c = 1 \quad \text{(I.5c)}$$

which makes $a = b = l = m = \frac{1}{2}$. In addition, it is of interest to point out some possible inconsistencies which we might face if we blithely try to introduce the geostrophic approximation into the one-layer, homogeneous atmosphere equations (1a) (or (1b)) and (2) by merely neglecting the inertia terms in (2),

$$gh_x - fu = 0, \quad \text{(I.6a)}$$

$$gh_y + fv = 0. \quad \text{(I.6b)}$$

The implication of alternative (i) above and (8) (*cf.* discussion following (8)) is that f = constant is consistent with (1a) (or (1b)) and (2). The consequences of taking f = constant and (I.6) are quite drastic; it follows then that div $(u, v) = 0$ which implies from (1a) that $dh/dt = 0$. Then, from (1b), $dh/dt = 0$ implies that $d(\zeta + f)/dt = d\zeta/dt = 0$ which implies (1b) *but not the converse.* Hence, we conclude that (I.6) and (1b) *do not form a consistent system of equations*, and the approximation procedure described in §3 yielding (to lowest order) the geostrophic conservation equation (8) or (11) remains the only consistent way we know of introducing the geostrophic approximation in (1a) (or (1b)) and (2).

APPENDIX II. Derivation[16] of the equations of motion of geostrophic point vortices

Consider each of n vortex points P_i covered by a smooth symmetrical distribution function[17] $F(r_{xx_i})$, where $r_{xx_i} = [(x - x_i(\tau))^2 + (y - y_i(\tau))^2]^{\frac{1}{2}}$, over a finite circular area R_i, with $F = 0$ outside R_i. It is implied that these P_i-lumped distributions do not overlap for all $\tau \geq 0$. Then the solution of

$$(\Delta - \kappa^2)\psi = \sum_{i=1}^{n} \gamma_i F(r_{xx_i}) \quad \text{(II.1)}$$

is

$$\psi = \sum_{i=1}^{n} \gamma_i \Psi(r_{xx_i}) \quad \text{(II.2a)}$$

where

$$\Psi(r_{xx_i}) = \frac{-1}{2\pi} \iint_{R_i} K_0(\kappa r_{x\xi}) \cdot F(r_{\xi x_i}) d\xi d\eta \quad \text{(II.2b)}$$

is also a smooth symmetrical function with respect to each P_i; γ_i is the vortex strength (assumed to be constant). By the geostrophic conservation equation (11a) and (II.1), we formally write

$$\frac{d^{(1)}}{d\tau}\left\{ \sum_{i=1}^{n} \gamma_i F(r_{xx_i}) \right\} = 0 \quad \text{(II.3a)}$$

or

$$\sum_{i=1}^{n} \frac{\gamma_i F'(r_{xx_i})}{r_{xx_i}} \left\{ [(x - x_i)u^{(1)} + (y - y_i)v^{(1)}] - \left[(x - x_i)\frac{dx_i}{d\tau} + (y - y_i)\frac{dy_i}{d\tau} \right] \right\} = 0. \quad \text{(II.3b)}$$

Since, by construction, $F'(r_{xx_i}) = 0$ for all points outside of R_i, (II.3b) is identically satisfied there; and we need only consider (x, y) inside R_k, for instance, for the k-th vortex point P_k. Then (II.3b) becomes

$$\frac{\gamma_k F'(r_{xx_k})}{r_{xx_k}} \left\{ [(x - x_k)u^{(1)} + (y - y_k)v^{(1)}] - \left[(x - x_k)\frac{dx_k}{d\tau} + (y - y_k)\frac{dy_k}{d\tau} \right] \right\} = 0. \quad \text{(II.4)}$$

Now, by equations (11b, c) and (II.2), we can write

$$[(x - x_k)u^{(1)} + (y - y_k)v^{(1)}] = \left\{ (y - y_k)\frac{\partial}{\partial x} - (x - x_k)\frac{\partial}{\partial y} \right\} \sum_{i=1}^{n} \gamma_i \Psi(r_{xx_i}). \quad \text{(II.5)}$$

But, since $\Psi(r_{xx_k})$ is a symmetrical function about the point P_k independent of θ_k, we have

[16] With B. Zumino.

[17] A more manageable symbolic notation is used here for the distance $r_{xx_i} \equiv |r - r_i|$ (*cf.* (17)).

$$\left\{ (y - y_k) \frac{\partial}{\partial x} - (x - x_k) \frac{\partial}{\partial y} \right\} \Psi(r_{xx_k}) = - \frac{\partial}{\partial \theta_k} \Psi(r_{xx_k}) = 0. \quad \text{(II.6)}$$

Hence, in equation (II.4), we can replace $(u^{(1)}, v^{(1)})$ by $\left(\underset{(k)}{\overset{(1)}{u}}, \underset{(k)}{\overset{(1)}{v}}\right)$ where

$$\underset{(k)}{\overset{(1)}{u}} = - \frac{\partial}{\partial y} \psi_{(k)}, \quad \underset{(k)}{\overset{(1)}{v}} = \frac{\partial}{\partial x} \psi_{(k)} \quad \text{(II.7a)}$$

and $\psi_{(k)}$ is the "regular part" of ψ; *i.e.*,

$$\psi_{(k)} = \sum_{\substack{i=1 \\ i \neq k}}^{n} \gamma_i \Psi(r_{xx_i}). \quad \text{(II.7b)}$$

Thus, equation (II.4) becomes

$$\gamma_k F'(r_{xx_k}) \left\{ \left(\underset{(k)}{\overset{(1)}{u}} - \frac{dx_k}{d\tau} \right) \cos \theta_k + \left(\underset{(k)}{\overset{(1)}{v}} - \frac{dy_k}{d\tau} \right) \sin \theta_k \right\} = 0. \quad \text{(II.8)}$$

If we now choose F to be such a function that $\lim_{R_b \to 0} F(r_{xx_k}) = \delta(r_{xx_k})$, the Dirac δ-function, then equation (II.8) is satisfied by

$$\underset{(k)}{\overset{(1)}{u}} - \frac{dx_k}{d\tau} = 0, \quad \text{(II.9a)}$$

$$\underset{(k)}{\overset{(1)}{v}} - \frac{dy_k}{d\tau} = 0 \quad \text{(II.9b)}$$

where $\left(\underset{(k)}{\overset{(1)}{u}}, \underset{(k)}{\overset{(1)}{v}}\right)$ is evaluated at $(x, y) = (x_k, y_k)$. The symmetry of the δ-function is implied by the symmetry of $\Psi(r_{xx_k})$ which, by equations (II.1) and (II.2), becomes $\Psi(r_{xx_k}) = -(1/2\pi)K_0(\kappa r_{xx_k})$. Thus, by choosing the distribution function F properly, we have replaced the geostrophic conservation equation (11) by equations (II.9) describing the motion of geostrophic point vortices.

APPENDIX III. Geostrophic conservation equations for a two-layer atmosphere[18]

To be concise, the equations describing the two-layer atmosphere are given here in conservation vector form.

Continuity:

$$\frac{d_i}{dt} (h_i) + h_i \operatorname{div} \vec{q}_i = 0. \quad \text{(III.1)}$$

Vorticity:

$$\frac{d_i}{dt} \left(\frac{\zeta_i + f}{h_i} \right) = 0. \quad \text{(III.2)}$$

[18] With S. C. Lowell.

Momentum:

$$\frac{d_i}{dt} (\vec{q}_i) + g \operatorname{grad} (h_1 + \beta_i h_2) + f[\vec{k} \times \vec{q}_i] = 0. \quad \text{(III.3)}$$

Characteristics:

$$\left(\frac{d_1\Phi}{dt}\right)\left(\frac{d_2\Phi}{dt}\right)\left\{\left[\left(\frac{d_1\Phi}{dt}\right)^2 - gh_1|\nabla\Phi|^2\right]\left[\left(\frac{d_2\Phi}{dt}\right)^2 - gh_2|\nabla\Phi|^2\right] - \beta_1(gh_1)(gh_2)|\nabla\Phi|^4\right\} = 0 \quad \text{(III.4)}$$

where

$$\frac{d_i}{dt}(\ ) = (\ )_t + u_i \cdot (\ )_x + v_i \cdot (\ )_y \quad \text{(III.5)}$$

and subscripts $i = 1, 2$ refer to the lower and upper layers respectively (no summation); $\vec{q}_i = (u_i, v_i)$; $\zeta_i = (v_i)_x - (u_i)_y$; $\beta_1 = \rho_2/\rho_1 < 1$, $\beta_2 = 1$; $\vec{k}$ is the unit vector in the vertical direction z.

It is a simple matter to follow the same ordering approximation procedure outlined in §3. By defining

$$\psi_i = \frac{g}{f} (h_1^{(1)} + \beta_i h_2^{(1)}), \quad \text{(III.6)}$$

which corresponds to (10) in the one-layer case, we obtain

$$\frac{d_i^{(1)}}{d\tau} \{\Delta\psi_i - \kappa_i^2(\psi_i - \beta_i\psi_2)\} = 0, \quad \text{(III.7a)}$$

$$\vec{q}_i^{(1)} - [\vec{k} \times \nabla\psi_i] = 0 \quad \text{(III.7b)}$$

where

$$\frac{d_i^{(1)}}{d\tau}(\ ) = (\ )_\tau + u_i^{(1)} \cdot (\ )_x + v_i^{(1)} \cdot (\ )_y \quad \text{(III.7c)}$$

and

$$\kappa_i^2 = \frac{f^2}{gh_{0i}(1 - \beta_1)}. \quad \text{(III.7d)}$$

Equations (III.7a) are the two-layer geostrophic conservation equations, a system of two (coupled) third-order, non-linear differential equations for $\psi_1(x, y, \tau)$ and $\psi_2(x, y, \tau)$, which correspond to (11a) in the one-layer case. (III.7b) are the geostrophic conditions (*cf.* (11b)) and the vorticity $\xi_i^{(1)} = \Delta\psi_i$. For this coupled system, the sixth-degree characteristics equation (III.4) degenerates (to this order of approximation) to the quadratic equation

$$\left(\frac{d_1^{(1)}\Phi^{(1)}}{d\tau}\right)\left(\frac{d_2^{(1)}\Phi^{(1)}}{d\tau}\right) = 0 \quad \text{(III.8)}$$

for the two particle paths in each layer (*cf.* first factor in (8g)).

REFERENCES

1. Blinova, E. N., and I. A. Kibel, 1957: Hydrodynamical methods of the short and long-range weather forecasting in the USSR. *Tellus*, **9**, 447–463.
2. Bolin, B., 1955: Numerical forecasting with the barotropic model. *Tellus*, **7**, 27–49.
3. Bolin, B., 1956: An improved barotropic model and some aspects of using the balance equation for three-dimensional flow. *Tellus*, **8**, 61–75.
4. Charney, J. G., 1948: On the scale of atmospheric motions. *Geofys. Publ.* **17**, No. 2, 17 pp.
5. Charney, J. G., 1949: On a physical basis for numerical prediction of large-scale motions in the atmosphere. *J. Meteor.*, **6**, 371–385.
6. Charney, J. G. 1955: The use of the primitive equations of motion in numerical forecasting. *Tellus*, **7**, 22–26.
7. Charney, J. G., R. Fjørtoft, and J. von Neumann, 1950: Numerical integration of the barotropic vorticity equation. *Tellus*, **2**, 237–254.
8. Courant, R., and D. Hilbert, 1937: *Methoden der Mathematischen Physik, Vol. II.* Berlin, Julius Springer, 549 pp.
9. Friedrichs, K. O., 1948: On the derivation of the shallow water theory (Appendix to "The formation of breakers and bores" by J. J. Stoker). *Comm. Appl. Math.*, **1**, 81–85.
10. Friedrichs, K. O., 1955: Asymptotic phenomena in mathematical physics. *Bull. Amer. math. Soc.*, **61**, 485–504.
11. James, R. W., 1950: On the theory of large scale vortex motion in the atmosphere. *Quart. J. r. meteor. Soc.*, **76**, 255–276.
12. Kasahara, A., 1957: The numerical prediction of hurricane movement with the barotropic model. *J. Meteor.*, **14**, 386–402.
13. Keller, J. B., 1948: The solitary wave and periodic waves in shallow water. *Comm. Appl. Math. and Mech.*, **1**, 323–329.
14. Lamb, H., 1932: *Hydrodynamics*, sixth ed. Cambridge Univ. Press, 738 pp.
15. Lichtenstein, L., 1930: Über einige Existenzprobleme der Hydrodynamik. *Math. Zeits.*, **32**, 608–640.
16. Lin, C. C., 1943: *On the motion of vortices in two dimensions.* Appl. Math. Series No. 5, Toronto, Univ. Toronto Press, 39 pp.
17. Morikawa, G. K., 1957: Non-linear diffusion of flood waves in rivers. *Comm. Pure and Appl. Math.*, **10**, 291–303.
18. Phillips, N. A., 1951: A simple three-dimensional model for the study of large-scale extratropical flow patterns. *J. Meteor.*, **8**, 381–394.
19. Prandtl, L., 1952: *Essentials of fluids dynamics.* New York, Hafner, 452 pp.
20. Rossby, C. G., 1940: Planetary flow patterns in the atmosphere. *Quart. J. r. meteor. Soc.*, **66**, supplement, 68–87.
21. Sasaki, Y., and K. Miyakoda, 1954: *Prediction of typhoon tracks on the basis of numerical weather forecasting methods.* Proc. UNESCO Sympos. on Typhoons, 221–233.
22. Sensenig, C., 1958: *Existence and uniqueness for a third order, non-linear partial differential equation.* Unpubl. Ph.D. thesis, New York Univ.
23. Stoker, J. J., 1956: *Water waves.* New York, Interscience Pub. Co., 567 pp.
24. Syono, S., 1955: On the motion of a typhoon. *J. meteor. Soc. Japan*, Series II, **33**, 245–261.
25. Thompson, P. D., 1953: On the theory of large-scale disturbances in a two-dimensional baroclinic equivalent of the atmosphere. *Quart. J. r. meteor. Soc.*, **79**, 51–69.
26. Thompson, P. D., 1956: A theory of large-scale disturbances in non-geostrophic flow. *J. Meteor.*, **13**, 251–261.