A SIMPLIFIED LINEAR THEORY OF EQUATORIAL EASTERLY WAVES

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ABSTRACT

If the perturbation of the zonal wind component and the Coriolis term which arises in the zonal equation of motion as a result of vertical motions are neglected, the linearized vorticity equation and the continuity equation (when written in pressure coordinates) become a complete set for the meridional wind and vertical motion perturbations. This set is solved for a class of easterly waves which reach their maximum intensity at the equator and dampen poleward.

The theoretical streamlines and the theoretical field of divergence both agree quite well with their empirical counterparts. On the other hand, the theoretical isotachs are somewhat distorted and the theoretical phase speed is a bit low.

1. Introduction

Analytic treatment of very-low-latitude, time-dependent flow is exceedingly difficult even after the governing equations have been linearized. The difficulties stem mainly from the need to allow the Coriolis parameter to vary during integration as well as in differentiation. This is in contrast to higher latitude dynamics where realistic solutions may often be obtained by holding the Coriolis parameter constant during integration. As pointed out by Schmidt [1], a completely variable Coriolis parameter generally leads, after linearization and separation of the independent variables, to higher-order ordinary differential equations with variable coefficients. There are no general methods for solving such equations. Solutions, when attainable, are usually in the form of infinite series. Schmidt [1] illustrates the solution procedure for wave motions which propagate in the meridional direction and which are independent of longitude.

Solutions in closed form may, however, be obtained when certain other simplifications are made. Sellick [2] was able to obtain closed solutions (from the non-linear equations of motion in spherical coordinates) after assuming the motion to be horizontal and non-divergent and with the neglect of certain small terms. Sherman [3] obtained closed solutions for permanent type motions from the linearized vector vorticity equation and the continuity equation (spherical coordinates). In this work, however, Sherman used the device of completely specifying one of the dependent variables. The remaining variables were then obtained by direct integration.

In an earlier paper [4], Sherman obtained the phase speed of zonally propagating waves in a zonal basic current (which varied as the cosine of the latitude). Here, Sherman linearized the scalar vorticity and continuity equations, used a spherical coordinate system and neglected the zonal velocity perturbation as well as the compressibility of the fluid.

In the present paper, the mechanics of a class of equatorial waves, whose empirical characteristics have been described by Palmer [5], will be explored. Figs. 1, 2 and 3, taken from [5], illustrate, respectively, the basic current, the horizontal velocity perturbation and the vertical velocity perturbation. As Palmer [5] points out, these diagrams are based on synoptic experience in the central parts of the Pacific Ocean. The disturbances under discussion generally have a wave length of around 15 deg long and a phase speed of from 10 to 15 kn [5].

The system illustrated by figs. 1, 2 and 3 is one in which the meteorological equator coincides with the geographical equator. This simplification will also be made in the theoretical model presented in the next section. As will be noted from the diagrams, waves of this type show their maximum amplitude at the equator and dampen poleward.

The method of attack used here is similar to that of Sherman [4] in that the perturbation of the zonal wind component is neglected. This may be partially justified by noting that recent computations of wind spectra at equatorial stations [6] seem to indicate that waves of the type considered here manifest themselves mainly in the meridional wind component. The present work differs from [4] in several respects. Sherman assumed, at the start, that the amplitude of the meridional wind perturbation vanishes at some latitude. In our development, the poleward damping of the waves is a natural consequence of the theory. Also, we go beyond the frequency equation and obtain

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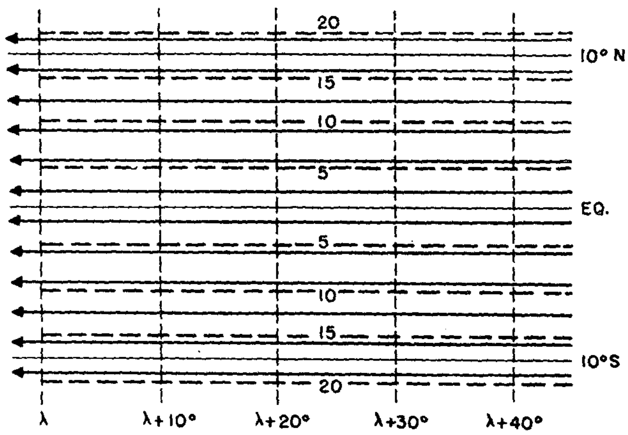

FIG. 1. Idealized basic current, Central Pacific (after C. E. Palmer). Solid lines are streamlines. Dashed lines are isotachs (knots).

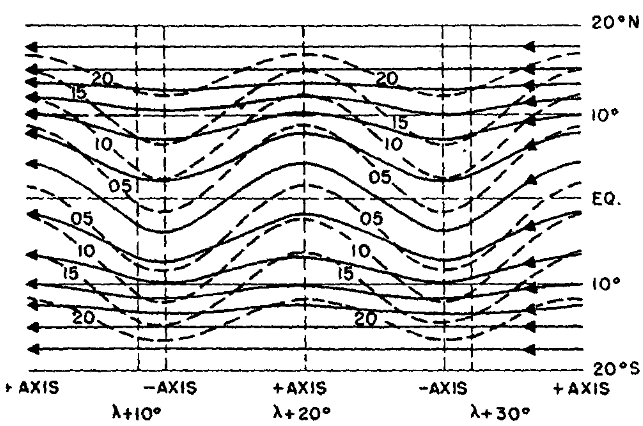

FIG. 2. Basic current of fig. 1 disturbed by an equatorial wave (after C. E. Palmer). Solid lines are streamlines. Dashed lines are isotach (knots).

expressions for the meridional wind and vertical-motion perturbations. We have, on the other hand, made assumptions not found in Sherman's development. We assume a quasi-hydrostatic atmosphere and use pressure coordinates (Sherman used level coordinates but neglected the solenoid term in the vorticity equation and the individual rate of change of density in the continuity equation). We have neglected the curvature of the earth through the use of cartesian coordinates. This is probably not a severe restriction. We have also neglected the Coriolis term which arises in the zonal equation of motion as a result of vertical motion. This may very well be a serious restriction since this term depends, partially, on the cosine of the latitude which, of course, reaches a maximum at the equator. Finally we assume the Coriolis parameter to be a linear function of meridional distance.

2. Theory

If the basic current is assumed to be zonal and a function of latitude only, then, with the assumptions

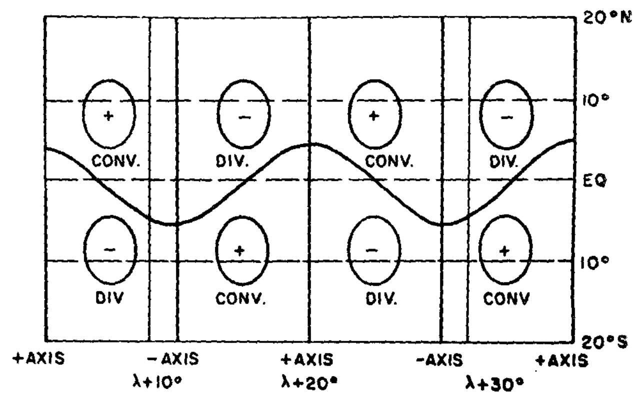

FIG. 3. Qualitative estimate of the vertical velocity in the lower atmosphere (after C. E. Palmer).

outlined above and with the neglect of friction, the linearized vorticity equation and the continuity equation, in pressure coordinates, may be written as

$$\frac{\partial^2 v}{\partial x t} - E\frac{\partial^2 v}{\partial x^2} + \left(\beta y + \frac{dE}{dy}\right)\frac{\partial v}{\partial y} + \left(\beta + \frac{d^2E}{dy^2}\right)v = 0 \quad (1)$$

$$\frac{\partial \omega}{\partial p} = -\frac{\partial v}{\partial y}. \quad (2)$$

Here, v is the perturbation of the meridional velocity component, E is the zonal (easterly) current, β is the rate of change of the Coriolis parameter with meridional distance (assumed constant), ω is the individual rate of change of pressure, x is zonal distance (positive eastward), y is meridional distance (positive northward, origin at the equator) and p is pressure.

We assume solutions of the form

$$v = G(p)F(y)\sin k(x + ct) \quad (3)$$

$$\omega = H(p)M(y)\sin k(x + ct) \quad (4)$$

where k is the wave number (2π per wave length) and c is the phase speed. It will be noted that in our coordinate system a positive value of c corresponds to a wave which propagates westward. When (3) is substituted into (1), we obtain

$$\left(\beta y + \frac{dE}{dy}\right)\frac{dF}{dy} + \left[\beta + \frac{d^2E}{dy^2} - k^2c + k^2E\right]F = 0. \quad (5)$$

We are concerned with waves whose amplitude is a maximum at the equator. A reasonable side condition is then

$$\left(\frac{dF}{dy}\right)_{y=0} = 0. \quad (6)$$

Thus, at the equator, eq (5) reduces to

$$F(0)\left[\beta + \left(\frac{d^2E}{dy^2}\right)_{y=0} - k^2c + k^2E(0)\right] = 0. \quad (7)$$

Since $F(0) \neq 0$, (7) leads to

$$c = E(0) + \left[\beta + \left(\frac{d^2E}{dy^2}\right)_{y=0}\right]k^{-2}, \quad (8)$$

which is the frequency equation for the waves under discussion. We see, at once, that c can never be complex; hence, the waves (under our assumptions and side conditions) are always of neutral stability. If the zonal current is not a function of y, eq (8) reduces to the Rossby equation for non-divergent waves in an easterly current.

We will assume $E(y)$ to be a parabola,

$$E(y) = E_0 + Sy^2, \ (E_0 \geq 0, S \geq 0). \quad (9)$$

This is in fair agreement with fig. 1. When (9) is substituted into (5) and (8), we obtain, respectively,

$$y(\beta + 2S)\frac{dF}{dy} + [\beta + 2S - k^2c + k^2E_0 + k^2Sy^2]F = 0 \quad (10)$$

and

$$c = E_0 + (\beta + 2S)k^{-2}. \quad (11)$$

Combination of (10) and (11) yields, after minor rearrangement,

$$d(\ln F) + k^2S(\beta + 2S)^{-1}y\,dy = 0. \quad (12)$$

Integration of (12) gives

$$F = K_1 \exp\{-k^2S[2(\beta + 2S)]^{-1}y^2\}, \quad (13)$$

where K_1 is a constant of integration. We note that the perturbation dampens exponentially as the square of distance from the equator.

The remaining unknown functions, G, H and M, may now be found by substituting (3) and (4) into (2) to obtain

$$M\frac{dH}{dp} = -G\frac{dF}{dy}. \quad (14)$$

Division of (14) by the product MG separates the independent variables and leads to the two equations

$$\frac{dH}{dp} = \epsilon G \quad (15)$$

and

$$\frac{dF}{dy} = -\epsilon M \quad (16)$$

where ϵ is the separation constant. From (16) and (13), we obtain

$$M = 2\delta y\epsilon^{-1}K_1e^{-\delta y^2}, \quad (17)$$

where

$$\delta = k^2S/2(\beta + 2S). \quad (18)$$

The only restrictions on H and G are that they satisfy (15) and proper boundary conditions. We take the following boundary conditions.

$$\omega = 0, \quad p = p_1 = 100 \text{ mb (the tropopause)}$$

and

$$\omega = 0, \quad p = p_0 = 1000 \text{ mb}, \quad (19)$$

where it is agreed that the solution is only valid in the troposphere. These conditions are satisfied if we take

$$H = K_2 \sin\frac{\pi}{2}\left(\frac{p_0 - p}{p_0 - p^*}\right) \quad (20)$$

and

$$G = -\frac{K_2\pi}{2\epsilon(p_0 - p^*)}\cos\frac{\pi}{2}\left(\frac{p_0 - p}{p_0 - p^*}\right), \quad (21)$$

where $p^* = 550$ mb, K_2 is a constant, and $p \geq p_1$.

From (3), (4), (13), (17), (20) and (21), we obtain

$$v = -\frac{K_1K_2\pi}{2\epsilon(p_0 - p^*)}e^{-\delta y^2}\cos\frac{\pi}{2}\left(\frac{p_0 - p}{p_0 - p^*}\right)\sin k(x + ct) \quad (22)$$

and

$$\omega = \frac{2\delta K_1K_2}{\epsilon}ye^{-\delta y^2}\sin\frac{\pi}{2}\left(\frac{p_0 - p}{p_0 - p^*}\right)\sin k(x + ct). \quad (23)$$

If we designate $v = v_0$ at $y = 0$, $p = p_0$, $\sin k(x + ct) = 1$, then, from (22),

$$\frac{K_1K_2}{\epsilon} = -\frac{2v_0(p_0 - p^*)}{\pi}, \quad (24)$$

so that (22) and (23) may be written as

$$v = v_0e^{-\delta y^2}\cos\frac{\pi}{2}\left(\frac{p_0 - p}{p_0 - p^*}\right)\sin k(x + ct) \quad (25)$$

and

$$\omega = -\frac{4v_0(p_0 - p^*)\delta}{\pi}ye^{-\delta y^2}\sin\frac{\pi}{2}\left(\frac{p_0 - p}{p_0 - p^*}\right) \times \sin k(x + ct) \quad (26)$$

for $p \geq p_1$.

3. Discussion of the solutions

Figs. 4 and 5 show the theoretical results which are to be compared to the empirical models found in figs. 2 and 3. The following realistic values of the parameters were selected for construction of figs. 4 and 5: $v_0 = 3 \text{ m sec}^{-1}$, $E_0 = 1.5 \text{ m sec}^{-1}$, $L = 1700$ km, $S = 3.6 \times 10^{-12} \text{ m}^{-1} \text{ sec}^{-1}$. The value of S given above

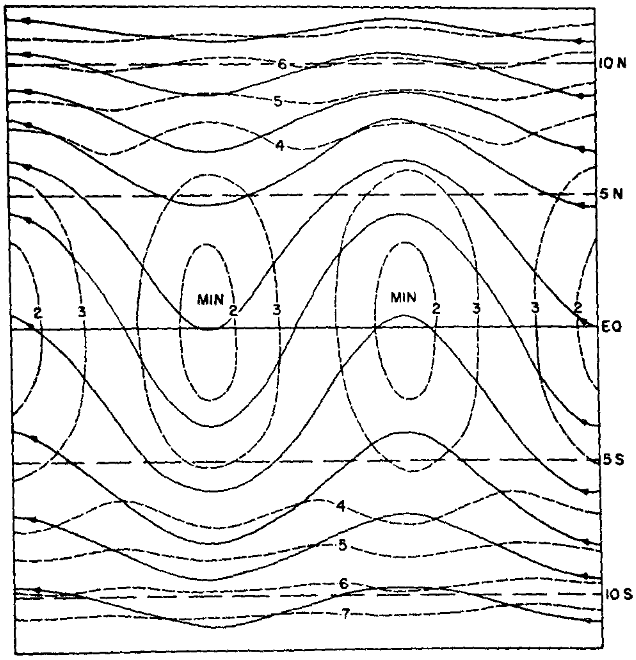

FIG. 4. Theoretical streamlines (solid) and isotachs (dashed) at 1000 mb. See text for values of the parameters. Speeds are given in meters per second.

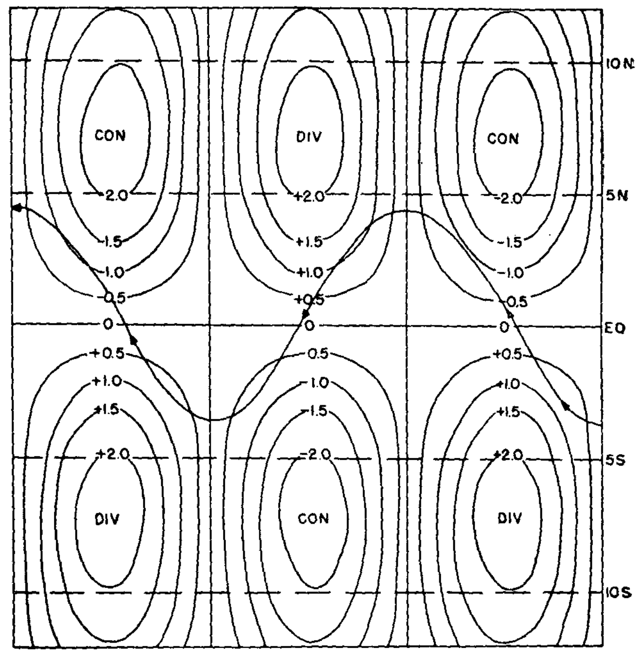

FIG. 5. Theoretical field of horizontal divergence at 1000 mb. Values are 10^{-6} seconds^{-1}. See text for values of the parameters.

corresponds to a shear in $E(\Delta E)$ of 10 m sec^{-1} between the equator and 15 deg lat.

It will be noted that the streamlines of fig. 4 have a very strong resemblance to the empirical streamlines (fig. 2). Although the poleward damping of the theoretical streamlines is, in part, predetermined by the choice of basic current, the result that the function $F(y)$ is a maximum at the equator (rather than a minimum) comes directly from the theory, and, without this result, the agreement between theoretical and empirical streamlines would not be nearly so good.

The theoretical isotachs of fig. 4 are not in good agreement with the empirical isotachs. While the empirical isotachs show a line of minimum speed oscillating about the equator and in phase with the streamlines, the theoretical isotachs show a series of speed minima centered on the equator and separated from each other by narrow necks of higher speeds. The theoretical isotachs tend to stretch in the meridional direction, whereas the empirical isotachs are elongated, essentially, in the zonal direction. The distortion of the theoretical isotachs must be attributed to the form of the solution assumed for v. A form, more realistic than (3), would be

$$v = G(p)N(x,y) \sin k(x + ct). \qquad (27)$$

With (27), a side condition of the type

$$y = \Gamma \sin kx \quad \text{when} \quad \frac{\partial N}{\partial y} = 0 \qquad (28)$$

would have been more realistic than (6). However, the use of (27) and (28) in place of (3) and (6) would have rendered the problem hopelessly complex.

Fig. 5 shows the theoretical field of horizontal divergence at 1000 mb. Comparison with fig. 3 shows that theory and nature agree quite well here. The magnitude of the divergences given by the theory is in good agreement with those found in Palmer's paper [5].

Fig. 6 illustrates the poleward damping of the v perturbation (at $p = p_0$, $\sin k(x + ct) = 1$) for various wave lengths. The parameter S was evaluated for a ΔE of 10 m sec^{-1}. For wave lengths of the magnitude given in Palmer's model (15 deg long $\approx$ 1700 km (at the equator)), fig. 4 shows that the waves dampen very rapidly with latitude. At 20 deg lat, the amplitudes are less than one tenth of their equatorial values. For longer wave lengths, the damping is less rapid.

If (25) is differentiated twice with respect to y and if the resulting equation is set equal to zero, we can compute the latitude (y_m) at which the divergence and vertical motion have their greatest magnitudes. Such a computation gives

$$y_m = \pm (2\delta)^{-\frac{1}{2}}, \ p \neq p^*, \ \sin k(x + ct) \neq 0. \qquad (29)$$

For $\Delta E = 10$ m sec^{-1} and for a wave length of 1700 km, the divergence has its greatest magnitude at 7 deg lat. This is in quite good agreement with fig. 3. The magnitude of the maximum divergence

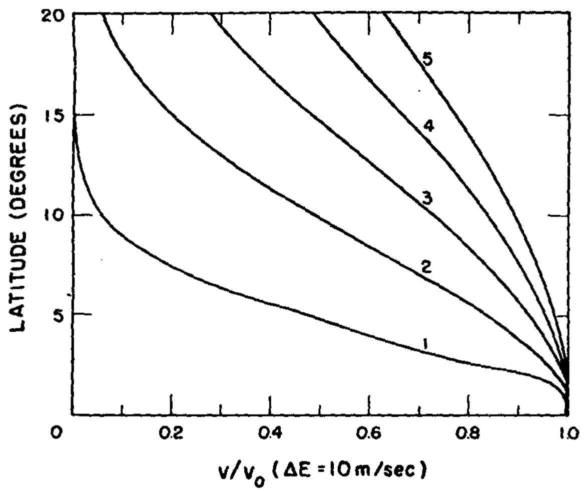

FIG. 6. Latitudinal damping of v perturbation for: $\sin k(x + ct) = 1$, $p = 1000$ mb, $\Delta E = 10$ m-sec^{-1}. Isopleth labels are wave length in thousands of kilometers.

($|\partial v/\partial y|_m$) can be computed from

$$\left|\frac{\partial v}{\partial y}\right|_m = v_0(2\delta)^{\frac{1}{2}} e^{-\frac{1}{2}}. \tag{30}$$

For $\Delta E = 10$ m sec^{-1} and for a wave length of 1700 km, $|\partial v/\partial y|_m$ is about 2×10^{-6} sec^{-1} when v_0 is 3 m sec^{-1}. As noted earlier, the order of magnitude is consistent with that given in Palmer's paper [5]. A simple calculation shows that the maximum perturbation vorticity is about an order of magnitude greater than $|\partial v/\partial y|_m$ for $v_0 = 3$ m sec^{-1} and $L = 1700$ km.

With regard to the phase speed of these waves, for a wave length of 1700 km and $\Delta E = 10$ m sec^{-1}, $k^{-2}(\beta + 2S)$ is only 3 m sec^{-1} or about 6 kn. If $E_0 = 1.5$ m sec^{-1}, the waves propagate at about 9 kn. This value is a little on the low side.

From eq (11), we see that at the equator the waves propagate more rapidly than the speed of the basic current. However the zonal current increases with latitude; hence, one eventually reaches latitudes where the wave propagates less rapidly than the speed of the current. It is easy to show that the latitude where $c - E = 0$ is the same as the latitude of maximum divergence (eq 29).

4. Summary

A simplified linear theory of equatorial waves has been presented. Major assumptions of the theory include (1) neglect of the zonal perturbation, (2) neglect of the Coriolis term which arises due to vertical motions in the zonal equation of motion, (3) a Coriolis parameter which is a linear function of meridional distance, (4) a zonal current which varies parabolically with meridional distance, (5) neglect of the earth's curvature and (6) neglect of friction.

The theoretical streamlines and low-level divergences are in excellent agreement with the corresponding empirical distributions. The theoretical speed field is, on the other hand, greatly distorted and a poor representation of the empirical speed field.

REFERENCES

1. Schmidt, F. H., 1949: On the theory of small disturbances in equatorial regions. *J. Meteor.*, **6**, 427–428.
2. Sellick, N. P., 1950: Equatorial circulations. *Quart. J. r. meteor. Soc.*, **76**, 89–94.
3. Sherman, L., 1951: On permanent perturbations of zonal atmospheric motion, with applications to low latitudes. *J. Meteor.*, **8**, 84–94.
4. Palmer, C. E., and L. Sherman, 1950: Seventh report (January 1950) of the tropical Pacific project. Inst. Geophys., Univ. Calif. at Los Angeles, 9 pp.
5. Palmer, C. E., 1952: Tropical meteorology. *Quart. J. r. meteor. Soc.*, **78**, 126–163.
6. Rosenthal, S. L., 1959: Some estimates of the power spectra of large-scale disturbances in lower altitudes. *J. Meteor.*, **17**, 259–263.